RG Equation for Chiral Theories in DimReg & BMHV scheme: Application to χQED at 2 loops

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Based on [arXiv:2208.09006], [arXiv:2109.11042 (JHEP 11 (2021) 159)], and [arXiv:2004.14398 (JHEP 08 (2020) 024)], with Amon Ilakovac, Marija Mađor-Božinović (PMF Zagreb), Paul Kühler, Dominik Stöckinger (IKTP, TU Dresden)

Outline

1. Previous results: 1 and 2-loop singular and finite BRST-restoring counterterms
   - χQED action in d dimensions
   - 1-loop $S_{\text{sct}}^{(1)}$ and finite BRST-restoring $S_{\text{fct}}^{(1)}$
   - 2-loop $S_{\text{sct}}^{(2)}$ and finite BRST-restoring $S_{\text{fct}}^{(2)}$
   - Observations

2. RG Equation in BRST-restored DimReg
   - Usual formulation; Problems
   - Modified "Multiplicative Renormalization"
   - Resolution in Algebraic Renormalization
   - Solving at $\hbar^1$ and $\hbar^2$ orders

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Chiral Right-handed QED (χQED) in d dimensions

χQED tree action in d dimensions:

$$S_0 = \int d^d x \left(i\overline{\psi}_i \not{D}_{ij}\psi_j - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - \bar{c}\partial^2 c + \mathcal{L}_{\text{ext}} \right) .$$

$$S_0 = (\overline{S_{\bar{\psi}\psi}} + \widehat{\mathbf{S}_{\bar{\psi}\psi}} + \overline{S_{\overline{\psi_R}A\psi_R}}) + S_{AA} + S_{\text{g-fix}} + S_{\bar{c}c} + S_{\rho c} + S_{\bar{R}c\psi} + S_{\bar{\psi}cR} .$$

- Field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$,
  $\boldsymbol{R_\xi}$ **gauge-fixing** $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \sim \frac{\xi}{2}B^2 + B\partial^\mu A_\mu$; **Ghost field** c.
- RH fermions $\psi_{Ri} \equiv \mathbb{P}_{\mathsf{R}}\psi_i$, $U(1)$ "Generators" $\mathcal{Y}_{Rij}$.
  d*-D** Fermion kinetic + ***fully **R-chiral** interaction terms:

$$i\overline{\psi}_i \not{D}_{ij}\psi_j = i\overline{\psi}_i \not{\partial}\psi_i + e\mathcal{Y}_{Rij}\overline{\psi}_i \mathbb{P}_{\mathsf{L}} \not{A} \mathbb{P}_{\mathsf{R}}\psi_j \equiv \overline{S_{\bar{\psi}\psi}} + \widehat{\mathbf{S}_{\bar{\psi}\psi}} + \overline{S_{\overline{\psi_R}A\psi_R}} .$$

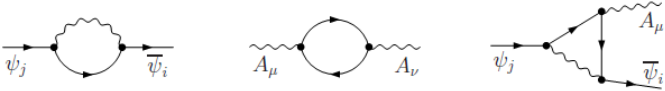
  (Also, $\overline{S_{\bar{\psi}\psi}} \equiv \overline{S_{\overline{\psi}\psi_R}} + \overline{S_{\overline{\psi}\psi_L}}$)
- **External BRST sources** $\boldsymbol{K_\phi}$ sourcing BRST transformations

$$\mathcal{L}_{\text{ext}} = \rho^\mu s_d A_\mu + \bar{R}^i s_d \psi_{Ri} + s_d \overline{\psi_{Ri}} R^i .$$

$$sA_\mu = \partial_\mu c , \qquad s\psi_{Ri} = ie\, c\, \mathcal{Y}_{Rij}\psi_{Rj} ,$$
$$s\bar{c} = B \equiv -\partial A/\xi , \qquad s\overline{\psi_{Ri}} = ie\, \overline{\psi_{Rj}} c \mathcal{Y}_{Rji} .$$

1-loop singular counterterm $S^{(1)}_{\text{sct}}$ action

1-loop SCT action evaluated from 1-loop diagrams (self-energies, vertices):



$$S^{(1)}_{\text{sct}} = -\Gamma^{(1)}_{\text{DReg}}\big|_{\text{div}} = S^{(1)}_{\text{sct,inv}} + S^{(1)}_{\text{sct,evan}} \,.$$

$S^{(1)}_{\text{sct,inv}}$ arises from usual renormalization transformation, $S_{0,\text{inv}} \longrightarrow S_{0,\text{inv}} + S_{\text{ct,inv}}$:

$$S^{(1)}_{\text{sct,inv}} = \frac{-\hbar\, e^2}{16\pi^2\epsilon}\left(\frac{2\,\mathrm{Tr}[\mathcal{Y}_R^2]}{3} S_{AA} + \xi \sum_j (\mathcal{Y}_R^j)^2 \left(\overline{S^j_{\overline{\psi}\psi_R}} + \overline{S^j_{\overline{\psi}_R A \psi_R}}\right)\right) .$$

Second term *specific to BMHV scheme*, arises from fermion loops, & *evanescent*:

$$S^{(1)}_{\text{sct,evan}} = \frac{-\hbar}{16\pi^2\epsilon}\frac{e^2\,\mathrm{Tr}[\mathcal{Y}_R^2]}{3}\left(2\left(\overline{S_{AA}} - S_{AA}\right) + \int \mathrm{d}^d x\, \frac{1}{2}\bar{A}_\mu \widehat{\partial}^2 \bar{A}^\mu\right) .$$

(Specific case of $SU(N)$ result [arXiv:2004.14398])

Anomalies $= 0$; **Finite counterterms $S_{\text{fct}}^{(1)}$**

$$(\mathcal{S}\Gamma_{\text{ren}})^{(1)} = \underset{d\to4}{\text{LIM}}\{\widehat{\Delta}\cdot\Gamma_{\text{DReg}}\big|^{(1)}_{\text{div.}} + \Delta^{(1)}_{\text{sct}}\} + N[\widehat{\Delta}]\cdot\Gamma_{\text{ren}}\big|^{(1)} + \Delta^{(1)}_{\text{fct},4} \stackrel{?}{=} 0\,.$$

Relevant anomalies? $\frac{-\hbar}{16\pi^2}\frac{e^3}{3}\int \mathrm{d}^4x\, \epsilon^{\mu\nu\rho\sigma} c\, d_\psi (\partial_\rho A_\mu)(\partial_\sigma A_\nu)$, with:
$d_\psi = 2\,\mathrm{Tr}[\mathcal{Y}_R^3]$ anomaly coefficient, **chosen** $= \mathbf{0}$, e.g. SM with correct hypercharges.

Finite $\mathcal{O}(\hbar)$ counterterms $S_{\text{fct}}^{(1)}$ such that $\Delta^{(1)}_{\text{fct},4} = s_4 S^{(1)}_{\text{fct}} = -N[\widehat{\Delta}]\cdot\Gamma_{\text{ren}}\big|^{(1)}$:

$$S_{\text{fct}}^{(1)} = \frac{\hbar\, e^2}{16\pi^2}\left\{\int \mathrm{d}^4x\, \left(\frac{-\mathrm{Tr}[\mathcal{Y}_R^2]}{3\times2}\bar{A}_\mu\bar{\partial}^2\bar{A}^\mu + \frac{e^2\,\mathrm{Tr}[\mathcal{Y}_R^4]}{3\times4}(\bar{A}^2)^2\right) + \frac{\xi+5}{6}\sum_j (\mathcal{Y}_R^j)^2\,\overline{S^j_{\psi\psi_R}}\right\}$$

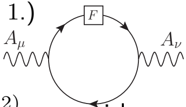
+ any BRST-invariant term .

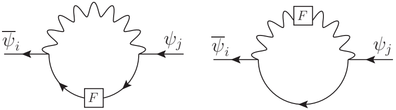
(Specific case of $SU(N)$ result [arXiv:2004.14398])

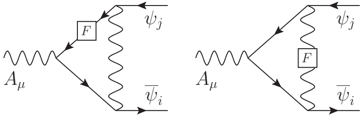
Not gauge-invariant! ($\equiv$ −breaking) and **non-vanishing**.

2-loop singular counterterm $S^{(2)}_{\text{sct}}$ action (1/2)

Photon, fermion SE and **fermion-photon vertex**. Additional diagrams:
(**Note:** In Feynman gauge $\xi = 1$.)







$S^{(2)}_{\text{sct}} = -\Gamma^{(2)}_{\text{div}} = S^{(2)}_{\text{sct,inv}} + S^{(2)}_{\text{sct,break}}$, with:

$$S^{(2)}_{\text{sct,inv}} = -\frac{\hbar^2 e^4}{256\pi^4} \frac{2\,\mathrm{Tr}[\mathcal{Y}_R^4]}{3\epsilon} S_{AA} + \frac{\hbar^2 e^4}{256\pi^4} \frac{(\mathcal{Y}_R^j)^2}{\epsilon} \left[\left(\frac{1}{2\epsilon} + \frac{17}{12} \right) (\mathcal{Y}_R^j)^2 - \frac{1}{9} \mathrm{Tr}[\mathcal{Y}_R^2] \right] \left(\overline{S^j_{\overline{\psi}\psi_R}} + \overline{S^j_{\overline{\psi_R} A \psi_R}} \right),$$

and:

$$S^{(2)}_{\text{sct,break}} = -\frac{\hbar^2 e^4}{256\pi^4} \frac{\mathrm{Tr}[\mathcal{Y}_R^4]}{3\epsilon} \left[2 \left(\overline{S_{AA}} - S_{AA} \right) + \left(\frac{1}{2\epsilon} - \frac{17}{24} \right) \int \mathrm{d}^d x \, \frac{1}{2} \bar{A}_\mu \widehat{\partial}^2 \bar{A}^\mu \right] - \frac{\hbar^2 e^4}{256\pi^4} \frac{(\mathcal{Y}_R^j)^2}{3\epsilon} \left(\frac{5}{2} (\mathcal{Y}_R^j)^2 - \frac{2}{3} \mathrm{Tr}[\mathcal{Y}_R^2] \right) \overline{S^j_{\overline{\psi}\psi_R}}.$$

2-loop singular counterterm $S_{\text{sct}}^{(2)}$ action (2/2)

From the pure 2-loop sub-renormalized diagrams:

$$S_{\text{sct}}^{(2,\,2)} = -\frac{\hbar^2}{256\pi^4}\frac{e^4}{3\epsilon}\,\text{Tr}[\mathcal{Y}_R^4]\left(6\overline{S_{AA}} + \left(\frac{1}{2\epsilon} + \frac{55}{24}\right)\int \mathrm{d}^d x\, \frac{1}{2}\bar{A}_\mu \widehat{\partial}^2 \bar{A}^\mu\right)$$
$$+ \frac{\hbar^2}{256\pi^4}\frac{e^4}{3\epsilon}\sum_j (\mathcal{Y}_R^j)^2 \left[\left(\frac{3}{2\epsilon} - \frac{7}{4}\right)(\mathcal{Y}_R^j)^2 + \frac{2}{3}\,\text{Tr}[\mathcal{Y}_R^2]\right]\left(\overline{S_{\overline{\psi}\psi_R}^j} + \overline{S_{\overline{\psi_R}A\psi_R}^j}\right)$$
$$+ \frac{\hbar^2}{256\pi^4}\frac{e^4}{3\epsilon}\sum_j (\mathcal{Y}_R^j)^2 \left(\frac{1}{2}(\mathcal{Y}_R^j)^2 + \frac{2}{3}\,\text{Tr}[\mathcal{Y}_R^2]\right)\overline{S_{\overline{\psi}\psi_R}^j}\,.$$

and the 1-loop diagrams with $S_{\text{fct}}^{(1)}$ insertions:

$$S_{\text{sct}}^{(2,\,1)} = -\left(S_{\text{fct}}^{(1)} \cdot \Gamma_{\text{ren}}^{=1}\right)^{\text{div}} = \frac{\hbar^2}{256\pi^4}\frac{e^4}{3\epsilon}\left\{\text{Tr}[\mathcal{Y}_R^4]\left(4\overline{S_{AA}} + 3\int \mathrm{d}^d x\, \frac{1}{2}\bar{A}_\mu \widehat{\partial}^2 \bar{A}^\mu\right)\right.$$
$$\left. + \sum_j \left(6(\mathcal{Y}_R^j)^4 - \text{Tr}[\mathcal{Y}_R^2](\mathcal{Y}_R^j)^2\right)\left(\overline{S_{\overline{\psi}\psi_R}^j} + \overline{S_{\overline{\psi_R}A\psi_R}^j}\right) - 3\sum_j (\mathcal{Y}_R^j)^4 \overline{S_{\overline{\psi}\psi_R}^j}\right\}.$$

Finite counterterms $S_{\text{fct}}^{(2)}$

$$(\mathcal{S}\Gamma_{\text{ren}})^{(2)} = \underset{d\to 4}{\text{LIM}}\{\Delta_d^{(1)} \cdot \Gamma_{\text{DReg}}|^{(2)}_{\text{div.}} + \Delta_{\text{sct}}^{(2)}\} + N[\Delta_d^{(1)}] \cdot \Gamma_{\text{ren}}|^{(2)} + \Delta_{\text{fct},4}^{(2)} \stackrel{?}{=} 0\,.$$

(with: $\Delta_d^{(1)} \equiv \widehat{\Delta} + \Delta_{\text{ct}}^{(1)}$)

Finite $\mathcal{O}(\hbar^2)$ counterterms $S_{\text{fct}}^{(2)}$ such that $\Delta_{\text{fct},4}^{(2)} = s_4 S_{\text{fct}}^{(2)} = -\ldots$:

$$S_{\text{fct}}^{(2)} = \left(\frac{\hbar}{16\pi^2}\right)^2 e^4 \int \mathrm{d}^4 x \left\{ \frac{11\,\mathrm{Tr}[\mathcal{Y}_R^4]}{24\times 2} \bar{A}_\mu \bar{\partial}^2 \bar{A}^\mu + \frac{3e^2\,\mathrm{Tr}[\mathcal{Y}_R^6]}{2\times 4} (\bar{A}^2)^2 \right.$$
$$\left. - \sum_j (\mathcal{Y}_R^j)^2 \left(\frac{127}{36}(\mathcal{Y}_R^j)^2 - \frac{1}{27}\,\mathrm{Tr}[\mathcal{Y}_R^2]\right) \overline{S^j_{\bar{\psi}\psi_R}} \right\}$$

+ any BRST-invariant term .

