Notes on continuous tensor network states

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Abstract

A set of biased notes on continuous tensor networks. As of September 30, 2021 it only covers the basics of CMPS.

1 Introduction

Tensor network state methods are powerful to solve many-body quantum problems on the lattice, and it is tempting to extend this success to the continuum. Abstracting away the particular optimization algorithms, tensor networks work by constructing an appropriate variational submanifold $\mathcal{M}_D$ of the many-body Hilbert space $\mathscr{H}$, on which one can *e.g.* minimize a Hamiltonian to find the ground state, or approximate a real-time evolution. This submanifold grows larger as the *bond dimension* D is increased, which increases precision, until the whole Hilbert space is exactly covered (for exponentially large values of D). In contrast with the dimension of the Hilbert space $\dim \mathscr{H}$ that grows exponentially in the number of lattice sites N, tensor network states provide extensive representations: their number of parameters grows only linearly in N (and polynomially in D). In the translation invariant case, one can even work directly in the thermodynamic limit. Finally, the compression provided by tensor network states is efficient for low energy states of local Hamiltonians in the sense that the approximation error for local observables on these states decreases superpolynomially in the bond dimension D.

A first method to deal with continuum problems is to bring the models to the method: tensor network states work well already for lattice problems, so simply discretize the models and use all the tools available on the lattice (this is the same philosophy as lattice Monte-Carlo). This is sometimes called the *numerical continuum limit*, because one takes the limit of the lattice spacing going to zero at the very end, on results obtained numerically. This method is extremely powerful and has been used on a wide variety of problems. In many cases, the results are currently state of the art. So why bother with anything else?

The reason is that the numerical continuum limit is not without problems. The main one, in my opinion, is that one generally needs to extrapolate the final results to the continuum limit, because one cannot reach an arbitrarily small lattice spacing. Indeed, one typically observes that the approximation error increases as the lattice spacing is reduced for fixed D (or equivalently, one needs to increase D as the lattice spacing is reduced to get the same error). The large number of sites to consider, which are the consequence of a vanishing lattice spacing are not the root of the problem, as one can take the thermodynamic limit easily with tensor networks. Rather, the difficulty comes from the generically singular short distance behavior of field theories: as the lattice spacing is reduced, proportionally more variational parameters need to be dedicated. The problem with extrapolations is that they usually contain a degree of arbitrariness in the choice of functions to fit, and making the right guesses requires a good theoretical understanding of the problem. Extrapolations also break the variational nature of the results: one *e.g.* no longer gets rigorous

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upper bounds on the energy of the initial continuum model, which was a convenient property of tensor network states on the lattice. There are other difficulties with the numerical continuum limit I will not elaborate on, but an one to mention for a theoretical physicist (even if it is perhaps the weakest) is aesthetic. If it is possible to deal with the continuum directly, it feels more satisfying and parcimonious to do so.

This brings us to the second approach, the *analytical continuum limit*, which is the focus of these notes. The idea is to keep the problem as it is, but instead to take the continuum limit of tensor network states themselves analytically, hoping that a large fraction of the theoretical results and numerical methods carry through. This is a seducing program that is still largely in progress, with interesting problems even in the simplest settings. I will start with the most stabilized theory in sec 2, the continuous matrix product states for non-relativistic quantum field theories in $d = 1$ space dimension. Since CMPS are less trivial to master at first than their discrete counterpart, I will go through the needed computations slowly. I will then move to a fairly recent relativistic extension of these states, still in $d = 1$ space dimension. Finally, I will discuss continuous tensor network states in $d \geq 2$ space dimensions, where the situation is more dire: part of the theory already exists, but no general purpose algorithm for optimization has been implemented yet. I will try to mention the (numerous) open problems along the way.

2 Continuous matrix product states (CMPS)

2.1 Hilbert space of non-relativistic field theories

Before defining the CMPS, it is helpful to say what it is an ansatz for: non-relativistic quantum field theories in 1 space dimension. To avoid complications, we will stick to the bosonic case, and briefly mention fermionic subtleties at the end. The corresponding Hilbert space $\mathscr{H}$ is the symmetric Fock space $\mathcal{F}[L^2(I)]$ where L^2 is the space of square integrable functions on I, which is the space interval on which the field theory is defined (*e.g.* $[0, L]$ or $\mathbb{R}$). Formally, the Fock space is simply the direct sum of 1, 2, 3, .. to arbitrarily many (symmetric) particle states

$$\mathcal{F}[L^2(I)] = \bigoplus_{n=0}^{+\infty} S_+ L^2(I)^{\otimes n} \,, \tag{1}$$

where S_+ means taking the symmetric subspace. More concretely, a state in this Hilbert space can be represented in second quantized form as

$$|\Psi\rangle = \sum_{n=0}^{+\infty} \int_{I^n} \mathrm{d}x_1 \cdots \mathrm{d}x_n \; \varphi_n(x_1, \cdots, x_n)\, \hat{\psi}^\dagger(x_1) \cdots \hat{\psi}^\dagger(x_n) \, |\Omega\rangle \;. \tag{2}$$

In this expression, $\hat{\psi}^\dagger(x)$ is the bosonic creation operator in x which verifies the canonical commutation relations $[\hat{\psi}(x), \hat{\psi}^\dagger(y)] = \delta(x - y)$. The state $|\Omega\rangle$ is the Fock vacuum associated to $\hat{\psi}$, *i.e.* the state without particles, verifying $\forall x$, $\hat{\psi}(x)\,|\Omega\rangle = 0$. The functions $\varphi_n(x_1, \cdots, x_n)$ are the (symmetric) n-particle wavefunctions. Note that the sum in (2) is allowed to be infinite, but a rather strong constraint is that the state has to be normalizable, *i.e.* $\langle\psi|\,\psi\rangle < +\infty$.

It helps to have in mind a prototypical Hamiltonian that naturally lives in such a Hilbert space. A good example is the (grand canonical) Lieb-Liniger Hamiltonian

$$H_{\mathrm{LL}} = \int_I \partial_x \hat{\psi}^\dagger \partial_x \hat{\psi} + c\, \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi} - \mu \hat{\psi}^\dagger \hat{\psi} \,, \tag{3}$$

which happens to be exactly solvable, and thus provides a convenient benchmark. The first term is the kinetic term, the second is a repulsive contact interaction, while the last one corresponds to a chemical potential forcing a non-zero particle density (otherwise the ground state would be empty). More generally, one could consider higher order 3 point interactions $\propto \hat{\psi}^{\dagger 3} \hat{\psi}^3$, interactions

extending over some range $\propto \int \mathrm{d}y\, V(x-y)\hat{\psi}^\dagger(x)\hat{\psi}^\dagger(y)\hat{\psi}(x)\hat{\psi}(y)$, or even terms breaking particle conservation *e.g.* $\propto \hat{\psi}^{\dagger 2} + \hat{\psi}^2$ which would no longer yield an integrable Hamiltonian but for which the methods we will discuss should apply. However, in this section, one should keep the kinetic term $\partial_x\hat{\psi}^\dagger\partial_x\hat{\psi}$, which is the second quantized rewriting of the standard Schrödinger Hamiltonian (with $\hbar^2/(2m) = 1$), and that is necessary to get a physically reasonable non-relativistic model.