Same structure as $S_{\text{fct}}^{(1)}$. Is it always true @ any order?
(See [arXiv:2205.10381 Cornella, Feruglio, Vecchi] and talk by Luca Vecchi.)

Observations

Dimensional Regularization/Renormalization has some **freedom** in definitions:

- When extending the model to d dimensions:
  $\Rightarrow$ Different possible fermion-gauge-boson chiral interactions;
- Any additional ***finite BRST-invariant*** terms in the $S_{\text{fct}}^{(1),(2),\cdots}$:
  $\Rightarrow$ Different choices would modify calculations at higher-orders.

$\Rightarrow$ Different dimensional BMHV "schemes"!
Each of these choices ***needs to be explicitly mentioned*** for accurateness!

Very **small set** of finite counterterms needed for restoring BRST symmetry at any order. (Compare with manual BRST/Ward ID restoration for individual diagrams...)

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Renormalization Group Equation (Usual form)

- 1-PI effective action Γ: functional of fields ϕ and parameters. (χQED: $A_\mu, \psi, \overline{\psi}, c, \bar{c}, B$; ext. BRST sources $\rho^\mu, R, \overline{R}$; coupling e, gauge parameter ξ.)
- ***Renormalization "mass-scale" μ*** dependence for field renormalizations $Z_\phi^{1/2}$ and parameters $\rightsquigarrow$ $\Gamma[\{\phi(\mu)\}; e(\mu), \xi(\mu), \mu]$.
- Total invariance of Γ under μ:

$$\mu \frac{\mathrm{d}\,\Gamma}{\mathrm{d}\,\mu} = 0 = \mu \frac{\partial \Gamma}{\partial \mu} + \left(\beta_e \frac{\partial}{\partial e} + \beta_\xi \frac{\partial}{\partial \xi} - \sum_\phi \gamma_\phi N_\phi \right) \Gamma \,.$$

$\mu\partial/\partial\mu$: *RGE differential operator*.
$\beta_{e,\xi} = \mu \,\mathrm{d}(e,\xi)/\mathrm{d}\,\mu$: β-funcs. for *each* parameter (incl. gauge parameter ξ).
$\gamma_\phi = \mu \,\mathrm{d} \ln Z_\phi^{1/2} / \mathrm{d}\,\mu$: Anomalous dimensions for χQED fields+sources ϕ.
N_ϕ: field-numbering ("leg-counting") diff. operators:

$$N_\phi \equiv \int \mathrm{d}^d x \; \phi_i(x) \delta/\delta\phi_i(x) \,,$$

$$N_\psi^R \equiv \int \mathrm{d}^d x \; (\mathbb{P}_\mathsf{R} \psi_i(x)) \delta/\delta\psi_i(x) \,, \qquad N_{\overline{\psi}}^L \equiv \int \mathrm{d}^d x \; (\overline{\psi}_i(x) \mathbb{P}_\mathsf{L}) \delta/\delta\overline{\psi}_i(x) \,.$$

Usual paradigm (['t Hooft–1973, Machacek,Vaughn–1983,'84,'85]) (1/2)

(In $d = 4 - 2\epsilon$ dims) Laurent ϵ-expansions for:

- *(Divergent)* bare couplings x_k^0 (mass-dimensionality η_k), wrt. renormalized x_k:

$$x_k^0 \mu^{-\eta_k \epsilon} = x_k + \sum_{n=1}^{+\infty} C_k^{(n)}(\{x_l\})/\epsilon^n (\equiv Z_{x_k} x_k) \,.$$

- *(Divergent)* wave-function renormalization factors Z_ϕ (recall: $\phi^0 = Z_\phi^{1/2} \phi$):

$$Z_\phi = 1 + \sum_{n=1}^{+\infty} C_\phi^{(n)}(\{x_l\})/\epsilon^n \,.$$

  - β_k functions for couplings x_k:

$$\beta_k = \mu \frac{\mathrm{d}\, x_k}{\mathrm{d}\, \mu}\bigg|_{\epsilon \to 0} = -\eta_k\, C_k^{(1)}(\{x_l\}) + \sum_{x_l} \eta_l\, x_l \frac{\partial C_k^{(1)}(\{x_l\})}{\partial x_l} \,,$$

  - and γ_ϕ anomalous dimension for ϕ:

$$\gamma_\phi = \frac{1}{2} \mu \frac{\mathrm{d} \ln Z_\phi}{\mathrm{d}\, \mu}\bigg|_{\epsilon \to 0} = \frac{-1}{2} \sum_{x_l} \eta_l\, x_l \frac{\partial C_\phi^{(1)}(\{x_l\})}{\partial x_l} \,.$$

Usual paradigm (['t Hooft–1973, Machacek,Vaughn–1983,'84,'85]) (2/2)

Expansions readable from the (order-by-order) ϵ-expansions from diagrams (equivalently, the singular counterterms):

- div($\Gamma_{\phi_1\phi_2}$): give expansion terms of $Z^{-1}_{\phi_1\phi_2}$
- div($\Gamma_{\phi_1\ldots\phi_n}$): give expansion terms of $Z^{-1}_{x_n} Z^{-1/2}_{\phi_1} \ldots Z^{-1/2}_{\phi_n}$, and $Z_{x_n} \sim x^0_k / x_k$.

Streamlined when singular CT structure follows the one of the tree-level action: "symmetric-invariant" CTs (respect the fundamental symmetries: BRST, ...) $\longrightarrow$ ***"Multiplicative Renormalization"***.

Pitfalls

Suppose gauge anomalies are cancelled (e.g. through suitable field contents).

- $S_{\text{sct,evan}}^{(1),(2)}$ and $S_{\text{fct}}^{(1),(2)}$ are $\neq 0$. No effect on 1-loop-level RGEs. However they matter for renormalization at ≥ 2-loop order, from insertion in loop diagrams.
- **Problem:** $S_{\text{sct,evan}}^{(1),(2)} \neq 0 \longrightarrow$ Cannot use straightforwardly the technique with bare φ's & e's, and Z renormalization factors for defining the β_e and γ_φ functions, because we have (non-physical) evanescent operators.

    - **RGEs** for the **DimReg** theory: define $\beta_{\widehat{\mathcal{O}}}$ for the (non-physical) evanescent operators $\Longrightarrow$ **All** β-functions need to be considered for consistency.
    - **RGEs** for the **renormalized 4D** theory: using the ***Algebraic Renormalization*** framework.

Modified "Multiplicative Renormalization" approach (1/4)

DimRen χQED generates singular (and finite) CTs with new structures.
Following [Bos–1987,Schubert–1989], associate auxiliary couplings $\rho_{\mathcal{O}}$ to these operators $\mathcal{O}$, and define new action

$$S_0^* = S_0 + \int \mathrm{d}^d x \, \rho_{\mathcal{O}} \mathcal{O}(x) \,.$$

Example for χQED:

$$\begin{aligned} S_0 \to S_0^* &= S_0 + \sigma_1 \widehat{S_{\overline{\psi}\psi}} + \sigma_2 \widehat{S_{AA}} + \int \mathrm{d}^d x \, \left(\sigma_3 \frac{1}{2} \bar{A}_\mu \widehat{\partial}^2 \bar{A}^\mu + \rho_1 \frac{1}{2} \bar{A}_\mu \overline{\partial}^2 \bar{A}^\mu + \rho_2 \frac{e^2}{4} (\bar{A}^2)^2 \right) \\ &= \overline{S_{\overline{\psi}\psi}} + (1+\sigma_1) \widehat{S_{\overline{\psi}\psi}} + \overline{S_{\overline{\psi}A\psi_R}} + \overline{S_{AA}} + (1+\sigma_2) \widehat{S_{AA}} + S_{\text{g-fix}} + S_{\bar{c}c} + S_{\rho c} + S_{\bar{R}c\psi} + S_{\overline{\psi}cR} \\ &\qquad + \int \mathrm{d}^d x \, \left(\sigma_3 \frac{1}{2} \bar{A}_\mu \widehat{\partial}^2 \bar{A}^\mu + \rho_1 \frac{1}{2} \bar{A}_\mu \overline{\partial}^2 \bar{A}^\mu + \rho_2 \frac{e^2}{4} (\bar{A}^2)^2 \right) \,, \end{aligned}$$

$S_{\text{Bare}}^* = S_0^* + S_{\text{sct}}^* + S_{\text{fct}}^* \Longrightarrow$ modified effective action $\Gamma_{\text{DReg}}^*[e, \{\sigma_i\}, \{\rho_i\}]$.
Aux.couplings: unphysical, absent in the renormalized theory. Their renormalized values are $= 0$. Original theory recovered when $\sigma_i \to 0$, $\rho_i \to 0$.

Modified "Multiplicative Renormalization" approach (2/4)

Example for $\widehat{S_{\overline{\psi}\psi}}$:

$$(1+\sigma_1)\widehat{S_{\overline{\psi}\psi}} \to Z_{\widehat{\overline{\psi}\psi}}(1+\sigma_1)\widehat{S_{\overline{\psi}\psi}} = (1 + Z_{\sigma_1}\sigma_1\mathcal{Z}_{\psi_R}^{-1})\mathcal{Z}_{\psi_R}\widehat{S_{\overline{\psi}\psi}}\,,$$

or:

$$(1+\sigma_1)\widehat{S_{\overline{\psi}\psi}} \to (1 + \underbrace{(\sigma_1+\delta_{\sigma_1})\mathcal{Z}_{\psi_R}^{-1}}_{=\sigma_1^0})\mathcal{Z}_{\psi_R}\widehat{S_{\overline{\psi}\psi}} = (\underbrace{\sigma_1}_{\to 0} + \underbrace{\mathcal{Z}_{\psi_R}+\delta_{\sigma_1}}_{=\,1\text{ because no SCT}})\widehat{S_{\overline{\psi}\psi}}\,.$$

(Both Lorentz and gauge invariance broken $\to$ *additive renormalization*.)
$\mathcal{Z}_{\psi_R}$: R-fermion wave-function renormalization, has both singular and finite contributions (from $S^{(1),(2)}_{\text{sct,fct}}$).

Here, $\mathcal{Z}_{\psi_R}+\delta_{\sigma_1}=1$, because no $\widehat{S_{\overline{\psi}\psi}}$ in S_{sct} (only "tree-level" contribution).
Obtain: $\sigma_1^0 = (\sigma_1+\delta_{\sigma_1})\mathcal{Z}_{\psi_R}^{-1} = \cdots = \delta_{\sigma_1}\mathcal{Z}_{\psi_R}^{-1} = \mathcal{Z}_{\psi_R}^{-1} - 1$, and define a $\widetilde{\beta_{\sigma_1}}$.
Similar logic for the other operators / aux.couplings.

Modified "Multiplicative Renormalization" approach (3/4)

Modified RGE:

$$\mu \frac{\partial \Gamma^*_{\text{DReg}}}{\partial \mu} = \left(-\widetilde{\beta_e} e \frac{\partial}{\partial e} - \widetilde{\beta_\xi} \frac{\partial}{\partial \xi} - \widetilde{\beta_{\sigma_i}} \frac{\partial}{\partial \sigma_i} - \widetilde{\beta_{\rho_i}} \frac{\partial}{\partial \rho_i} + \sum_\phi \widetilde{\gamma_\phi} N_\phi \right) \Gamma^*_{\text{DReg}} \,.$$

- $\widetilde{\beta}$, $\widetilde{\gamma_\phi}$: ***NOT*** the true beta-functions/anomalous dimensions of the renormalized theory $\longrightarrow$ ***auxiliary intermediate*** quantities. (Example: $\widetilde{\beta_{\sigma_1}}$ defined out of δ_{σ_1}.)
- True β, γ_ϕ functions function of $\widetilde{\beta}$, $\widetilde{\gamma_\phi}$. Make sense only in renormalized theory.