Remark 1 (The continuous Fock space is mostly empty). Fock spaces in the continuum are empty compared to lattice ones, which will later yield simplifications. Indeed, for N bosons on the line, the discrete analog of (2) is

$$|\psi\rangle = \sum_{i_1,\cdots,i_N\in\mathbb{N}} c_{i_1,i_2,\cdots i_N} a_1^{\dagger i_1} a_2^{\dagger i_2}\cdots a_N^{\dagger i_N}\,|0\rangle\ . \tag{4}$$

In such a state, there can be non-zero components with 1 excitation everywhere, or even k excitations everywhere for bosons. In contrast, in the continuum, one fills a continuous interval with a countable number of particles, and thus it is mostly empty if seen as a continuum of harmonic oscillators. For example, the state naively defined with one excitation in every infinitesimal $\mathrm{d}x$, *i.e.* informally "$\prod_x \hat{\psi}^\dagger(x)\,|0\rangle$" is ill-defined in the continuum and does not belong in the Fock space.

2.2 Definition and basic properties

Let us consider a translation invariant quantum field model for simplicity. Let Q and R be two arbitrary $D\times D$ complex matrices. A continuous matrix product states (CMPS) is defined as

$$|Q,R\rangle = \mathrm{tr}\left\{\mathcal{P}\exp\left[\int_I \mathrm{d}x\, Q\otimes\mathbb{1} + R\otimes\hat{\psi}^\dagger(x)\right]\right\}|\Omega\rangle \tag{5}$$

where $\mathcal{P}$ is the path ordering operator, which puts largest arguments on the right[1] *i.e.*

$$\mathcal{P}\mathcal{O}(x_1)\mathcal{O}(x_2) = \begin{cases} \mathcal{O}(x_1)\mathcal{O}(x_2) \text{ if } x_2 > x_1 \\ \mathcal{O}(x_2)\mathcal{O}(x_2) \text{ if } x_2 < x_1 \end{cases} \tag{6}$$

The trace in (5) is taken over the auxiliary $D\times D$ matrix space and D is the bond dimension, which is *really* the direct analog of the bond dimension of MPS. The two matrices Q and R contain the free parameters of the state, that need to be adjusted *e.g.* to find the ground state of a given Hamiltonian. Note that the state produced at least formally belongs to the appropriate field theory Hilbert space: $|Q,R\rangle = \mathrm{tr}[U_{0,L}]\,|\Omega\rangle$, where $U_{0,L}$ is an operator acting on $\mathcal{F}[L^2(I)]\otimes\mathbb{C}^D$. Taking a trace gives an operator acting on the Fock space, and thus acting on $|\Omega\rangle$ we ultimately get a state in the Fock space. To make this more explicit, we can write the CMPS (5) in the wave-function representation of (2).

Fact 1 (Wave-function representation of the CMPS)**.** *For $I=[0,L]$ (for simplicity) CMPS admit the expression*

$$|Q,R\rangle = \sum_{n=0}^{+\infty}\int_{0<x_1<x_2<\cdots<x_n<L} \mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_n\ \varphi_n(x_1,\cdots,x_n)\,\hat{\psi}^\dagger(x_1)\cdots\hat{\psi}^\dagger(x_n)\,|\Omega\rangle$$

with

$$\varphi_n(x_1,\cdots,x_n) = \mathrm{tr}\left[\mathrm{e}^{Qx_1}R\mathrm{e}^{Q(x_2-x_1)}R\cdots\mathrm{e}^{Q(x_n-x_{n-1})}R\mathrm{e}^{Q(L-x_n)}\right] \tag{7}$$

This can be seen as an alternative definition of CMPS. This representation has the advantage of being more explicit in terms of the decomposition of $\mathcal{F}[L^2(I)]$ into a direct sum, but makes the connection with matrix product states less transparent. Let us prove the equivalence.

[1] In this convention, the path-ordering operator $\mathcal{P}$ gives a reversed order compared to the one obtained with the standard time-ordering operator. It is however merely a matter of convention, which allows to have the expression of wavefunctions and correlation functions ordered with arguments increasing from left to right, mimicking the physical ordering.

Proof. The simplest method is to rewrite the original definition (5) of $|Q,R\rangle$ in an "interacting" representation. To this end, let us define $U_{x,y} = \mathcal{P}\exp\left[\int_x^y \mathrm{d}u\, Q\otimes\mathbb{1} + R\otimes\hat{\psi}^\dagger(u)\right]$. We can interpret U as the solution of a dynamical evolution in space

$$\partial_y U_{x,y} = U_{x,y}\left[Q + R\hat{\psi}^\dagger(y)\right] \tag{8}$$

Defining an interaction representation version of U as $\tilde{U}_{x,y} = U_{x,y}\mathrm{e}^{Q(x-y)}$, we see that it obeys

$$\partial_y \tilde{U}_{x,y} = \tilde{U}_{x,y}\tilde{R}(y)\hat{\psi}^\dagger(y) \tag{9}$$

with $\tilde{R}(y) = \mathrm{e}^{Qy}R\mathrm{e}^{-Qy}$. This can be integrated into $\tilde{U}_{x,y} = \mathcal{P}\exp\left[\int_x^y \mathrm{d}u\tilde{R}(u)\,\hat{\psi}^\dagger(u)\right]$. Finally we have

$$\begin{aligned}
|Q,R\rangle &= \mathrm{tr}\left[U_{0,L}\right]|\Omega\rangle && (10)\\
&= \mathrm{tr}\left[\tilde{U}_{0,L}\mathrm{e}^{QL}\right]|\Omega\rangle && (11)\\
&= \mathrm{tr}\left\{\mathcal{P}\exp\left[\int_0^L \mathrm{d}u\,\tilde{R}(u)\,\hat{\psi}^\dagger(u)\right]\mathrm{e}^{QL}\right\}|\Omega\rangle && (12)\\
&= \sum_{n=0}^{+\infty}\mathrm{tr}\left[\tilde{R}(x_1)\tilde{R}(x_2)\cdots\tilde{R}(x_n)\mathrm{e}^{QL}\right]\hat{\psi}^\dagger(x_1)\hat{\psi}^\dagger(x_2)\cdots\hat{\psi}(x_n)\,|\Omega\rangle && (13)
\end{aligned}$$