4-dim renormalized effective action $\Gamma_{\text{ren}} \equiv \Gamma$ defined by

$$\Gamma[e] = \underset{d \to 4}{\text{LIM}} \lim_{\substack{\sigma_i \to 0 \\ \rho_i \to 0}} \Gamma^*_{\text{DReg}}[e, \{\sigma_i\}, \{\rho_i\}] \,,$$

i.e. in the limit ($\text{LIM}_{d \to 4}$) where: *(i)* divergences are MS-subtracted, and *(ii)* $d \to 4$, with *(iii)* remaining (finite) evanescent quantities set to zero.

Modified "Multiplicative Renormalization" approach (4/4)

"True" RGE for the *renormalized* effective action Γ would be:

$$\mu \frac{\partial \Gamma}{\partial \mu} = \left(-\beta_e e \frac{\partial}{\partial e} - \beta_\xi \frac{\partial}{\partial \xi} + \sum_\phi \gamma_\phi N_\phi \right) \Gamma \quad \sim \quad \underset{d\to 4}{\text{LIM}} \lim_{\substack{\sigma_i \to 0 \\ \rho_i \to 0}} \mu \frac{\partial \Gamma^*_{\text{DReg}}}{\partial \mu} .$$

The effects of the evanescent operators dilute into the non-evanescent ones [Bos–1987,Schubert–1989].
Evaluate

$$\left(-\widetilde{\beta_{\sigma_i}} \frac{\partial}{\partial \sigma_i} - \widetilde{\beta_{\rho_i}} \frac{\partial}{\partial \rho_i} \right) \Gamma^*_{\text{DReg}}[e, \{\sigma_i\}, \{\rho_i\}] \Big|_{\substack{\sigma_i \to 0 \\ \rho_i \to 0}} .$$

Via the Regularized Action Principle [Breitenlohner,Maison–1977] evaluate insertions:

$$\frac{\partial \Gamma^*_{\text{DReg}}}{\partial \rho_{\mathcal{O}}} \Big|_{\rho_{\mathcal{O}} \to 0} = \frac{\partial (S^*_0 + S^*_{\text{fct}})}{\partial \rho_{\mathcal{O}}} \cdot \Gamma^*_{\text{DReg}} \Big|_{\rho_{\mathcal{O}} \to 0} = \left(\mathcal{O} + \frac{\partial S^*_{\text{fct}}}{\partial \rho_{\mathcal{O}}} \right) \cdot \Gamma^*_{\text{DReg}} \Big|_{\rho_{\mathcal{O}} \to 0} .$$

We don't continue pursuing this approach here...

Algebraic Properties of the RGE

The RGEs for the *renormalized* Γ must satisfy [Piguet,Sorella–1995]:

- BRST-compatibility: $\mu\partial_\mu(\mathcal{S}\Gamma) = \mathcal{S}_\Gamma(\mu\partial_\mu\Gamma) = 0$.
- *Gauge-fixing condition*:

$$\frac{\delta\Gamma}{\delta B} = \xi B + \partial^\mu A_\mu = 0 \quad \text{at all orders} \qquad \longrightarrow \qquad \frac{\delta}{\delta B}\,\mu\partial_\mu\Gamma = 0\,,$$

- *Ghost equation*:

$$\mathcal{G}\Gamma = 0 \quad \text{at all orders} \quad \longrightarrow \quad \mathcal{G}\,\mu\partial_\mu\Gamma = 0\,, \quad \text{with} \quad \mathcal{G} = \frac{\delta}{\delta\bar{c}} + \partial^\mu\frac{\delta}{\delta\rho^\mu}\,.$$

Solution: linear combination of functionals $\partial_e\Gamma \equiv \partial\Gamma/\partial e$ and $\mathcal{N}_\varphi\Gamma$ that satisfy the previous conditions. ($\mathcal{N}_\varphi$: linear combinations of N_φ):

$$\mu\partial_\mu\Gamma = \left(-\beta_e\, e\partial_e + \sum_\varphi \gamma_\varphi\mathcal{N}_\varphi\right)\Gamma\,,$$

defining the "true" β_e, γ_φ functions.

The linear-BRST invariants $\mathcal{N}_\varphi$

"Curly" $\mathcal{N}_\varphi$ defined from invariants under the linear BRST operator $b_d := \mathcal{S}_{S_0}$:

$$
\begin{aligned}
L_A &= b_d \int \mathrm{d}^d x\, \tilde{\rho}^\mu A_\mu = \left(N_A - N_{\bar{c}} - N_B - N_\rho + 2\xi \frac{\partial}{\partial \xi}\right) S_0 \equiv \mathcal{N}_A S_0 \\
&= 2 S_{AA} + \overline{S_{\overline{\Psi} A \Psi}} - S_{\bar{c}c} - S_{\rho c}\,, \\
L_c &= -b_d \int \mathrm{d}^d x\, \zeta_a c^a = (N_c - N_\zeta) S_0 \equiv \mathcal{N}_c S_0 \\
&= S_{\bar{c}c} + S_{\rho c} + S_{\zeta cc} + S_{\bar{R} c \psi} + S_{\bar{\psi} c R}\,, \\
L_{\Psi_R} &= -b_d \int \mathrm{d}^d x\, (\bar{R}^i \mathbb{P}_\mathsf{R} \Psi_i + \overline{\Psi}_i \mathbb{P}_\mathsf{L} R^i) = (N_\Psi^R + N_{\overline{\Psi}}^L - N_{\bar{R}} - N_R) S_0 \equiv \mathcal{N}_\psi S_0 \\
&= 2 \int \mathrm{d}^d x\, i \overline{\Psi}_i \overline{\partial\!\!\!/} \mathbb{P}_\mathsf{R} \Psi_i + \overline{S_{\overline{\Psi} A \Psi}} + \widehat{S_{\bar{\psi}\psi}}\,,
\end{aligned}
$$

Other b_d invariants: pure Yang-Mills term L_{F^2} ; coupling variation L_e:

$$
L_{F^2} = \frac{-1}{4} \int \mathrm{d}^d x\, F^a_{\mu\nu} F^{a\,\mu\nu} = S_{AA}\,, \qquad L_e = L_A + L_c - 2 L_{F^2} \equiv e \frac{\partial S_0}{\partial e}\,.
$$

Resolution in Algebraic Renormalization (1/2)

Quantum Action Principle [Lowenstein–1971, Piguet,Sorella–1995, Piguet,Rouet–1981] re-expresses variations (DVOs) of Γ as insertions:

$$e\partial_e \Gamma = N\left[e\partial_e S_{\text{DReg}}^{\text{fin.}}\right] \cdot \Gamma \quad \text{(for all physical parameters)}\,,$$
$$\mathcal{N}_\varphi \Gamma = N\left[\mathcal{N}_\varphi S_{\text{DReg}}^{\text{fin.}}\right] \cdot \Gamma \quad \text{(for all fields)}\,.$$

Uses the finite part of the DimReg action: $S_{\text{DReg}}^{\text{fin.}} = S_0 + S_{\text{fct}}$.

Cannot do the same for $\mu\partial_\mu\Gamma$: renormalization scale μ ***not introduced*** in $S_{\text{DReg}}^{\text{fin.}}$, but as a modification of the loop integration: $\mu^\epsilon \int \mathrm{d}^d x$.
$\Longrightarrow$ **Bonneau Identities** [Bonneau–1980]:

$$\mu\partial_\mu\Gamma = \sum_{N_l \geq 1} N_l \, N[\text{r.s.p.}\, \Gamma_{\text{DReg}}^{N_l \text{ loops}}] \cdot \Gamma\,.$$

"r.s.p.": residue of simple pole in $\nu = 4 - d = 2\epsilon$.
$\Gamma_{\text{DReg}}^{N_l \text{ loops}}$: DReg'ed sub-renormalized 1-PI diagrams with precisely N_l loops.
(Their $\hbar$-counting can be $> N_l$ if they contain S_{fct} insertions.)

Resolution in Algebraic Renormalization (2/2)

RGE $\mu\frac{\partial\Gamma}{\partial\mu}$ becomes, with $S_{\text{DReg}}^{\text{fin.}} = S_0 + S_{\text{fct}}$, the equality:

$$\sum_{N_l \geq 1} N_l \, N[\text{r.s.p.}\, \Gamma_{\text{DReg}}^{N_l \text{ loops}}] \cdot \Gamma = -\beta_e \, N\left[e\partial_e S_{\text{DReg}}^{\text{fin.}}\right] \cdot \Gamma + \sum_{\varphi} \gamma_\varphi N\left[\mathcal{N}_\varphi S_{\text{DReg}}^{\text{fin.}}\right] \cdot \Gamma \,.$$

$d = 4 + (d-4)$-dimensional insertions. The $(d-4)$ insertions $\widehat{\mathcal{O}}$ are ***not*** independent. Expand them in terms of 4-dimensional insertions (Bonneau IDs):

$$N[\widehat{\mathcal{O}}] \cdot \Gamma = \sum_i c_{\mathcal{O},i} N[\overline{\mathcal{M}_i}] \cdot \Gamma \,.$$

Re-express all insertions into a basis of (independent) 4-dimensional operator insertions:

$$\mu\partial_\mu \Gamma = \sum_i r_i N[\overline{\mathcal{M}_i}] \cdot \Gamma = \sum_i \left(-\beta_e \, e \times w_{e,i} + \sum_\varphi \gamma_\varphi \times w_{\varphi,i}\right) N[\overline{\mathcal{M}_i}] \cdot \Gamma \,,$$

System of equations for the β_e's and the γ_φ's to be solved.

Reminder about the χQED tree action

Starting-point tree action in $d = 4$:

$$S_0^{(4D)} \equiv \overline{S_0} = \int \mathrm{d}^4 x \left(i\overline{\overline{\psi}_R}_i \overline{\not{D}}_{ij} \psi_{Rj} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 - \bar{c} \partial^2 c + \mathcal{L}_{\text{ext}} \right)$$
$$= (\overline{S_{\bar{\psi}\psi_R}} + \overline{S_{\overline{\psi_R} A \psi_R}}) + \overline{S_{AA}} + S_{\text{g-fix}} + \overline{S_{\bar{c}c}} + \overline{S_{\rho c}} + \overline{S_{\bar{R}c\psi}} + \overline{S_{\bar{\psi}cR}} \,.$$

Extended d-D action:

$$S_0 = (\overline{S_{\bar{\psi}\psi}} + \color{red}{\widehat{\mathbf{S}_{\bar{\psi}\psi}}} + \overline{S_{\overline{\psi_R} A \psi_R}}) + S_{AA} + S_{\text{g-fix}} + S_{\bar{c}c} + S_{\rho c} + S_{\bar{R}c\psi} + S_{\bar{\psi}cR} \,.$$

Separates into 4D $\overline{S_0}$ and evanescent $\widehat{S_0}$ actions:

$$\widehat{S_0} \equiv \color{blue}{\overline{S_{\overline{\psi}\psi_L}}} + \color{red}{\widehat{S_{\overline{\psi}\psi}}} + \widehat{S_{AA}} + \widehat{S_{\text{g-fix}}} + \widehat{S_{\bar{c}c}} + \widehat{S_{\rho c}} \,,$$

where: $\widehat{S_{\mathcal{O}}} := S_{\mathcal{O}} - \overline{S_{\mathcal{O}}}$. (For those operators with tensorial Lorentz structures.)
Note: we include there the 4D dummy left-handed fermion kinetic term $\color{blue}{\overline{S_{\overline{\psi}\psi_L}}}$.

One-loop ($\hbar^1$) RGE

$$\mu\partial_\mu\Gamma|^{=1} = -\beta_e^{(1)} e\partial_e \overline{S_0} + \sum_{\phi=A,\psi,c} \gamma_\phi^{(1)} \mathcal{N}_\phi \overline{S_0}$$

$$= 2\gamma_A^{(1)}\overline{S_{AA}} + \sum_i \left(2\gamma_{\psi_i}^{(1)}\overline{S_{\overline{\psi}\psi}^i} + \left(2\gamma_{\psi_i}^{(1)} + \gamma_A^{(1)} - \beta_e^{(1)}\right)\overline{S_{\overline{\psi_R}A\psi_R}^j}\right)$$

$$+ \left(\gamma_c^{(1)} - \gamma_A^{(1)}\right)\left(\overline{S_{\bar{c}c}} + \overline{S_{\rho c}}\right) + \left(\gamma_c^{(1)} - \beta_e^{(1)}\right)\left(\overline{S_{\bar{R}c\psi}} + \overline{S_{\overline{\psi}cR}}\right),$$

also equal to:

$$\text{r.s.p.}\,\overline{\Gamma_{\text{DReg}}^{\text{1-loop}}} \equiv -\text{r.s.p.}\,\overline{S_{\text{sct}}^{(1)}}$$

$$= \frac{\hbar}{16\pi^2}\frac{4e^2\,\text{Tr}[\mathcal{Y}_R^2]}{3}\overline{S_{AA}} + \frac{\hbar}{16\pi^2}2e^2\xi\sum_j(\mathcal{Y}_R^j)^2\left(\overline{S_{\overline{\psi}\psi_R}^j} + \overline{S_{\overline{\psi_R}A\psi_R}^j}\right).$$

Obtain system of equations for independent operators and solve:

$$\overline{S_{AA}} \to 2\gamma_A^{(1)}, \quad \overline{S_{\overline{\psi}\psi_R}^i} \to 2\gamma_{\psi_i}^{(1)},$$

$$\overline{S_{\overline{\psi_R}A\psi_R}^i} \to 2\gamma_{\psi_i}^{(1)} + \gamma_A^{(1)} - \beta_e^{(1)},$$

$$\overline{S_{\bar{c}c}}, \overline{S_{\rho c}} \to \gamma_c^{(1)} - \gamma_A^{(1)} = 0,$$

$$\overline{S_{\bar{R}c\psi}}, \overline{S_{\overline{\psi}cR}} \to \gamma_c^{(1)} - \beta_e^{(1)} = 0.$$

$$\beta_e^{(1)} = \gamma_A^{(1)} = \gamma_c^{(1)} = \frac{\hbar}{16\pi^2}e^2\frac{2\,\text{Tr}[\mathcal{Y}_R^2]}{3},$$

$$\gamma_{\psi_i}^{(1)} = \frac{\hbar}{16\pi^2}e^2\xi(\mathcal{Y}_R^i)^2.$$

~~Two-loop~~ $\hbar^2$-order RGE: Structure (1/2)

Expand $\mu\partial_\mu\Gamma = -\beta_e\, N\left[e\partial_e S^{\text{fin.}}_{\text{DReg}}\right]\cdot\Gamma + \sum_\varphi \gamma_\varphi N\left[\mathcal{N}_\varphi S^{\text{fin.}}_{\text{DReg}}\right]\cdot\Gamma$ at $\mathcal{O}(\hbar^2)$:

$$
\begin{aligned}
\mu\partial_\mu\Gamma\big|^{=2} = & -\beta_e^{(1)} N[e\partial_e\overline{S_0}]\cdot\Gamma^{=1} + \sum_{\phi=A,\psi,c}\gamma_\phi^{(1)} N[\mathcal{N}_\phi\overline{S_0}]\cdot\Gamma^{=1} && \rightsquigarrow \quad \mathfrak{W}_1 \\
& -\beta_e^{(1)} N[e\partial_e\widehat{S_0}]\cdot\Gamma^{=1} + \sum_{\phi=A,\psi,c}\gamma_\phi^{(1)} N[\mathcal{N}_\phi\widehat{S_0}]\cdot\Gamma^{=1} && \rightsquigarrow + \mathfrak{W}_2 \\
& -\beta_e^{(1)} e\partial_e\overline{S^{(1)}_{\text{fct}}} + \sum_{\phi=A,\psi,c}\gamma_\phi^{(1)}\mathcal{N}_\phi\overline{S^{(1)}_{\text{fct}}} && \rightsquigarrow + \mathfrak{W}_3 \\
& -\beta_e^{(2)} e\partial_e\overline{S_0} + \sum_{\phi=A,\psi,c}\gamma_\phi^{(2)}\mathcal{N}_\phi\overline{S_0} && \rightsquigarrow + \mathfrak{W}_4\,.
\end{aligned}
$$

$N[\mathcal{O}]\cdot\Gamma^{=1}$: Insertion of $\mathcal{O}$ in 1-loop 1-PI diagrams and renormalized.
(Notation "$=1$" and $\sum_{\phi=A,\psi,c}\dots$ understood in the next slides.)

~~Two-loop~~ $\hbar^2$-order RGE: Structure (2/2)

Expand $\mu\partial_\mu\Gamma = \sum_{N_l \geq 1} N_l\, N[\text{r.s.p.}\, \Gamma^{N_l \text{ loops}}_{\text{DReg}}] \cdot \Gamma$ at $\mathcal{O}(\hbar^2)$:

$$
\begin{aligned}
\mu\partial_\mu\Gamma\big|^{=2} = \quad & N[\text{r.s.p.}\, \overline{\Gamma^{\text{1-loop}}_{\text{DReg, No } S^{(1)}_{\text{fct}}}}] \cdot \Gamma^{=1} && \rightsquigarrow \quad \mathfrak{R}_1 = -N[\text{r.s.p.}\, \overline{S^{(1)}_{\text{sct}}}] \cdot \Gamma^{=1} \\
+\, & N[\text{r.s.p.}\, \widehat{\Gamma^{\text{1-loop}}_{\text{DReg, No } S^{(1)}_{\text{fct}}}}] \cdot \Gamma^{=1} && \rightsquigarrow + \mathfrak{R}_2 = -N[\text{r.s.p.}\, \widehat{S^{(1)}_{\text{sct}}}] \cdot \Gamma^{=1} \\
+\, & \text{r.s.p.}\, \overline{S^{(1)}_{\text{fct}} \cdot \Gamma^{=1}} && \rightsquigarrow + \mathfrak{R}_3 = -\text{r.s.p.}\, \overline{S^{(2,1)}_{\text{sct}}} \\
+\, & 2\, \text{r.s.p.}\, \overline{\Gamma^{\text{2-loops}}_{\text{DReg, No } S^{(1),(2)}_{\text{fct}}}} && \rightsquigarrow + \mathfrak{R}_4 = -2\, \text{r.s.p.}\, \overline{S^{(2,2)}_{\text{sct}}} \,.
\end{aligned}
$$

- As in the previous slide, we have evanescent insertions (terms $\mathfrak{W}_2$ and $\mathfrak{R}_2$).
- Those quantities correspond to already-evaluated counterterms. Only evanescent insertions are "new" and require a separate calculation.

$\mathfrak{W}_1$ and $\mathfrak{R}_1$ (4-dimensional insertions)

$$\mathfrak{W}_1 = -\beta_e^{(1)} N[e\partial_e \overline{S_0}] \cdot \Gamma + \gamma_\phi^{(1)} N[\mathcal{N}_\phi \overline{S_0}] \cdot \Gamma \,,$$

and

$$\mathfrak{R}_1 = -N[\text{r.s.p.}\, \overline{S_{\text{sct}}^{(1)}}] \cdot \Gamma \,.$$

- This is the 1-loop 4-dim RGE, but as an insertion $N[\ldots] \cdot \Gamma$.
- Since the (non-evanescent) 4-dim classical operators $\left\{ \overline{S_{AA}}, \overline{S^j_{\overline{\psi}\psi_R}}, \overline{S^j_{\overline{\psi_R}A\psi_R}} \right\}$ present in the 1-loop RGE constitute an operator basis, their quantum insertions $N[\ldots] \cdot \Gamma$ ***also*** form an *operator basis*.
- Therefore, $\mathfrak{W}_1 = \mathfrak{R}_1$ trivially, from the 1-loop RGE solution.

$\hbar^2$-order RGE: $\mathfrak{W}_4$ and $\mathfrak{R}_4$ (Genuine 2-loop contribs) (1/2)

$\mathfrak{W}_4$ "defines" the $\hbar^2$-order β and γ functions:

$$\begin{aligned}\mathfrak{W}_4 &= -\beta_e^{(2)} e\partial_e \overline{S_0} + \gamma_\phi^{(2)} \mathcal{N}_\phi \overline{S_0} \\ &= 2\gamma_A^{(2)} \overline{S_{AA}} + \sum_i \left(2\gamma_{\psi_i}^{(2)} \overline{S^i_{\overline{\psi}\psi}} + \left(2\gamma_{\psi_i}^{(2)} + \gamma_A^{(2)} - \beta_e^{(2)} \right) \overline{S^j_{\overline{\psi_R} A \psi_R}} \right) \\ &\quad + \left(\gamma_c^{(2)} - \gamma_A^{(2)} \right) \left(\overline{S_{\bar{c}c}} + \overline{S_{\rho c}} \right) + \left(\gamma_c^{(2)} - \beta_e^{(2)} \right) \left(\overline{S_{\bar{R}c\psi}} + \overline{S_{\overline{\psi}cR}} \right).\end{aligned}$$

$\mathfrak{R}_4$ comes from the "pure 2-loop" singular CTs (from usual QED-like diagrams):

$\left(\frac{\hbar}{16\pi^2}\right)^2$ factored out!

$$\begin{aligned}\mathfrak{R}_4 &= -2\,\text{r.s.p.}\, \overline{S^{(2,2)}_{\text{sct}}} \\ &= \frac{4e^4}{3} 6\,\text{Tr}[\mathcal{Y}_R^4] \overline{S_{AA}} - \frac{4e^4}{3} \sum_j (\mathcal{Y}_R^j)^2 \left(\frac{4}{3} \text{Tr}[\mathcal{Y}_R^2] - \frac{5}{4} (\mathcal{Y}_R^j)^2 \right) \overline{S^j_{\overline{\psi}\psi_R}} \\ &\quad - \frac{4e^4}{3} \sum_j (\mathcal{Y}_R^j)^2 \left(\frac{2}{3} \text{Tr}[\mathcal{Y}_R^2] - \frac{7}{4} (\mathcal{Y}_R^j)^2 \right) \overline{S^j_{\overline{\psi_R} A \psi_R}}.\end{aligned}$$

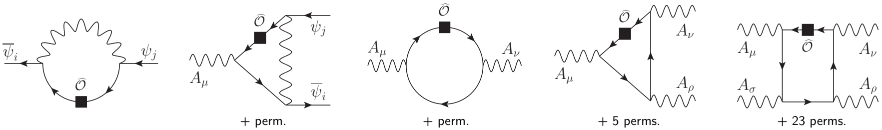
$\hbar^2$-order RGE: $\mathfrak{W}_4$ and $\mathfrak{R}_4$ (Genuine 2-loop contribs) (2/2)

We observe:

- $\gamma_A^{(2)}$ (associated with $\overline{S_{AA}}$) not the "expected" value.
- No ghost contributions $\longrightarrow \beta_e^{(2)} = \gamma_A^{(2)} = \gamma_c^{(2)}$.
- So both terms $\overline{S_{\overline{\psi}\psi}}$ and $\overline{S_{\overline{\psi}A\psi_R}}$ must have same coefficient: $2\gamma_{\psi_i}^{(2)}$.
  But not the case.

$\Longrightarrow$ Other contributions are needed!

$\mathfrak{W}_2$ and $\mathfrak{R}_2$ (Evanescent insertions)



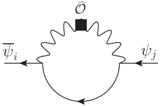
$$\left(\frac{\hbar}{16\pi^2}\right)^2 \text{ factored out!}$$

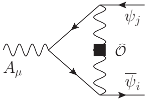
$$\mathfrak{W}_2 = \gamma_\phi^{(1)} N[\mathcal{N}_\phi \widehat{S_0}] \cdot \Gamma$$

$$= 2\gamma_A^{(1)} N[\widehat{S_{AA}}] \cdot \Gamma + \sum_i \overbrace{1 \times \gamma_{\psi_i}^{(1)} N[\widehat{S^i_{\overline{\psi}\psi}}] \cdot \Gamma}^{=2\gamma_{\psi_i}^{(1)} N[\overline{S^i_{\overline{\psi}\psi_L}} + \widehat{S^i_{\overline{\psi}\psi}}]\cdot\Gamma} + \underbrace{\left(\gamma_c^{(1)} - \gamma_A^{(1)}\right)}_{=0} N[\widehat{S_{\bar{c}c}} + \widehat{S_{\rho c}}] \cdot \Gamma$$

$$\overset{\xi=1}{\longrightarrow} \frac{4e^4}{3} \operatorname{Tr}[\mathcal{Y}_R^4] \overline{S_{AA}} - 2e^4 \sum_j (\mathcal{Y}_R^j)^4 \overline{S^j_{\overline{\psi}\psi_R}}$$

$$+ \frac{4e^4}{9} \sum_j \left(\operatorname{Tr}[\mathcal{Y}_R^2](\mathcal{Y}_R^j)^2 + 3(\mathcal{Y}_R^j)^4\right) \left(2\overline{S^j_{\overline{\psi}\psi_R}} + \overline{S^j_{\overline{\psi}_R A \psi_R}}\right) .$$





$\mathfrak{R}_2$ from evanescent part of $S^{(1)}_{\text{sct}}$:

$$\mathfrak{R}_2 = -N[\text{r.s.p. } \widehat{S^{(1)}_{\text{sct}}}] \cdot \Gamma = \frac{\hbar}{16\pi^2} \frac{2e^2 \operatorname{Tr}[\mathcal{Y}_R^2]}{3} N[\int d^d x \, \frac{1}{2} \bar{A}_\mu \widehat{\partial}^2 \bar{A}^\mu] \cdot \Gamma$$

$$\overset{\xi=1}{\longrightarrow} \frac{2e^4}{9} \sum_j \operatorname{Tr}[\mathcal{Y}_R^2](\mathcal{Y}_R^j)^2 \left(2\overline{S^j_{\overline{\psi}\psi_R}} + \overline{S^j_{\overline{\psi}_R A \psi_R}}\right) .$$

$\mathfrak{W}_3$ and $\mathfrak{R}_3$ (Finite CTs contribs)

$$\mathfrak{W}_3 = -\beta_e^{(1)} e\partial_e \overline{S_{\text{fct}}^{(1)}} + \gamma_\phi^{(1)} \mathcal{N}_\phi \overline{S_{\text{fct}}^{(1)}}$$

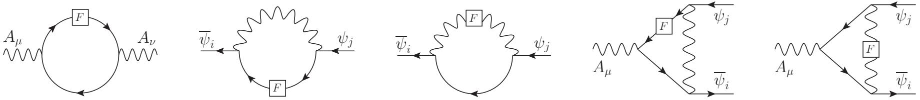
$$\equiv \underbrace{\left(\gamma_A^{(1)} - \beta_e^{(1)}\right)}_{=0} N_A \overline{S_{\text{fct}}^{(1)}} + \sum_i \left(\gamma_{\psi_i}^{(1)} + \gamma_A^{(1)} \xi \frac{\partial}{\partial \xi} - \beta_e^{(1)}\right) \mathcal{N}_{\psi_i} \overline{S_{\text{fct}}^{(1)}}$$

$$\xrightarrow{\xi=1} 2e^4 \sum_j \left(\frac{-5}{9} \mathrm{Tr}[\mathcal{Y}_R^2](\mathcal{Y}_R^j)^2 + (\mathcal{Y}_R^j)^4\right) \overline{S_{\overline{\psi}\psi_R}^j} .$$

$\left(\frac{\hbar}{16\pi^2}\right)^2$ factored out!