which is exactly the expression in (7). □

Remark 2 (Beyond translation invariance). To simplify, I considered a translation invariant state with periodic boundary conditions in (5). Both can be relaxed, exactly as with discrete MPS: one can make Q and R depend on x and add a boundary matrix B in the trace to get

$$|Q,R\rangle = \mathrm{tr}\left\{B\,\mathcal{P}\exp\left[\int_I \mathrm{d}x\, Q(x)\otimes\mathbb{1} + R(x)\otimes\hat{\psi}^\dagger(x)\right]\right\}|\Omega\rangle\ . \tag{14}$$

Technically, the number of free parameters becomes infinite (a set $Q(x), R(x)$ per point of space x) and one needs to choose a functional ansatz for $Q(x)$ and $R(x)$ (*e.g.* a finite sum of Hermite functions, a finite Fourier sum or a spline) to recover a finite number of parameters that fit in a computer.

Remark 3 (Only two matrices). A striking aspect of the CMPS ansatz is that it requires *only two* $D\times D$ matrices (at least in the translation invariant case). For matrix product states, the number of matrices is given by the local physical dimension (*e.g.* 2 for spin 1/2, 3 for spin 1, etc.), which is technically infinite for a chain of bosons. In the discrete, this requires artificially truncating at (potentially very large) local physical dimensions. This is the illustration of the drastic simplification (or compression) brought by the continuum limit for bosonic systems[2]. This is in fact related to the previous remark 1 that the Fock space is almost empty and that, as a result, double occupation in an elementary volume $\mathrm{d}x$ need not be considered. We will see this more clearly in when considering the connection with the discrete.

The fact that the ansatz is parameterized by 2 matrices only in the translation invariant case (instead of infinitely many as one would naively guess from the discrete) could be the sign that we are simply missing a part of the Hilbert space (a bit like Gaussian states do not capture all states). The following fact shows that this is not the case, and that the CMPS ansatz is maximally expressive, simply because it contains sums of coherent states.

Fact 2. *CMPSs are dense in the Fock space.*

[2]In fact, in the continuum, the bosonic Fock space is not bigger than the fermionic Fock space.

Proof. The proof is actually quite simple and works by induction. Note first that for $D = 1$ with $Q = 0$ and $R(x) = r(x) \in \mathbb{C}$, a CMPS is simply a field coherent state

$$|0, R\rangle = \exp\left[\int_I \mathrm{d}x\, r(x)\hat{\psi}^\dagger(x)\right]|\Omega\rangle \tag{15}$$

The sum of two CMPSs of bond dimension D is not a CMPS of bond dimension D, but it is easy to see that it is a CMPS of bond dimension $2 \times D$:

$$|Q_1, R_1\rangle + |Q_2, R_2\rangle = \left|\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}, \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}\right\rangle . \tag{16}$$

This easily generalizes to an arbitrarily linear combination by reweighing Q_1 and Q_2. By induction, this shows that we can represent an arbitrary finite sum of field coherent states as a CMPS (potentially of very large but finite D). Field coherent states form an overcomplete basis of the Fock space, and thus can be used to approximate arbitrarily well any state in the Fock space. □

This does not mean that CMPSs are good at approximating states of physical interest with a low (or slowly growing) bond dimension D. CMPS are observed to work well in practice in the continuum, in the same situations where MPS work well in the discrete. However, to my knowledge, an equivalent of the strong approximation theorems known for MPS, which *e.g.* guarantee us that MPS approximate the ground states of local gapped Hamiltonian with an error decaying superpolynomially in D^{-1} (see *e.g.* [1]), do not yet exist for CMPS.

Remark 4 (The previous proof is disappointing). The previous proof is in many ways too trivial to yield real insight. The CMPS obtained from this construction has a very large bond dimension, and does not exploit what makes the power of CMPS: non-commuting R and Q. The density of field coherent states in the Fock space requires space-dependent field coherent states, hence this construction does not tell us, for example, that we can approximate arbitrary translation invariant space in the Fock space with a translation invariant CMPS (which is what we do in practice). It is most likely possible, and intuitively one should be able to reproduce any wavefunction φ_n with the equation (7). However, to my knowledge, no such proof exists.

2.3 From CMPS to MPS

The easiest way to connect CMPS to their discrete MPS counterpart is simply to cut space into small intervals. Let us do it in a pedestrian yet reasonably rigorous way. Let $\varepsilon \ll L$ be the lattice spacing, $x_j = j\varepsilon$ for $0 \leq j \leq L/\varepsilon = N$ the discretized positions. The path-ordered exponential can be approximated by a finite product when ε is small enough:

$$|Q, R\rangle = \mathrm{tr}\left\{\prod_{j=0}^{L/\varepsilon-1} \exp\left[\varepsilon Q + R\int_{j\varepsilon}^{(j+1)\varepsilon} \hat{\psi}^\dagger(x)\right]\right\}|\Omega\rangle + O(\varepsilon)\,. \tag{17}$$

We now introduce a new discrete (or coarse grained) bosonic operator b_j:

$$b_j := \frac{1}{\sqrt{\varepsilon}}\int_{j\varepsilon}^{(j+1)\varepsilon} \hat{\psi}(x)\,. \tag{18}$$

Let us verify its commutation relations. Clearly, $\forall j \neq k,\ [b_j, b_k^\dagger] = 0$

$$[b_j, b_j^\dagger] = \frac{1}{\varepsilon}\int_{j\varepsilon}^{(j+1)\varepsilon} \mathrm{d}x \int_{j\varepsilon}^{(j+1)\varepsilon} \mathrm{d}y\, [\hat{\psi}(x), \hat{\psi}^\dagger(y)] \tag{19}$$

$$= \frac{1}{\varepsilon}\int_{j\varepsilon}^{(j+1)\varepsilon} \mathrm{d}x \int_{j\varepsilon}^{(j+1)\varepsilon} \mathrm{d}y\, \delta(x-y) \tag{20}$$

$$= \frac{1}{\varepsilon} \times \varepsilon = 1 \tag{21}$$

Hence $[b_j, b_k^\dagger] = \delta_{j,k}$. Note the sneaky factor $\sqrt{\varepsilon}$ in (18) that is crucial to go from the continuum Dirac δ to the discrete Kronecker δ. We will encounter it many times again. We now have

$$|Q, R\rangle = \mathrm{tr}\left\{\prod_{j=1}^{L/\varepsilon} \exp\left[\varepsilon Q + \sqrt{\varepsilon} R\, b_j^\dagger\right]\right\} |\Omega\rangle + O(\varepsilon) \tag{22}$$

This is exactly an MPS. Indeed in the number basis $|n_1, n_2, \cdots, n_N\rangle = b_1^{\dagger n_1} b_2^{\dagger n_2} \cdots b_N^{\dagger n_N} |\Omega\rangle$ we have

$$|Q, R\rangle = \sum_{n_1, n_2, \cdots, n_N \in \mathbb{N}} \mathrm{tr}[A_{n_1} A_{n_2} \cdots A_{n_n}] \, |n_1, n_2, \cdots n_N\rangle + O(\varepsilon) \tag{23}$$

where the $D \times D$ complex matrices A_n are obtained (at leading non-trivial order in ε) by expanding the exponential in (22)

$$\begin{aligned} A_0 &= \mathbb{1} + \varepsilon Q \\ A_1 &= \sqrt{\varepsilon} R \\ A_2 &= \varepsilon R^2/2 \\ A_n &= o(\varepsilon) \text{ for } n \geq 3 \end{aligned} \tag{24}$$

Although the A_n for $n \geq 3$ can in principle be computed, they all give subleading contributions. In fact, our starting point (18) (the discretization of the path-ordered exponential) is accurate only to leading order in ε, and thus one is free to choose the subleading terms (for example by fixing them to 0) without changing the continuum limit $\varepsilon \to 0$.

2.4 Computing expectation values

We saw previously in (7) that we could very explicitly write down the CMPS "wavefunction", that is, its decomposition in the natural Fock space basis. However, in general, having an explicit expression for a state does not imply we can have compact expressions for expectation values of local operators. Indeed, the state lives in an infinite dimensional Hilbert space, and computing expectation values a priori requires summing over infinitely many basis elements. With matrix product states, we know there are efficient routines to evaluate expectation values that do not require the full summation (finite in this context, but still exponential in the system size). In the continuum, the same fortunately remains true, and evaluating expectation values remains cheap (polynomial cost in D).

2.4.1 The norm

Before computing expectation values of local observables, let us consider the simplest expectation value, that of the identity (aka. the norm of $|Q, R\rangle$).

Fact 3 (Norm of a CMPS)**.** *For a translation invariant CMPS on the interval $I = [0, L]$:*

$$\langle Q, R | \, Q, R\rangle = \mathrm{tr}_{\mathbb{C}^D \otimes \mathbb{C}^D} [\exp(L\mathbb{T})] \tag{25}$$

with the transfer operator $\mathbb{T}$

$$\mathbb{T} = Q \otimes \mathbb{1} + \mathbb{1} \otimes Q^* + R \otimes R^* \tag{26}$$

The generalization to space dependent Q, R and a non-trivial boundary matrix B is straightforward.

There are many proofs of this result, one that relies on the discretization but is quite transparent, one that works directly in the continuum but is less transparent, and finally one that is is both directly in the continuum and transparent but requires more advanced mathematics (quantum Itô calculus).

Proof strategy 1: discretizing. From (22) we have

$$|Q,R\rangle \simeq \mathrm{tr}\left\{\prod_{j=1}^{L/\varepsilon} \exp\left[\varepsilon Q + \sqrt{\varepsilon} R\, b_j^\dagger\right]\right\} |\Omega\rangle \tag{27}$$

hence

$$\langle Q,R|\,Q,R\rangle = \langle\Omega|\,\mathrm{tr}\left\{\prod_{j=1}^{L/\varepsilon} \exp\left[\varepsilon Q^* + \sqrt{\varepsilon} R^*\, b_j\right]\right\} \mathrm{tr}\left\{\prod_{j=1}^{L/\varepsilon} \exp\left[\varepsilon Q + \sqrt{\varepsilon} R\, b_j^\dagger\right]\right\} |\Omega\rangle \tag{28}$$

$$= \mathrm{tr}\left\{\prod_{j=1}^{L/\varepsilon} \langle\Omega_j|\exp\left[\varepsilon \mathbb{1}\otimes Q^* + \sqrt{\varepsilon}\mathbb{1}\otimes R^*\, b_j\right] \exp\left[\varepsilon Q\otimes\mathbb{1} + \sqrt{\varepsilon} R\otimes\mathbb{1}\, b_j^\dagger\right] |\Omega_j\rangle\right\} \tag{29}$$

We could put everything into the same trace using that $\mathrm{tr}[A\otimes B] = \mathrm{tr}[A]\mathrm{tr}[B]$. We also used that the bosonic vacuum is simply a tensor product of states $|\Omega_j\rangle$ annihilated by the b_j. Finally, expanding the exponentials to order $O(\varepsilon)$

$$\langle Q,R|\,Q,R\rangle \simeq \mathrm{tr}\left\{\prod_{j=1}^{L/\varepsilon} \langle\Omega_j|\,1 + \varepsilon(\mathbb{1}\otimes Q^* + Q\otimes\mathbb{1} + R\otimes R^* b_j b_j^\dagger) + \sqrt{\varepsilon}(\mathbb{1}\otimes R^*\, b_j + R\otimes\mathbb{1}\, b_j^\dagger)\,|\Omega_j\rangle\right\} \tag{30}$$

$$\simeq \mathrm{tr}\left\{\prod_{j=1}^{L/\varepsilon} 1 + \varepsilon\mathbb{T}\right\} \simeq \mathrm{tr}\left\{\prod_{j=1}^{L/\varepsilon} \exp\left(\varepsilon\mathbb{T}\right)\right\} \simeq \mathrm{tr}\left[\exp L\mathbb{T}\right] \tag{31}$$

□

Remark 5 (Direct MPS version). In fact, although it is written in a slightly more cumbersome way as if we knew nothing about tensor networks, this first proof is the direct translation of the standard one for MPS. $\Phi = \exp[\varepsilon\mathbb{T}]$

$$\Phi := \sum_{i\geq 0} A_i\otimes A_i^* \tag{32}$$

$$= \mathbb{1} + \varepsilon\left(Q\otimes\mathbb{1} + \mathbb{1}\otimes Q^* + R\otimes R^*\right) + \varepsilon^2(Q\otimes Q^*) + \cdots \tag{33}$$

$$= \exp(\varepsilon\mathbb{T}) + o(\varepsilon) \tag{34}$$

this allows to understand the role of the factor $\sqrt{\varepsilon}$

Proof strategy 2: using the continuum wave-function. We can evaluate the norm directly in the continuum using the wavefunction expression of the CMPS (7):

$$\langle Q,R\,|Q,R\rangle = \sum_{n=0}^{+\infty} \int_{0<x_1<x_2<\cdots<x_n<L} \mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_n\;\varphi_n(x_1,\cdots,x_n)\varphi_n^*(x_1,\cdots,x_n) \tag{35}$$