Important gauge dependence from $\mathcal{N}_{\psi_i} \overline{S_{\text{fct}}^{(1)}}$!!

$\mathfrak{R}_3$ from previously-evaluated $S_{\text{sct}}^{(2,1)}$:



$$\mathfrak{R}_3 = -\text{r.s.p.}\ \overline{S_{\text{sct}}^{(2,1)}} = \text{r.s.p.}\ \overline{S_{\text{fct}}^{(1)} \cdot \Gamma^{=1}}$$

$$\xrightarrow{\xi=1} -\frac{8e^4}{3} \mathrm{Tr}[\mathcal{Y}_R^4] \overline{S_{AA}} + 2e^4 \sum_j (\mathcal{Y}_R^j)^4 \overline{S_{\overline{\psi}\psi_R}^j}$$

$$-\frac{2e^4}{3} \sum_j \left(6(\mathcal{Y}_R^j)^4 - \mathrm{Tr}[\mathcal{Y}_R^2](\mathcal{Y}_R^j)^2\right) \left(\overline{S_{\overline{\psi}\psi_R}^j} + \overline{S_{\overline{\psi}_R A\psi_R}^j}\right) .$$

Summary of contributions (1/2)

$$\mathfrak{W}_4 = \mathfrak{R}_2 + \mathfrak{R}_3 + \mathfrak{R}_4 - \mathfrak{W}_2 - \mathfrak{W}_3 \,.$$

Contrib. from $(\mathfrak{R}_i - \mathfrak{W}_i)$	Contributions to operators from (normalised) $\mathfrak{W}_4$: $\left(\frac{\hbar}{16\pi^2}\right)^{-2} e^{-4} \times \left(-\beta_e^{(2)} e\partial_e + \gamma_\phi^{(2)} \mathcal{N}_\phi\right) \overline{S_0}$		
	$\overline{S_{AA}} \rightsquigarrow 2\gamma_A^{(2)}$	$\overline{S_{\bar{c}c}} + \overline{S_{\rho c}} \rightsquigarrow -\gamma_A^{(2)} + \gamma_c^{(2)}$	$\overline{S_{\bar{R}c\psi}} + \overline{S_{\overline{\psi}cR}} \rightsquigarrow -\beta_e^{(2)} + \gamma_c^{(2)}$
$\mathfrak{R}_4 = -2\,\text{r.s.p.}\, \overline{S_{\text{sct}}^{(2,2)}} \longrightarrow$	$\frac{24}{3}\,\text{Tr}[\mathcal{Y}_R^4]$	0	
$\mathfrak{R}_2 = -N[\text{r.s.p.}\, \widehat{S_{\text{sct}}^{(1)}}] \cdot \Gamma \longrightarrow$	0	0	
$\mathfrak{R}_3 = -\text{r.s.p.}\, \overline{S_{\text{sct}}^{(2,1)}} \longrightarrow$	$\frac{-8}{3}\,\text{Tr}[\mathcal{Y}_R^4]$	0	
$-\mathfrak{W}_2 = -\left(-\beta_e^{(1)} N[e\partial_e \widehat{S_0}] \cdot \Gamma + \gamma_\phi^{(1)} N[\mathcal{N}_\phi \widehat{S_0}] \cdot \Gamma\right) \longrightarrow$	$\frac{-4}{3}\,\text{Tr}[\mathcal{Y}_R^4]$	0	
$-\mathfrak{W}_3 = -\left(-\beta_e^{(1)} e\partial_e \overline{S_{\text{fct}}^{(1)}} + \gamma_\phi^{(1)} \mathcal{N}_\phi \overline{S_{\text{fct}}^{(1)}}\right) \longrightarrow$	0	0	

Summary of contributions (2/2)

$$\mathfrak{W}_4 = \mathfrak{R}_2 + \mathfrak{R}_3 + \mathfrak{R}_4 - \mathfrak{W}_2 - \mathfrak{W}_3 \,.$$

Contrib. from $(\mathfrak{R}_i - \mathfrak{W}_i)$	Contributions to operators from (normalised) $\mathfrak{W}_4$: $\left(\frac{\hbar}{16\pi^2}\right)^{-2} e^{-4} \times \left(-\beta_e^{(2)} e\partial_e + \gamma_\phi^{(2)} \mathcal{N}_\phi\right) \overline{S_0}$: $\overline{S^j_{\overline{\psi}\psi_R}} \rightsquigarrow 2\gamma_{\psi_i}^{(2)}$	$\overline{S^j_{\overline{\psi_R} A \psi_R}} \rightsquigarrow -\beta_e^{(2)} + \gamma_A^{(2)} + 2\gamma_{\psi_i}^{(2)}$
$\mathfrak{R}_4 = -2\,\text{r.s.p.}\,\overline{S_{\text{sct}}^{(2,2)}} \longrightarrow$	$\frac{-16}{9}\,\text{Tr}[\mathcal{Y}_R^2](\mathcal{Y}_R^j)^2 + \frac{5}{3}(\mathcal{Y}_R^j)^4$	$\frac{-8}{9}\,\text{Tr}[\mathcal{Y}_R^2](\mathcal{Y}_R^j)^2 + \frac{7}{3}(\mathcal{Y}_R^j)^4$
$\mathfrak{R}_2 = -N[\text{r.s.p.}\,\widehat{S_{\text{sct}}^{(1)}}]\cdot\Gamma \longrightarrow$	$\frac{4}{9}\,\text{Tr}[\mathcal{Y}_R^2](\mathcal{Y}_R^j)^2$	$\frac{2}{9}\,\text{Tr}[\mathcal{Y}_R^2](\mathcal{Y}_R^j)^2$
$\mathfrak{R}_3 = -\text{r.s.p.}\,\overline{S_{\text{sct}}^{(2,1)}} \longrightarrow$	$\frac{2}{3}\,\text{Tr}[\mathcal{Y}_R^2](\mathcal{Y}_R^j)^2 - 2(\mathcal{Y}_R^j)^4$	$\frac{2}{3}\,\text{Tr}[\mathcal{Y}_R^2](\mathcal{Y}_R^j)^2 - 4(\mathcal{Y}_R^j)^4$
$-\mathfrak{W}_2 = -\Big(-\beta_e^{(1)} N[e\partial_e \widehat{S_0}]\cdot\Gamma + \gamma_\phi^{(1)} N[\mathcal{N}_\phi \widehat{S_0}]\cdot\Gamma\Big) \longrightarrow$	$\frac{-8}{9}\,\text{Tr}[\mathcal{Y}_R^2](\mathcal{Y}_R^j)^2 - \frac{2}{3}(\mathcal{Y}_R^j)^4$	$\frac{-4}{9}\,\text{Tr}[\mathcal{Y}_R^2](\mathcal{Y}_R^j)^2 - \frac{4}{3}(\mathcal{Y}_R^j)^4$
$-\mathfrak{W}_3 = -\Big(-\beta_e^{(1)} e\partial_e \overline{S_{\text{fct}}^{(1)}} + \gamma_\phi^{(1)} \mathcal{N}_\phi \overline{S_{\text{fct}}^{(1)}}\Big) \longrightarrow$	$\frac{10}{9}\,\text{Tr}[\mathcal{Y}_R^2](\mathcal{Y}_R^j)^2 - 2(\mathcal{Y}_R^j)^4$	0

$\hbar^2$-order Solution (Feynman gauge $\xi = 1$)

Obtain system of equations for independent operators and solve:

$$\overline{S_{AA}} \to 2\gamma_A^{(2)}\,, \quad \overline{S^i_{\overline{\psi}\psi_R}} \to 2\gamma_{\psi_i}^{(2)}\,, \qquad \overline{S_{\bar{c}c}}, \overline{S_{\rho c}} \to \gamma_c^{(2)} - \gamma_A^{(2)} = 0\,,$$

$$\overline{S^i_{\overline{\psi}_R A \psi_R}} \to 2\gamma_{\psi_i}^{(2)} + \gamma_A^{(2)} - \beta_e^{(2)}\,, \qquad \overline{S_{\bar{R}c\psi}}, \overline{S_{\overline{\psi}cR}} \to \gamma_c^{(2)} - \beta_e^{(2)} = 0\,.$$

$$\beta_e^{(2)} = \gamma_A^{(2)} = \gamma_c^{(2)} = \left(\frac{\hbar}{16\pi^2}\right)^2 2e^4\,\mathrm{Tr}[\mathcal{Y}_R^4]\,,$$

$$\gamma_{\psi_i}^{(2)} = -\left(\frac{\hbar}{16\pi^2}\right)^2 e^4 \left(\frac{2}{9}\,\mathrm{Tr}[\mathcal{Y}_R^2](\mathcal{Y}_R^i)^2 + \frac{3}{2}(\mathcal{Y}_R^i)^4\right)\,.$$

Now: $\gamma_A^{(2)}$: "expected" value.
We really have: $\beta_e^{(2)} = \gamma_A^{(2)}$.

Comparison with literature: [Machacek,Vaughn–1983,'84,'85]

$\kappa = 1/2$

$$(4\pi)^4\gamma_A|_{2\text{-loop}} = -\tfrac{34}{3}g^4[C_2(G)]^2 + \kappa g^4\left[4C_2(F) + \tfrac{20}{3}C_2(G)\right]S_2(F)$$

$+$ *(Yukawa contribs.)*, (5.5)

$$(4\pi)^4\gamma^F|_{2\text{-loop}} = \textit{(Yukawa contribs.)}$$
$$+ g^4 C_2(F)\left[\tfrac{1}{2}(17 - 5\alpha + \tfrac{1}{2}\alpha^2)C_2(G) - 2\kappa S_2(F) - [illegible]\,S_2(S)\right]$$
$$- \tfrac{3}{2}g^4[C_2(F)]^2,$$ (4.4)

OK *except* for the $C_2(F)S_2(F)$ term in γ^F.

Outline

Summary

- DimReg + BMHV scheme applied to the **massless χQED** at one [arXiv:2004.14398] and two loops [arXiv:2109.11042], obtaining local **singular, evanescent**, and **BRST-restoring finite counterterms**.
- CT structure ***not "symmetric"***: d-dim. Lorentz and gauge invariance broken $\rightarrow$ "naive" multiplicative renormalization (with Z factors) not applicable for deriving RGEs. Modified version possible but cumbersome.
- Using ***Algebraic Renormalization*** framework we derive $\mathcal{O}(\hbar^2)$-order RGEs, reproducing known β_e and γ_ϕ [arXiv:2208.09006], from a simple linear system of equations (per-operator).

Possible future investigations:

- Massive case, non-zero VEV? (1-loop Abelian-Higgs by [Sanchez-Ruiz–2002].)
- Generalization to 2-loop YM / Standard Model? Higher-order results?
- Application to (SM)EFT?

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Thank you!

But wait, there is more! SMEFT-Tools (1/2)

An (SM)EFT example:

$$\mathcal{L}_{\text{EFT}} = \mathcal{L}_{\text{SM}} + \frac{\mathcal{C}^{(6)}}{\Lambda^2}\mathcal{O}^{(6)} + \frac{\mathcal{C}^{(8)}}{\Lambda^4}\mathcal{O}^{(8)} + \dots .$$

- Slavnov-Taylor Identity double-expansion *(i)* in $\hbar$, *(ii)* in $1/\Lambda$:

$$\mathcal{S}(\Gamma) = \int \frac{\delta\Gamma}{\delta K_\phi}\frac{\delta\Gamma}{\delta\phi} \equiv (\Gamma;\Gamma) = (\Gamma^{(4)};\Gamma^{(4)}) + \frac{1}{\Lambda^2}\left\{(\Gamma^{(4)};\Gamma^{(6)}) + (\Gamma^{(6)};\Gamma^{(4)})\right\} + \frac{1}{\Lambda^4}\left\{(\Gamma^{(4)};\Gamma^{(8)}) + (\Gamma^{(8)};\Gamma^{(4)}) + (\Gamma^{(6)};\Gamma^{(6)})\right\} + \mathcal{O}(\frac{1}{\Lambda^6}) .$$

- BRST restoration at each order in *(i)* $\hbar$, and *(ii)* $1/\Lambda$. Finite BRST-restoring counterterms of dimension ≥ 6 ?

- RGE for the EFT in our described method $\rightsquigarrow$ expanded in non-redundant (by EOMs) operator basis.

But wait, there is more! SMEFT-Tools (2/2)

Selection of EFT results that may be sensitive to this γ_5 discussion:

- One (and higher)-loop Fierz transformations [arXiv:2208.10513 Aebischer, Pesut]; Related: Basis transformations [arXiv:2202.01225 Aebischer, Buras, Kumar];
- Axion EFTs [arXiv:2112.00553 (JHEP 08 (2022) 137) Quevillon, Smith, P.N.H.Vuong] and [arXiv:2205.02248 Filoche et al.] – Functional formalism/Covariant Derivative Expansion; UOLEA: [arXiv:2006.16532 (JHEP 01 (2021) 049) Angelescu, Huang]; See papers by J.Quevillon, M.Krämer, B.Summ, ...
- SMEFT from 2HDM models, see e.g. [arXiv:2205.01561 Dawson, Fontes, Homiller, Sullivan]
- R_ξ gauge-fixing for EFTs: see [arXiv:1812.11513 Misiak, Paraskevas, Rosiek, Suxho, Zglinicki] "Effective Field Theories in R_ξ gauges".
- RGEs in generic EFTs: see talk by Mikolaj Misiak & Ignacy Nałęcz.

But wait, there is more! (SMEFT-) Tools

- Semi-automated calculations: `Mathematica` & manually.
- Model input: `FeynRules` [Christensen...–2009,Alloul...–2014] **(w/o BRST sources since unsupported)**. Manually patched for symbolic $SU(N)$.
- Loop diagrams **(w/o BRST sources)** generation: `FeynArts` [Hahn–2000]. Diagrams with sources manually generated. Amplitudes: `FeynCalc` [Mertig...–1990,Shtabovenko...–2016]; ϵ-expansion: interface `FeynHelpers` [Shtabovenko–2016] to `Package-X` [1] [Patel–2017] **(1 loop)**, or `TARCER` **(2 loops)**. (Development) versions of `FeynCalc` since ∼2020 should be OK (supports BMHV + fixes in 2-loop helpers).
- IRD method [Misiak...–1994, Chetyrkin...–1997], external momentum derivative, ...
- For $SU(N)$ model + scalars: Semi-automated group-invariants evaluation.

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1 Retired since July 1st, 2022!
See `https://github.com/FeynCalc/feyncalc/discussions/189`

Supplements

Dimensional Regularization and γ_5 (1/2)

- *Theory:* Divergent multi-loop Feynman integrals; Regularization.
  Experiment: Running of parameters; Renormalization.
  Fundamental QFT properties: Unitarity, Causality. Renormalizable gauge(-fixed) theories $\to$ **BRST symmetry** must **remain preserved**.

- **Dimensional Regularization** (DReg, ['t Hooft,Veltman–1972]...): $\mu^{4-d}\int \mathrm{d}^d x$ widely used in calculations / literature / automated codes, etc.: doesn't break gauge and Lorentz symmetries *(**as long as NO** γ_5, e.g. QCD)*.

$d = 4 - 2\epsilon$ "dimensions": $\mathbb{M}_d = \mathbb{M}_4 \oplus \mathbb{M}_{-2\epsilon}$.
Small $\epsilon > 0$ regularizes UV divergences ($\epsilon < 0$ for IR divs.).
Lorentz objects (metrics, ...): $X_{\mu\ldots} = \overline{X}_{\mu\ldots} + \widehat{X}_{\mu\ldots}$
$\widehat{X}_{\mu\ldots}$: ***evanescent*** objects.

$$g_{\mu\nu} g^{\nu\mu} = d\,, \qquad \bar{g}_{\mu\nu} \bar{g}^{\nu\mu} = 4\,, \qquad \hat{g}_{\mu\nu} \hat{g}^{\nu\mu} = -2\epsilon\,.$$

Dimensional Regularization and γ_5 (2/2)

- Observable nature **chiral** $\Rightarrow$ Realistic 4D models contain **chiral** fermions (e.g. Standard Model, ...) $\rightsquigarrow \mathbb{P}_{R/L} = (1 \pm \gamma_5)/2$. $\rightsquigarrow$ chiral anomaly, pion decay...
  - DimReg and Dirac γ^μ matrices? [Collins–1986] Intrinsic 4D objects γ_5, $\epsilon_{\mu\nu\rho\sigma}$?
- In 4D: $\{\gamma_5, \gamma^\mu\} = 0\,, \quad \mathrm{Tr}(\gamma_5 \gamma_{\mu_1} \cdots \gamma_{\mu_4}) = 4i\epsilon_{\mu_1\mu_2\mu_3\mu_4}\,, \quad \mathrm{Tr}(ab) = \mathrm{Tr}(ba)\,.$
  Inconsistent in d-D: $\mathrm{Tr}(\gamma_5 \gamma_{\mu_1} \cdots \gamma_{\mu_4}) \propto (d-4)\, \mathrm{Tr}(\gamma_5 \gamma_{\mu_1} \cdots \gamma_{\mu_4}) \underset{d\to 4}{=} 0.$
- Semi-"naive" γ_5 + manual traces fixes, syms. restoration (using Ward IDs, ...):

$$\{\gamma_5, \gamma^\mu\} = 0\,, \qquad \mathrm{Tr}(\gamma_5 \gamma_{\mu_1} \cdots \gamma_{\mu_4}) \neq 4i\epsilon_{\mu_1\mu_2\mu_3\mu_4}\,, \qquad \mathrm{Tr}(ab) = \mathrm{Tr}(ba)\,,$$

- Non-cyclicity schemes [Kreimer–1990,'94] ("reading-point prescription", ...):

$$\{\gamma_5, \gamma^\mu\} = 0\,, \qquad \mathrm{Tr}(\gamma_5 \gamma_{\mu_1} \cdots \gamma_{\mu_4}) = 4i\epsilon_{\mu_1\mu_2\mu_3\mu_4}\,, \qquad \mathrm{Tr}(ab) \neq \mathrm{Tr}(ba)\,.$$

- $\exists$ numerous other γ_5 schemes (see e.g. the reviews [Gnendiger...–2017,Bruque...–2018], and [Larin–1993,Trueman–1995,Jegerlehner–2000]).

Consistency wrt. unitarity/causality *not always clear* at high orders...

DimReg, BMHV scheme

't Hooft-Veltman-Breitenlohner-Maison ("BMHV") scheme

[Breitenlohner,Maison–1975, Breitenlohner,Maison–1977]

$\gamma_5 = (i/4!)\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$, $\{\gamma_5, \bar{\gamma}^\mu\} = 0$, but $[\gamma_5, \hat{\gamma}^\mu] = 0$,
and: $\{\gamma_5, \gamma^\mu\} = \{\gamma_5, \hat{\gamma}^\mu\}$, $[\gamma_5, \gamma^\mu] = [\gamma_5, \bar{\gamma}^\mu]$.
Cyclic trace, and $\mathrm{Tr}(\gamma_5\gamma_{\mu_1}\cdots\gamma_{\mu_4}) = 4i\epsilon_{\mu_1\mu_2\mu_3\mu_4}$.

Proven axiomatically consistent (*unitarity/causality*) at all orders.
$1/\epsilon$-pole (e.g. MS(bar)) subtraction ⇒ **"Dimensional renormalization"** (DimRen).

Extension to d dimensions

Bosonic fields in d-dims. Chiral fermions introduce **two problems**:

1. Kinetic term is **chiral** $\Rightarrow$ *non-regularized propagator* $\propto 1/\overline{\not p}$ in d-D [Bilal–2008]

> $\Rightarrow$ Need an actual d-D kinetic term:
> $\approx$ "left-handed inert" fermion component.
> (Inert because removed in interactions due to $\mathbb{P}_{\mathsf{R/L}}$.)
>
> $$i\overline{\psi}_i \not\partial \psi_i$$

2. How to promote in d-D the $\overline{\psi}\mathbb{P}_{\mathsf{L}} \not A \mathbb{P}_{\mathsf{R}}\psi$ interaction term $\propto \overline{\psi}\gamma^\mu\mathbb{P}_{\mathsf{R}}\psi$?
   $\gamma_\mu\mathbb{P}_{\mathsf{R}} = \mathbb{P}_{\mathsf{L}}\gamma_\mu = \mathbb{P}_{\mathsf{L}}\gamma_\mu\mathbb{P}_{\mathsf{R}}$ **only in 4D, *not* in d-D.**
   $\overline{\psi}\gamma^\mu\mathbb{P}_{\mathsf{R}}\psi \quad , \quad \overline{\psi}\mathbb{P}_{\mathsf{L}}\gamma^\mu\psi \quad , \quad \overline{\psi}\mathbb{P}_{\mathsf{L}}\gamma^\mu\mathbb{P}_{\mathsf{R}}\psi$
   $\Rightarrow$ **NO unique d-dimensional extension!**

> $\Rightarrow$ Use the interaction term that makes calculations **the most simple**:
>
> $$\overline{\psi}\mathbb{P}_{\mathsf{L}}\gamma^\mu\mathbb{P}_{\mathsf{R}}\psi$$
>
> **"symmetric chiral-projection"**
> (Explicitly conveys the fact that fermions are chiral.)
>
> $\equiv$ Larin symmetrization prescription $\frac{1}{2}\left(\gamma^\mu - \gamma_5\gamma^\mu\gamma_5\right)\mathbb{P}_{\mathsf{R}}$.

BRST symmetry: **Residual** symmetry *after fixing the gauge*
(≈ "generalized" version of gauge symmetry). [Becchi,Rouet,Stora–1975,Tyutin–1975]

Infinitesimal gauge transfo. of fields: $\varphi_i \to \delta_\alpha \varphi_i$ linear in the (small) gauge parameter α

$\overset{\alpha^a \to \theta c^a}{\Longrightarrow}$

θ: Grassmann parameter;
c^a: (anticommuting) ghost.
BRST transformation of φ:
$\delta_{\text{BRST}} \varphi = \theta s \varphi \equiv \delta_\alpha \varphi|_{\alpha^a \to \theta c^a}$.

All-loop order BRST invariance?

Aim: Verifying/enforcing BRST invariance $\forall$ orders of perturbation.
Algebraic renormalization framework.

BRST invariance for **quantum effective action** Γ (up to $\mathcal{O}(\hbar^n)$):
Functional Slavnov-Taylor Identities (STI) (~ Ward IDs (WTI) with gauge transfos.):

$$\mathcal{S}(\Gamma) \equiv \int \mathrm{d}\, x \left(\sum_{\Phi=A,\psi,\overline{\psi},c} \mathrm{Tr}\, \frac{\delta\Gamma}{\delta K_\Phi(x)} \frac{\delta\Gamma}{\delta\Phi(x)} + B(x) \frac{\delta\Gamma}{\delta\bar{c}(x)} \right) \overset{?}{=} 0\,.$$

($\mathcal{S}\Gamma_{\text{ren}}$: in 4 dims on renormalized Γ_{ren}; $\mathcal{S}_d\Gamma_{\text{DReg}}$: in DimReg on Γ_{DReg}.)

Quantum Action Principle [Lowenstein–1971, Piguet,Sorella–1995, Piguet,Rouet–1981]
$\Rightarrow$ BRST/ST breaking as a *local operator insertion* Δ in Γ:

$$\mathcal{S}(\Gamma) = \Delta \cdot \Gamma\,.$$

BRST restoration really matters only at the renormalized level (in 4D).

Effective action Γ: Interpretation & notation (1/2)

Effective action: Generating functional for 1-particle irreducible (1PI) Green's functions [Weinberg–1996]:

$$\Gamma[\Phi] = \sum_{n\geq 2} \frac{1}{|n|!} \int \left(\prod_{i=1}^{n} \mathrm{d}^4 x_i \; \phi_i(x_i) \right) \Gamma_{\phi_n \cdots \phi_1}(x_1, \ldots, x_n)$$

$$\left(\substack{\text{Fourier} \\ \text{transform}}\right) = \sum_{n\geq 2} \frac{1}{|n|!} \int \left(\prod_{i=1}^{n} \frac{\mathrm{d}^4 p_i}{(2\pi)^4} \widetilde{\phi}_i(p_i) \right) \Gamma_{\phi_n \cdots \phi_1}(p_1, \ldots, p_n) \overbrace{(2\pi)^4 \delta^4(\textstyle\sum_{j=1}^{n} p_j)}^{\text{Momentum conservation}} ,$$

$\Gamma_{\phi_n \cdots \phi_1}$ are the 1PI Green's functions defined by:

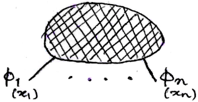
$$i\Gamma_{\phi_n \cdots \phi_1}(x_1, \ldots, x_n) = \left. \frac{i\delta^n \Gamma[\Phi]}{\delta\phi_n(x_n) \cdots \delta\phi_1(x_1)} \right|_{\phi_i = 0} = \langle \Omega | \mathbb{T}[\phi_n(x_n) \cdots \phi_1(x_1)] | \Omega \rangle^{\text{1PI}}$$

$$\equiv \langle \phi_n(x_n) \cdots \phi_1(x_1) \rangle^{\text{1PI}} ,$$

and $i\Gamma_{\phi_n \cdots \phi_1}(p_1, \ldots, p_n) \equiv \left\langle \widetilde{\phi_n}(p_n) \cdots \widetilde{\phi_1}(p_1) \right\rangle^{\text{1PI}}$ is defined similarly.