For $0<x_1<x_2<\cdots<x_n<L$:

$$\varphi_n\varphi_n^* = \mathrm{tr}\left[\mathrm{e}^{Qx_1} R\mathrm{e}^{Q(x_2-x_1)}\cdots R\mathrm{e}^{Q(L-x_n)}\right] \mathrm{tr}\left[\mathrm{e}^{Q^*x_1} R^*\mathrm{e}^{Q^*(x_2-x_1)}\cdots R^*\mathrm{e}^{Q^*(L-x_n)}\right] \tag{36}$$

$$= \mathrm{tr}\left[\left(\mathrm{e}^{Qx_1}\otimes\mathrm{e}^{Q^*x_1}\right)(R\otimes R^*)\left(\mathrm{e}^{Q(x_2-x_1)}\otimes\mathrm{e}^{Q^*(x_2-x_1)}\right)\cdots(R\otimes R^*)\left(\mathrm{e}^{Q(L-x_n)}\otimes\mathrm{e}^{Q^*(L-x_n)}\right)\right] \tag{37}$$

$$= \mathrm{tr}\left[\mathrm{e}^{(Q\otimes\mathbb{1}+\mathbb{1}\otimes Q^*)x_1}(R\otimes R^*)\,\mathrm{e}^{(Q\otimes\mathbb{1}+\mathbb{1}\otimes Q^*)(x_2-x_1)}\cdots(R\otimes R^*)\,\mathrm{e}^{(Q\otimes\mathbb{1}+\mathbb{1}\otimes Q^*)(L-x_n)}\right] \tag{38}$$

We can now re-exponentiate this expression using the same technique as for the derivation of the wavefunction representation of CMPS, with the substitution $Q\to Q\otimes\mathbb{1}+\mathbb{1}\otimes Q^*$, $R\to R\otimes R^*$, and $\psi^\dagger(x)\to\mathbb{1}$. □

This second proof has the advantage that it is directly in the continuum, but the fact that it works may seem a bit miraculous at first sight, and at least not obviously guessable from standard MPS techniques. There is, finally, a neat continuum proof that uses the formalism of quantum noise.

Proof strategy 3: quantum Itô calculus. Since the final result is the trace of an exponential, the easiest derivation would be by differentiation, to extract the generator $\mathbb{T}$ directly. The need for the formalism of quantum noise (or quantum Itô calculus) comes because differentiating expressions containing $\mathrm{d}\xi^\dagger := \psi^\dagger(x)\mathrm{d}x$ is non-trivial. Indeed, we have $\langle\Omega|\,\mathrm{d}\xi\mathrm{d}\xi^\dagger\,|\Omega\rangle = \mathrm{d}x$, which means that $\mathrm{d}\xi$ behaves more like the root of a differential rather than differential (this is the quantum equivalent of a white noise). Hence, when expressions depend on $\mathrm{d}\xi$, we need to push to second order to differentiate them, otherwise we miss squares of $\mathrm{d}\xi$ that are of order $\mathrm{d}x$ (this is the quantum equivalent of Itô's lemma). This is just the continuum manifestation of the factor $\sqrt{\varepsilon}$ instead of ε in front of R that we have seen over and over again.

Let us write down again the overlap

$$\langle Q,R\,|Q,R\rangle = \langle\Omega|\,\mathrm{tr}\underbrace{\left\{\mathcal{P}\exp\left[\int_I \mathrm{d}x\,Q^* + R^*\hat{\psi}(x)\right]\right\}}_{\tilde{U}_{0,L}}\mathrm{tr}\underbrace{\left\{\mathcal{P}\exp\left[\int_I \mathrm{d}x\,Q + R\hat{\psi}^\dagger(x)\right]\right\}}_{U_{0,L}}|\Omega\rangle \tag{39}$$

$$= \langle\Omega|\,\mathrm{tr}[\tilde{U}_{0,L}]\mathrm{tr}[U_{0,L}]\,|\Omega\rangle \tag{40}$$

$$= \mathrm{tr}[\langle\Omega|\,U_{0,L}\otimes\tilde{U}_{0,L}\,|\Omega\rangle]\,. \tag{41}$$

To obtain $\mathbb{T}$ we should now differentiate $\langle\Omega|\,U_{0,x}\otimes\tilde{U}_{0,x}\,|\Omega\rangle$ with respect to the endpoint x, going to second order in $\mathrm{d}\xi_x = \hat{\psi}(x)\mathrm{d}x$ (which is ultimately what quantum stochastic calculus boils down to):

$$\mathrm{d}\,\langle\Omega|\,U_{0,x}\otimes\tilde{U}_{0,x}\,|\Omega\rangle = \langle\Omega|\,U_{0,x}(Q\mathrm{d}x + R\mathrm{d}\xi)\otimes U_{0,x}^\dagger(Q^*\mathrm{d}x + R^*\mathrm{d}\xi_x^\dagger)\,|\Omega\rangle \tag{42}$$

$$= \langle\Omega|\left(U_{0,x}\otimes U_{0,x}^\dagger\right)\left(Q\otimes\mathbb{1}\mathrm{d}x + \mathbb{1}\otimes Q^*\mathrm{d}x + R\otimes R^*\,\mathrm{d}\xi_x\mathrm{d}\xi_x^\dagger\right)|\Omega\rangle \tag{43}$$

$$= \langle\Omega|\,U_{0,x}\otimes\tilde{U}_{0,x}\,|\Omega\rangle\;\mathbb{T}\,\mathrm{d}x \tag{44}$$

hence finally $\langle Q,R\,|Q,R\rangle = \mathrm{tr}[\exp(L\mathbb{T})]$ as advertised. □

Remark 6 (Subtlety: Itô vs Stratonovich). To go for (43) to (44) one needs to use the fact that $\mathrm{d}\xi_x$ acts infinitesimally after $U_{0,x}$, *i.e.* $U_{0,x}$ contains $\mathrm{d}\xi^\dagger$ only up to $x-\mathrm{d}x$. As a result the vacuum $|\Omega_x\rangle$ is not modified, and thus we can take the expectation value directly without worrying about U. This corresponds to the quantum Itô convention. In contrast taking the Stratonovich formulation amounts to having $\mathrm{d}\xi_x$ centered around the last x in $U_{0,x}$, which gives standard differentiation rules (no need to go to second order), but makes taking expectation values less trivial (this second order term has to be recovered somehow).

The reader should choose whatever proof they are comfortable with, but it is important to keep in mind that discretizing is not necessary.

Remark 7 (Overlap of two different CMPS). The 3 previous proofs generalize immediately to the computation of arbitrary overlaps of CMPS $\langle Q_1,R_1\,|Q_2,R_2\rangle$. One simply needs to formally substitute $Q^*,R^*\to Q_1,R_1$ and $Q,R\to Q_2,R_2$.

2.4.2 Normal-ordered correlation functions

Computing general correlation functions is easy once one understands the computation of the norm. The main tool is to introduce the generating functional of normal ordered correlation functions:

$$\mathcal{Z}_{j',j} = \frac{1}{\langle Q,R|\,Q,R\rangle}\,\langle Q,R|\exp\left[\int_I j'(x)\hat{\psi}^\dagger(x)\right]\exp\left[\int_I j(x)\hat{\psi}(x)\right]|Q,R\rangle\;. \tag{45}$$

Clearly, differentiating this function with respect to j, j' gives us access to all normal-ordered correlation functions, *e.g.*

$$\langle\hat{\psi}^\dagger(x)\hat{\psi}(y)\rangle_{Q,R} = \frac{\delta}{\delta j'(x)}\frac{\delta}{\delta j(y)}\mathcal{Z}_{j',j}\Big|_{j,j'=0} \tag{46}$$

Fact 4 (Generating functional)**.** *The generating functional admits the exact expression*

$$\mathcal{Z}_{j',j} = \frac{1}{\mathrm{tr}[\mathrm{e}^{L\mathbb{T}}]}\,\mathrm{tr}\left[\mathcal{P}\exp\left(\int_I \mathbb{T}_{j',j}\right)\right] \tag{47}$$

where

$$\mathbb{T}_{j',j}(x) = \mathbb{T} + j(x)R\otimes\mathbb{1} + j'(x)\mathbb{1}\otimes R^* \tag{48}$$

Proof. The first step is to rewrite $\mathcal{Z}_{j',j}$ by commuting the two exponentials containing j' and j. This is done easily using the Baker-Campbell-Hausdorff formula which gives, for $[X,Y]\propto\mathbb{1}$, $\mathrm{e}^X\mathrm{e}^Y = \mathrm{e}^{[X,Y]}\mathrm{e}^Y\mathrm{e}^X$:

$$\mathcal{Z}_{j',j} = \frac{1}{\langle Q,R|\,Q,R\rangle}\,\langle Q,R|\exp\left[\int_I j(x)\hat{\psi}(x)\right]\exp\left[\int_I j'(x)\hat{\psi}^\dagger(x)\right]|Q,R\rangle\exp\left(-\int_I j'j\right) \tag{49}$$

$$= \frac{\langle Q,R+j^*|\,Q,R+j'\rangle}{\langle Q,R|\,Q,R\rangle}\exp\left(-\int_I j'j\right) \tag{50}$$

$$= \frac{1}{\mathrm{tr}[\mathrm{e}^{L\mathbb{T}}]}\,\mathrm{tr}\left[\mathcal{P}\exp\left(\int_I \mathbb{T}_{j',j}\right)\right], \tag{51}$$

where the exponential of $j'j$ exactly cancels the quadratic term appearing in the overlap (obtained straightforwardly from remark 7). □

Now, from a straightforward functional differentiation of (47), we get the following fact.

Fact 5 (Normal-ordered correlation function)**.** *For $0<x_1<x_2<\cdots<x_n<L$:*

$$\langle:\hat{\psi}^{\delta_1}(x_1)\hat{\psi}^{\delta_2}(x_2)\cdots\hat{\psi}_n^{\delta_n}(x_n):\rangle_{Q,R} = \frac{1}{\mathrm{tr}[\mathrm{e}^{L\mathbb{T}}]}\mathrm{tr}\left[\mathrm{e}^{x_1\mathbb{T}}R^{(\delta_1)}\mathrm{e}^{(x_2-x_1)\mathbb{T}}\cdots R^{(\delta_n)}\mathrm{e}^{(L-x_n)\mathbb{T}}\right] \tag{52}$$

where δ_i can be either nothing or $\dagger$ and

$$R^{()} = R\otimes\mathbb{1} \tag{53}$$

$$R^{(\dagger)} = \mathbb{1}\otimes R^* \tag{54}$$

For example, this formula gives

$$\langle\hat{\psi}^\dagger(x)\hat{\psi}(y)\rangle_{Q,R} = \frac{1}{\mathrm{tr}[\mathrm{e}^{L\mathbb{T}}]}\mathrm{tr}\left[\mathrm{e}^{x\mathbb{T}}(\mathbb{1}\otimes R^*)\mathrm{e}^{(y-x)\mathbb{T}}(R\otimes\mathbb{1})\mathrm{e}^{(L-y)\mathbb{T}}\right] \tag{55}$$

Remark 8 (Exponentially decreasing). Formula (52) shows that, in complete analogy with MPS, correlation functions for CMPS are always exponentially decreasing (at least as long as the matrices Q, R are finite dimensional).

2.4.3 Simplifications: exploiting gauge freedom and taking the thermodynamic limit

The first thing to notice is that the norm of the CMPS looks ill-behaved in the continuum limit, and scales $\propto(\ell_1|r_1)\exp(\lambda_1 L)$ where λ_1, $|r_1)$, $|l_1)$ are the eivenvalue with largest real part of $\mathbb{T}$ and its associated right and left eigenvectors. We can cancel this behavior, without loss of generality,

by simply substituting $Q \to Q - \lambda_1 \mathbb{1}$. With this new choice, the leading eigenvalue of $\mathbb{T}$ is 0 and all the other ones have strictly negative real part. Diagonalizing $\mathbb{T}$ one gets:

$$\mathrm{e}^{L\mathbb{T}} = \sum_{j=1}^{D^2} \mathrm{e}^{\lambda_j} |r_j)(\ell_j| \underset{L\to\infty}{\longrightarrow} |r_1)(\ell_1| \tag{56}$$

and thus in the thermodynamic limit, the exponential of the transfer matrix becomes a simple rank-1 projector.