Effective action Γ: Interpretation & notation (2/2)

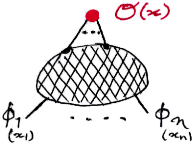
$$\Gamma[\Phi] = \sum_{n\geq 2} \frac{-i}{|n|!} \int \left(\prod_{i=1}^{n} \mathrm{d}^4 x_i \; \phi_i(x_i) \right) \langle \phi_n(x_n) \cdots \phi_1(x_1) \rangle^{\text{1PI}}$$

$$= \sum_{n\geq 2} \frac{-i}{|n|!}$$ 

.

Field-Operator insertion in $\Gamma[\Phi]$ [Piguet,Rouet–1981]:
(e.g. counterterm insertions in loop diagrams...)

$$\mathcal{O}(x) \cdot \Gamma[\Phi] = \sum_{n\geq 2} \frac{-i}{|n|!} \int \left(\prod_{i=1}^{n} \mathrm{d}^4 x_i \; \phi_i(x_i) \right) \langle \mathcal{O}(x) \phi_n(x_n) \cdots \phi_1(x_1) \rangle^{\text{1PI}}$$

$$= \sum_{n\geq 2} \frac{-i}{|n|!}$$ 

.

Notation:
$\mathcal{O} \cdot \Gamma[\Phi] = \int \mathrm{d}\, x \; \mathcal{O}(x) \cdot \Gamma[\Phi]$.

BRST invariance/breaking @ tree-level?

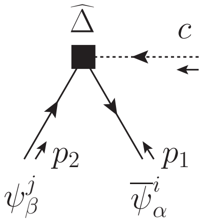
- χQED is BRST-invariant at tree-level in 4D due to gauge symmetry:

$$\mathcal{S}_4 S_0^{(4\mathrm{D})} = 0\,.$$

- Is it still so in d-dimensions? $\Rightarrow$ **No!** $\exists$ BRST breaking $\widehat{\Delta}$ at tree-level:

$$\mathcal{S}_d S_0 = {\color{red} s_d \widehat{S_{\bar\psi\psi}}} = \int \mathrm{d}^d x\; e\mathcal{Y}_{Rij} c \left\{ \overline{\psi}_i \left(\overleftarrow{\widehat{\not{\partial}}} \mathbb{P}_\mathsf{R} + \overrightarrow{\widehat{\not{\partial}}} \mathbb{P}_\mathsf{L} \right) \psi_j \right\} \equiv \widehat{\Delta}\,.$$

Interpreted as an interaction vertex whose Feynman rule is:



$$= e\mathcal{Y}_{Rij} \left(\widehat{\not{p}_1} \mathbb{P}_\mathsf{R} + \widehat{\not{p}_2} \mathbb{P}_\mathsf{L} \right)_{\alpha\beta}\,.$$

"Loop"-level BRST restoration; Renormalized action (1/2)

$\mathcal{S}\Gamma = \Delta \cdot \Gamma$ generalized for Γ_{DReg} with Regularized QAP [Breitenlohner,Maison–1977]:

using $\Delta_d \equiv \widehat{\Delta} + \Delta_{\text{ct}}$,

$$\mathcal{S}_d \Gamma_{\text{DReg}} = \Delta_d \cdot \Gamma_{\text{DReg}} \quad \underset{d\to 4}{\rightsquigarrow} \quad \mathcal{S}\Gamma_{\text{ren}} = \underset{d\to 4}{\text{LIM}}(\mathcal{S}_d \Gamma_{\text{DReg}}) = \Delta \cdot \Gamma_{\text{ren}} \,.$$

($\underset{d\to 4}{\text{LIM}}$: take $d \to 4$ and cancel evanescent structures. $\Gamma_{\text{ren}} \equiv \underset{d\to 4}{\text{LIM}}(\Gamma_{\text{DReg}})$.)

At $\mathcal{O}(\hbar^{n+1})$:

$$(\mathcal{S}\Gamma_{\text{ren}})^{(n+1)} = \underset{d\to 4}{\text{LIM}}\{\Delta_d^{(\leq n)} \cdot \Gamma_{\text{DReg}}|_{\text{div.}}^{(n+1)} + \Delta_{\text{sct}}^{(n+1)}\} + N[\Delta_d^{(\leq n)}] \cdot \Gamma_{\text{ren}}|^{(n+1)} + \Delta_{\text{fct,4}}^{(n+1)} \,,$$

$$\Delta_{\text{ct}} = \Delta_{\text{sct}} + \Delta_{\text{fct}} \equiv s_d S_{\text{sct}} + s_d S_{\text{fct}} \,, \quad \Delta_{\text{fct,4}} \equiv s_4 S_{\text{fct,4}} = \underset{d\to 4}{\text{LIM}}(s_d S_{\text{fct}}) \,.$$

S_{fct}: such that $\Delta_{\text{fct,4}}$ cancels the irrelevant anomalies from $N[\Delta_d] \cdot \Gamma_{\text{ren}}$.

Remove irrelevant anomalies if possible, with **Finite CT action** S_{fct}.
Relevant anomalies cannot be removed: ~~BRST symmetry~~, ~~renormalizability~~.

"Loop"-level BRST restoration; Renormalized action (2/2)

$$\Delta_d \equiv \widehat{\Delta} + \Delta_{\text{ct}} \; ; \qquad (\mathcal{S}\Gamma_{\text{ren}})^{(n+1)} = \underset{d\to 4}{\text{LIM}}\{\Delta_d^{(\leq n)} \cdot \Gamma_{\text{DReg}}|_{\text{div.}}^{(n+1)} + \Delta_{\text{sct}}^{(n+1)}\} + N[\Delta_d^{(\leq n)}] \cdot \Gamma_{\text{ren}}|^{(n+1)} + \Delta_{\text{fct,4}}^{(n+1)} \, .$$

General Procedure at a fixed $\hbar^{n+1}$ order:

1. Do the procedure at the previous order $\hbar^n$.
2. Evaluate $S_{\text{sct}}^{(n+1)}$ and $\Delta_{\text{sct}}^{(n+1)} = s_d S_{\text{sct}}^{(n+1)}$.
3. Evaluate $\Delta_d^{(\leq n)} \cdot \Gamma_{\text{DReg}}|^{(n+1)}$: loop diagrams with insertion of $\Delta_d^{(\leq n)}$.
4. Check whether their divergent part cancels with $\Delta_{\text{sct}}^{(n+1)}$ (breaking is finite). Evaluate their finite 4-dimensional part: $N[\Delta_d^{(\leq n)}] \cdot \Gamma_{\text{ren}}|^{(n+1)}$.
5. Define $S_{\text{fct,4}}^{(n+1)}$ such that $\Delta_{\text{fct,4}}^{(n+1)} = s_4 S_{\text{fct,4}}^{(n+1)} \overset{\text{def.}}{=} -N[\Delta_d^{(\leq n)}] \cdot \Gamma_{\text{ren}}|^{(n+1)}$ *("irrelevant anomalies")*, and verify the *absence of relevant anomalies*.

1-loop BRST restoration

$$(\mathcal{S}\Gamma_{\text{ren}})^{(1)} = \underset{d\to 4}{\text{LIM}}\{\widehat{\Delta} \cdot \Gamma_{\text{DReg}}|^{(1)}_{\text{div.}} + \Delta^{(1)}_{\text{sct}}\} + N[\widehat{\Delta}] \cdot \Gamma_{\text{ren}}|^{(1)} + \Delta^{(1)}_{\text{fct,4}} \, .$$

Procedure:

1. Evaluate $\Delta^{(1)}_{\text{sct}} = s_d S^{(1)}_{\text{sct}}$.
2. Evaluate $\widehat{\Delta} \cdot \Gamma_{\text{DReg}}|^{(1)}$: 1-loop diagrams with insertion of $\widehat{\Delta}$.
3. Check whether their divergent part cancels with $\Delta^{(1)}_{\text{sct}}$ (breaking is finite). Evaluate their finite 4-dimensional part: $N[\widehat{\Delta}] \cdot \Gamma_{\text{ren}}|^{(1)}$.
4. Define $S^{(1)}_{\text{fct}}$ such that $\Delta^{(1)}_{\text{fct,4}} = s_4 S^{(1)}_{\text{fct}} \overset{\text{def.}}{=} -N[\widehat{\Delta}] \cdot \Gamma_{\text{ren}}|^{(1)}$ *("irrelevant anomalies")*, and verify the *absence of relevant anomalies*.

2-loop BRST restoration

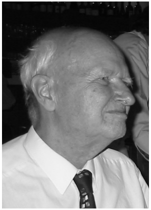
$$\Delta_d^{(1)} \equiv \widehat{\Delta} + \Delta_{\text{ct}}^{(1)} \; ; \qquad (\mathcal{S}\Gamma_{\text{ren}})^{(2)} = \underset{d\to 4}{\text{LIM}}\{\Delta_d^{(1)} \cdot \Gamma_{\text{DReg}}|_{\text{div.}}^{(2)} + \Delta_{\text{sct}}^{(2)}\} + N[\Delta_d^{(1)}] \cdot \Gamma_{\text{ren}}|^{(2)} + \Delta_{\text{fct},4}^{(2)} \, .$$

Procedure:

1. Evaluate $\Delta_{\text{sct}}^{(2)} = s_d S_{\text{sct}}^{(2)}$.
2. Evaluate $\Delta_d^{(1)} \cdot \Gamma_{\text{DReg}}|^{(2)}$: $\hbar^2$-order diagrams with insertion of $\Delta_d^{(1)}$.
3. Check whether their divergent part cancels with $\Delta_{\text{sct}}^{(2)}$ (breaking is finite). Evaluate the finite 4-dimensional part $N[\Delta_d^{(1)}] \cdot \Gamma_{\text{ren}}|^{(2)}$.
4. Define $S_{\text{fct}}^{(2)}$ such that $\Delta_{\text{fct},4}^{(2)} = s_4 S_{\text{fct}}^{(2)} \overset{\text{def.}}{=} -N[\Delta_d^{(1)}] \cdot \Gamma_{\text{ren}}|^{(2)}$ *("irrelevant anomalies")*.

Notation: "Normal Products" $N[\mathcal{O}(x)]$

Introduced by Zimmermann [Zimmermann–1973].
(See also [Lowenstein–1971].)
For a field-product operator $\mathcal{O}(x)$, a normal product $N[\mathcal{O}(x)]$ is defined as the "finite part" of $\mathcal{O}(x)$, i.e. via the finite part of the time-ordered Green's functions of $\mathcal{O}(x)$:
$\langle N[\mathcal{O}] \prod_i \phi_i(x_i) \rangle^{\text{1PI}} = \text{Fin.} \left(\langle \mathcal{O} \prod_i \phi_i(x_i) \rangle^{\text{1PI}} \right)$.
[Piguet,Rouet–1981]



They depend on the chosen renormalization scheme:

- In BPHZ renormalization (original): done by subtracting the first terms of a Taylor expansion of loop integrands up to a given order ("degree" of subtraction). $\rightarrow \exists$ different normal products associated to the choice of the "degree" of subtraction. [Piguet,Rouet–1981]
- In dimensional renormalization (DimRen): the normal products are defined with respect to the ϵ-pole subtraction. [Collins–1974]

Bonneau Identities, graphical interpretation (1/2)

In DimRen, normal products $N[\widehat{\mathcal{O}}]$ of evanescent operators $\widehat{\mathcal{O}}$ of the form $\widehat{\mathcal{O}} \equiv (\hat{g}_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu})\mathcal{O}_{\mu\nu\rho\dots}$ are interpreted [Bonneau–1980] as the difference between two ways of performing a "subtraction" in this renormalization scheme.
$\Rightarrow$ "Zimmermann-like" identities: **Bonneau Identities**.

$$N[\widehat{\mathcal{O}}] \cdot \Gamma_{\text{ren}} = -\sum_{n=2}^{n_{\text{max}}=4} \sum_{\substack{J=\{j_1,\cdots,j_n\},\\ 0\leq r\leq\delta(J)}} \sum_{\substack{\{i_1,\cdots,i_r\}/\\ 1\leq i_j\leq n}} \frac{(-i)^r}{r!} \frac{\partial^r}{\partial p_{i_1}^{\mu_1}\cdots\partial p_{i_r}^{\mu_r}} \cdot (-i\hbar)\text{r.s.p.} \left.\overline{\left\langle \prod_{i=1}^{n} \widetilde{\phi_{j_i}}(p_i) N[\check{\mathcal{O}}](q = -\textstyle\sum p_i) \right\rangle}^{\text{1PI}}\right|_{\substack{p_i=0\\ \check{g}=0}}$$
$$\times N\left[\frac{1}{n!}\prod_{k=n}^{1}\left(\prod_{\{\alpha/i_\alpha=k\}}\partial_{\mu_\alpha}\right)\phi_{j_k}\right] \cdot \Gamma_{\text{ren}} + \text{ similar with additional BV sources insertions}\,.$$