The second step, analogous to what is done with MPS, is to go to a super-operator representation of the transfer operator. In a nutshell we are mapping the tensor-product vector space $\mathbb{C}^D \otimes \mathbb{C}^D$ to the space of matrices $\mathcal{M}_D(\mathbb{C})$. We now look at a vector $|v)$ on which $\mathbb{T}$ acts as a matrix

$$|v) = \sum_{k,l} v_{k,l} \, |k\rangle \otimes |l\rangle \to v = \sum_{k,l} v_{k,l} \, |k\rangle \langle l| \ . \tag{57}$$

With this mapping, we can introduce the super-operator $\mathcal{L}$ which reproduces the action of $\mathbb{T}$ on v now written as a matrix

$$\mathbb{T}|v) \to \mathcal{L} \cdot v = Qv + vQ^\dagger + RvR^\dagger \, . \tag{58}$$

Remark 9 (Scaling). An obvious reason why this rewriting is advantageous is that it reduces the complexity of the operations we have to carry. Indeed, multiplying operators acting on $\mathbb{C}^D \otimes \mathbb{C}^D$, which is what is needed to compute correlation functions, naively scales like D^6 whereas composing superoperators acting simply on the left and right of a $D \times D$ matrix scales like D^3.

The third step, like in the discrete, is to note that there is a lot of redundancy in the parameterization of the CMPS. In particular, it is straightforward to see that conjugating the matrices Q, R with an invertible matrix U does not change the state

$$\left|U^{-1}QU, U^{-1}RU\right\rangle = |Q, R\rangle \ . \tag{59}$$

This can be seen either in the original definition of the CMPS (5) or in its wave-function representation (7). This can be exploited to fix properties of Q and R, to simplify computations without losing expressiveness. A particularly convenient choice is the so called *left canonical form* which is obtained by taking $U = \ell_1$ where $(\ell_1|$ is the leading left eigenvector of $\mathbb{T}$. By definition, this matrix verifies

$$\ell_1 Q + Q^\dagger \ell_1 + R^\dagger \ell_1 R = 0 \tag{60}$$

Now taking $Q_\ell = CQC^{-1}$ and $R_\ell = CRC^{-1}$ where $\ell_1 = C^\dagger C$ we get

$$Q_\ell + Q_\ell^\dagger + R_\ell^\dagger R_\ell = 0 \tag{61}$$

This implies that the identity matrix, once vectorized, is a left eigenvector of $\mathbb{T}$. Equivalently, this implies that $\mathcal{L}$ is of the Lindblad form. In practice, one can choose, without loss of generality, matrices verifying (61) from the beginning. Writing $Q_L = -iK - R_L^\dagger R_L/2$, equation (61) is equivalent to K being self-adjoint. One can thus parameterize the CMPS directly with K self-adjoint and R_L.

Finally, all these manipulations allow to rewrite correlation functions in the thermodynamic limit in a simpler way.

Fact 6 (Correlation functions in the thermodynamic limit in left-canonical gauge)**.** *For* $-\infty < x_1 < x_2 < \cdots < x_n < +\infty$*:*

$$\langle : \hat\psi^{\delta_1}(x_1)\hat\psi^{\delta_2}(x_2)\cdots\hat\psi_n^{\delta_n}(x_n) : \rangle_{Q,R} = \mathrm{tr}\left[R^{(\delta_1)} \cdot \mathrm{e}^{(x_2-x_1)\mathcal{L}} \cdot R^{(\delta_2)} \cdots \mathrm{e}^{(x_n-x_{n-1})\mathcal{L}} \cdot R^{(\delta_n)} \cdot \rho_0\right] \tag{62}$$

where ρ_0 *is the fixed point of* $\mathcal{L}$*, i.e.* $\mathcal{L} \cdot \rho_0 = 0$ *normalized to* $\mathrm{tr}[\rho_0] = 1$*,* $R^{(\dagger)} \cdot \rho := \rho R^\dagger$ *and* $R^{()} \cdot \rho = R\rho$*.*

This general formula gives *e.g.* the very compact expression for the particle density:

$$\langle \hat\psi^\dagger(x)\hat\psi(x)\rangle = \mathrm{tr}[R\rho_0 R^\dagger] \, . \tag{63}$$

2.5 Finding the ground state variationally

Once we have explicit expressions for simple correlation functions, we can evaluate the energy density for all non-relativistic Hamiltonians. If interactions are local, the formulas become fully local (they do not depend on $e^{x\mathbb{T}}$). Abstracting away the details, this gives an explicit function of the matrices Q, R

$$\langle h \rangle_{Q,R} = f(Q,R) = \mathrm{tr}[\cdots], \tag{64}$$

that one can feed to a simple minimizer (*e.g.* `scipy.optimize.minimize`) by collecting Q, R (or better K, R_L in left-canonical gauge) into a parameter vector x. As far as I know, this is simply how it was done in 2010 in the original paper on CMPS by Verstraete and Cirac [2].

This works fairly well for low bond dimension but suffers from 2 important drawbacks limiting the method to simple tests at low bond dimension ($D < 10$). There are essentially two reasons:

1. **Inefficient gradient:** For the minimization with gradient descent or its refinements, one needs the gradient. Most libraries will estimate it with finite differences from the energy. Aside from precision issues, this means the cost is now $\propto D^5$ ($\propto D^3$ cost for each evaluation of the energy multiplied by $\propto D^2$ components of the gradient). However, either through an explicit derivation or using backward differentiation methods (backpropagation), one can compute the full gradient exactly with the same asymptotic cost as the energy.

2. **Singular metric:** The space of CMPS is a manifold on which we are trying to follow the steepest descent direction for minimization. The notion of "steepest" depends on the metric on this manifold. Collecting the parameters K, R_L of the CMPS into a vector x and feeding it to a solver amounts to taking as metric the identity, *i.e.* we take the steepest descent for the scalar product $x \cdot y = \sum_i x_i y_i$. It turns out that this notion of distance is very different from the "natural" notion of distance in the space of CMPS, especially when D is large. In practice, this means that the energy goes down, but via a path that can be far from the steepest, and almost flat, leading to a drastic slow down at large D.

A more principled method to minimize the energy is to use the (imaginary time) time dependent variational principle (TDVP). The idea is simply to evolve the CMPS $|x\rangle := |Q,R\rangle$ in imaginary time

$$|x\rangle_\tau \simeq \frac{\exp(-\tau H)\,|x\rangle}{\sqrt{\langle x|\exp(-2\tau H)\,|x\rangle}} \tag{65}$$

where the denominator is just here to normalize the state. In differential form this gives:

$$\partial_\tau |x\rangle_\tau = -\left[H - \langle x|\,H\,|x\rangle_\tau\right]|x\rangle_\tau . \tag{66}$$

Imaginary time evolution should converge exponentially fast to the ground state, without plateaus or slow down. The difficulty is that it cannot be implemented exactly: after an infinitesimal time, the state is out of the CMPS manifold. We would like to find the "best" CMPS trajectory corresponding to this evolution. Equivalently we would like to know how much we should have moved the coefficients x to mimic the imaginary time evolution, *i.e.* have $|x_\tau\rangle \simeq |x\rangle_\tau$. In differential form, this means

$$\nabla_x |x_\tau\rangle \cdot \mathrm{d}x \simeq \left[H - \langle x_\tau|\,H\,|x_\tau\rangle\right]|x_\tau\rangle\,\mathrm{d}\tau \tag{67}$$

where ∇_x is the gradient with respect to all the coefficients (seen as real coefficients), and $\mathrm{d}x = u_x \mathrm{d}\tau$ is an infinitesimal move of the coefficients in the direction u_x (the descent direction). One way to give a precise meaning to (67) is to take $\mathrm{d}x$ such that the distance between the two sides of the equation is minimal

$$u_x = \mathrm{argmin}\,\|\nabla_x |x_\tau\rangle \cdot u_x - \left[H - \langle x_\tau|\,H\,|x_\tau\rangle\right]|x_\tau\rangle\,\|_2 \tag{68}$$

This corresponds to the TDVP prescription. Writing $H(x) = H - \langle x_\tau|\,H\,|x_\tau\rangle$ this gives

$$u_x = \mathrm{argmin}\left\{u_x \cdot \Re\mathrm{e}\left[(\nabla_x \langle x_\tau|)(\nabla_x |x_\tau\rangle)\right]\cdot u_x - 2\Re\mathrm{e}\left[\langle x_\tau|\,H(x)\nabla_x |x_\tau\rangle\right]\cdot u_x\right\}. \tag{69}$$

The matrix, or bilinear form $\Re\mathrm{e}\left[(\nabla_x \langle x_\tau|)(\nabla_x |x_\tau\rangle)\right] \equiv g(x_\tau)$ is the natural metric on the MPS tangent space induced by the Hilbert space scalar product. Assuming for simplicity that we work with CMPS written in the left-canonical gauge such the norm is always 1, the second term is simply the gradient of the energy. Finally we have

$$u_x = -g(x)^{-1}\nabla_x \langle H\rangle_x \tag{70}$$

Hence, imaginary time evolution is just gradient descent with the "natural" metric on the CMPS tangent space! The imaginary time evolution interpretation explains why this metric in particular makes the descent faster than a generic one. The gradient descent interpretation means one does not need to take small time steps: one can take large steps as long as the energy goes down[3]. Further, one could also use refinements of gradient descent like conjugate gradient or BFGS, provided the metric g is taken into account[4].

Remark 10 (Simplifications for local H). When the Hamiltonian is the integral of a local density, expressed simply in terms of $\hat{\psi}(x), \hat{\psi}^\dagger(x)$, one derive an explicit differential equation for R_τ, Q_τ in one step, instead of computing the gradient by backward differentiation and computing the metric. This substantially simplifies TDVP. This is the approach followed in the lecture notes of Vanderstraeten, Haegeman and Verstraete [3], illustrated on the Lieb-Liniger Hamiltonian.

2.6 Limitations and open problems

Continuous matrix product states are currently less developed than their discrete counterparts. There are consequently many things we understand with MPS, that we do not with MPS. Among those, I would like to mention three open problems.

Open Problem 1 (Parent Hamiltonian). For a MPS $|A\rangle$, under reasonably mild assumptions, one can construct a parent Hamiltonian. The parent Hamiltonian is a gapped local Hamiltonian H_A that has the MPS as unique ground state. This is a very useful theoretical result, that allows to reverse the logic in many instances, and start from states instead of Hamiltonians. *Can one find the same for CMPS?*

No equivalent is known currently and mimicking the discrete construction is not obvious. It is likely that the parent Hamiltonians of CMPS do not look "nice". They are probably not the types of local non-relativistic Hamiltonians we typically like to work with, made from a kinetic term $\partial_x\hat{\psi}^\dagger(x)\partial_x\hat{\psi}(x)$ and a potential $V[\hat{\psi}^\dagger(x), \hat{\psi}(x)]$. Indeed, the parent Hamiltonian should have $|Q, R\rangle$ has unique ground state, which means it should fully constrain Q, R (up to gauges). However expectation values of the kinetic term and generic potential only involve (traces of) the stationary state of the Lindbladian ρ_0, arbitrary polynomials in $R, R^\dagger$, and the commutators $[Q, R]$ and $[Q, R^\dagger]$. This seems to insufficiently constrain Q. Allowing less natural higher derivatives or having non-local potentials seems to be needed.

Open Problem 2 (Quality of the approximation). For MPS, we know that the error of the approximation of local observables decreases superpolynomially as a function of D [1] for local gapped Hamiltonians. Can one prove an equivalent result in the continuum? Do CMPS provide provably efficient approximations of (some) QFT ground states?

Open Problem 3 (Beyond translation invariance). In almost all of these notes, we assumed Q, R were constants and worked in the thermodynamic limit. With MPS, one can fairly easily deal with systems with open boundary conditions, without any translation invariance, using the density matrix renormalization group (DMRG). This is an extremely efficient method, because it acts locally on tensors, doing only linear algebraic operations (no gradient descent). Is there an equivalent approach for CMPS?

For this latter open problems, there has been very recent progress by Tuybens *et al.* [4], but the final optimization is still made with the TDVP.

[3] This is no longer true if one is interested in using TDVP for real-time evolution.

[4] In particular, one could make use of the package `OptimKit` developed in Ghent https://github.com/Jutho/OptimKit.jl

2.7 Further reading

I have skipped over regularity conditions, alternative gauges, and did not present the ansatz in its most general form (with many species, with different statistics). The interested reader should look at the very thorough description of Haegeman *et al.* [5]. For a very condensed introduction to CMPS, the historical paper by Verstraete and Cirac still reads very well. Finally, for the optimization of CMPS, one can look at the lecture notes of Vanderstraeten *et al.* [3], and an example of state of the art numerics is to be found in Rincon *et al.* [6]

3 (Relativistic) continuous matrix product states (RCMPS)

To be written. Before it is, one should check `arXiv:2102.07741` and `arXiv:2102.07741`

4 Continuous tensor network states in $d \geq 2$

To be written. Before it is, one should check `arXiv:1808.00976`

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