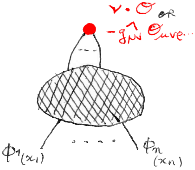
r.s.p.: residue of simple pole in $\nu = 2\epsilon = 4 - d$. Overline: 1PI minimally subtracted ("sub-renormalized").
$\check{g} \sim \hat{g}/\nu$, where this ν is not submitted to Laurent ν-expansion for the r.s.p..

$$N[\widehat{\mathcal{O}}] \cdot \Gamma_{\text{ren}} = \sum_{\{\overline{\mathcal{O}}\}_i} c_{\overline{\mathcal{O}}_i} N[\overline{\mathcal{O}}_i] \cdot \Gamma_{\text{ren}}\,.$$

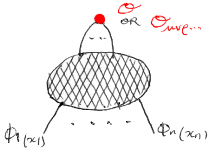
Expands evanescent operators $\widehat{\mathcal{O}}$ on a basis of quantum 4D operators of the renormalized 4D theory.

$$N[\widehat{\mathcal{O}}]\cdot\Gamma_{\text{ren}} = \sum_{\{\overline{\mathcal{O}}\}_i} c_{\overline{\mathcal{O}}_i} N[\overline{\mathcal{O}}_i]\cdot\Gamma_{\text{ren}}$$

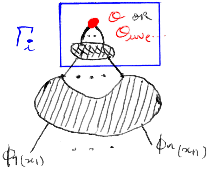
$\Bigg($



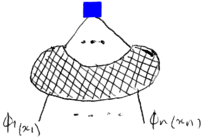
$- \left(\begin{matrix} \nu \text{ or} \\ (-\hat{g}_{\mu\nu}) \end{matrix} \right) \cdot$



$\Bigg) = \sum_{\Gamma_i}$



$$= \sum_{\{\overline{\mathcal{O}}\}_i} c_{\overline{\mathcal{O}}_i}$$



All sub-loops are sub-renormalized, including the loop containing the "special vertex" $\widehat{\mathcal{O}}$.

$$(\mathcal{S}\Gamma_{\text{ren}})^{(1)} = \underset{d\to 4}{\text{LIM}}\{\widehat{\Delta} \cdot \Gamma_{\text{DReg}}|^{(1)}_{\text{div.}} + \Delta^{(1)}_{\text{sct}}\} + N[\widehat{\Delta}] \cdot \Gamma_{\text{ren}}|^{(1)} + \Delta^{(1)}_{\text{fct,4}} \,.$$

$$N[\widehat{\Delta}] \cdot \Gamma_{\text{ren}}|^{(1)} = \underset{d\to 4}{\text{LIM}}[\widehat{\Delta} \cdot \Gamma^{(1)}]_{\text{fin}} \,,$$

finite part of $\widehat{\Delta} \cdot \Gamma_{\text{DReg}}$ after renormalization (removal of divs. and taking $\underset{d\to 4}{\text{LIM}}$).
@ Fixed $\hbar$ order: **limited finite number** of UV-singular diagrams.

Shown by interpreting $N[\widehat{\Delta}] \cdot \Gamma_{\text{ren}}|^{(1)}$ using ***Bonneau Identities*** [Bonneau–1980]:

At $\mathcal{O}(\hbar)$: $$N[\widehat{\Delta}] \cdot \Gamma_{\text{ren}}|^{(1)} = N\left[\text{r.s.p.}\left[N[-\check{\Delta}] \cdot \Gamma\right]^{(1)}_{\breve{g}=0}\right] \cdot \Gamma_{\text{ren}} \equiv \underset{d\to 4}{\text{LIM}}\left(\text{r.s.p.}\left[-\check{\Delta} \cdot \Gamma\right]^{(1)}_{\breve{g}=0}\right) .$$

"r.s.p.": residue of simple pole in $\nu = 4 - d = 2\epsilon$. $\check{\boldsymbol{\Delta}}$: $\widehat{\Delta}$ and formally replace $\hat{g}_{\mu\nu} \rightsquigarrow \breve{\mathbf{g}}_{\mu\nu}$ with:

$$\breve{g}_{\mu\nu} g^{\nu\rho} = \breve{g}_{\mu\nu} \hat{g}^{\nu\rho} = \breve{g}_{\mu}^{\ \rho} \,, \qquad \breve{g}_{\mu\nu} \bar{g}^{\nu\rho} = 0 \,, \qquad \breve{g}_{\mu}^{\ \mu} = 1 \,.$$

$\oplus$ no residual finite evanescent terms $\Rightarrow$ *Main advantage of this method.*

Evaluation of the finite part: $N[\Delta_d^{(1)}] \cdot \Gamma_{\text{ren}}|^{(2)}$: Bonneau IDs

$$(\mathcal{S}\Gamma_{\text{ren}})^{(2)} = \underset{d\to 4}{\text{LIM}}\{\Delta_d^{(1)} \cdot \Gamma_{\text{DReg}}|^{(2)}_{\text{div.}} + \Delta^{(2)}_{\text{sct}}\} + N[\Delta_d^{(1)}] \cdot \Gamma_{\text{ren}}|^{(2)} + \Delta^{(2)}_{\text{fct,4}} \,.$$

$$N[\Delta_d^{(1)}]\cdot\Gamma_{\text{ren}}|^{(2)} = N[\widehat{\Delta}+\Delta^{(1)}_{\text{ct}}]\cdot\Gamma_{\text{ren}}|^{(2)} = N[\widehat{\Delta}+\Delta^{(1)}_{\text{fct}}]\cdot\Gamma_{\text{ren}}|^{(2)} + \overbrace{\cancel{N[\Delta^{(1)}_{\text{sct}}]\cdot\Gamma_{\text{ren}}|^{(2)}}}^{=0 \text{ in } U(1) \text{ because non-1PI-insertable}} \,.$$

Interpret $N[\widehat{\Delta}] \cdot \Gamma_{\text{ren}}|^{(2)}$ using ***Bonneau Identities*** [Bonneau–1980]: at $\mathcal{O}(\hbar^2)$:

$$[N[\widehat{\Delta}] \cdot \Gamma_{\text{ren}}]^{(2)} = \underset{d\to 4}{\text{LIM}}\left(\text{r.s.p.}\left[N[-\check{\Delta}] \cdot \Gamma_{\text{DReg}}\right]^{(2)}_{\breve{g}=0}\right) + \underbrace{N\left[\text{r.s.p.}\left[N[-\check{\Delta}] \cdot \Gamma_{\text{DReg}}\right]^{(1)}_{\breve{g}=0}\right]}_{\equiv N[\widehat{\Delta}]\cdot\Gamma_{\text{ren}}|^{(1)} = -N\left[\Delta^{(1)}_{\text{fct}}\right]} \cdot\Gamma^{(1)}_{\text{ren}} \,.$$

@ Fixed $\hbar^2$ order: **limited finite number** of UV-singular diagrams.
Hence:

$$N[\Delta_d^{(1)}] \cdot \Gamma_{\text{ren}}|^{(2)} = \underset{d\to 4}{\text{LIM}}\left(\text{r.s.p.}\left[N[-\check{\Delta}] \cdot \Gamma_{\text{DReg}}\right]^{(2)}_{\breve{g}=0}\right) = \underset{d\to 4}{\text{LIM}}\left(\left[(\widehat{\Delta} + \Delta^{(1)}_{\text{ct}}) \cdot \Gamma_{\text{DReg}}\right]^{(2)}_{\text{fin}}\right) \,.$$

Model with generic gauge group $\mathcal{G}$ (usually $SU(N)$; can be something else...) with right-handed (RH) fermions in "right" (R) rep. of $\mathcal{G}$ and scalars in S rep. of $\mathcal{G}$, both coupling to gauge bosons.
Originally defined in 4 dimensions, using either Weyl, or Dirac fermions with projectors $\mathbb{P}_{\mathsf{R/L}} = (1 \pm \gamma_5)/2$.

$$S_0^{(4D)} = \int \mathrm{d}^4 x \, (\mathcal{L}_{\mathsf{YM}}^{(4D)} + \mathcal{L}_{\Psi}^{(4D)} + \mathcal{L}_{\Phi}^{(4D)} + \mathcal{L}_{\mathsf{Yuk}}^{(4D)} + \mathcal{L}_{\mathsf{gh}}^{(4D)} + \mathcal{L}_{\mathsf{g\text{-}fix}}^{(4D)}) \,,$$

with:

$$\mathcal{L}_{\mathsf{YM}}^{(4D)} = \frac{-1}{4} F_{\mu\nu}^a F^{a\,\mu\nu} \,,\ \mathcal{L}_{\Phi}^{(4D)} = \frac{1}{2}(D_\mu \Phi_m)^2 - \frac{\lambda_H^{mnop}}{4!} \Phi_m \Phi_n \Phi_o \Phi_p \,,$$

$$\mathcal{L}_{\Psi}^{(4D)} = i\overline{\Psi}_i \partial\!\!\!/ \mathbb{P}_{\mathsf{R}} \Psi_i + g_S T_{R\,ij}^{\,a} \overline{\Psi}_i G\!\!\!/^a \mathbb{P}_{\mathsf{R}} \Psi_j \equiv i\overline{\Psi}_i D\!\!\!\!/_R^{ij} \Psi_j \,,$$

$$\mathcal{L}_{\mathsf{Yuk}}^{(4D)} = -(Y_R)_{ij}^m/2\, \Phi_m \overline{\Psi}_i^C \mathbb{P}_{\mathsf{R}} \Psi_j + \mathsf{h.c.} \,,$$

$$\mathcal{L}_{\mathsf{gh}}^{(4D)} = \partial_\mu \bar{c}_a \cdot D^{ab\,\mu} c_b \,,\ \mathcal{L}_{\mathsf{g\text{-}fix}}^{(4D)} = \frac{\xi}{2} B^a B_a + B^a \partial^\mu G_\mu^a \,.$$

Note the Yukawa interaction with charge-conjugated fermion ($\neq$ Dirac model where left component couples to right component).

Nota about charge conjugation

While it is clear how to define the charge conjugation operation in 4D with e.g. an explicit construction: numerically $C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot\alpha\dot\beta} \end{pmatrix} \sim i\gamma^0\gamma^2$ with the good properties,
In d-D we can define a similar operation only by its action on the fermions – such that it turns fermions to their charge-conjugate and back: $\Psi^C = C\overline{\Psi}^T$ –, and its action on Dirac 4-spinor bilinears:

$$(\Psi^C)^C = \Psi \, , \, C^T = -C \, ;$$
$$\overline{\Psi}_i^C \Gamma \Psi_j^C = -\Psi_i^T C^{-1} \Gamma C \overline{\Psi}_j^T = \overline{\Psi}_j C \Gamma^T C^{-1} \Psi_i = \eta_\Gamma \overline{\Psi}_j \Gamma \Psi_i \, ,$$
$$\text{with: } \eta_\Gamma = \begin{cases} +1 & \text{for } \Gamma = \mathbb{1} \, , \, \gamma_5 \, , \, \gamma^\mu\gamma_5 \, , \\ -1 & \text{for } \Gamma = \gamma^\mu \, , \, \sigma^{\mu\nu} \, , \, \sigma^{\mu\nu}\gamma_5 \, . \end{cases}$$

(See e.g. Appendix A of [Tsai–2011].)

BRST transformations of fields of R-Model

The d-dimensional BRST transformations on the fields are as follows:

$$s_d G^a_\mu = D^{ab}_\mu c^b = \partial_\mu c^a + g_S f^{abc} G^b_\mu c^c \,,$$
$$s_d \Psi_i = s_d \Psi_{Ri} = i c^a g_S {T_R}^a_{ij} \Psi_{Rj} \,,\, s_d \overline{\Psi}_i = s_d \overline{\Psi_{Ri}} = +i \overline{\Psi_{Rj}} c^a g_S {T_R}^a_{ji} \,,$$
$$s_d \Phi_m = i c^a g_S \theta^a_{mn} \Phi_n \,,$$
$$s_d c^a = -\frac{1}{2} g_S f^{abc} c^b c^c \equiv i g_S c^2 \,,$$
$$s_d \bar{c}^a = B^a \,,\, s_d B^a = 0 \;\leftrightsquigarrow\; (\bar{c}^a, B^a) \text{ is a BRST doublet} \,,$$

with a similar form (noted s in what follows) in 4D.
The BRST operator s_d is nilpotent: $s_d(s_d \phi) = 0$, similarly to its 4D counterpart.

The completed R-Model defining action S_0 in d-D

Our complete defining action in d dimensions, including the antifields, reads:

$$S_0 = \int \mathrm{d}^d x \left(\mathcal{L}_{\mathsf{YM}} + \mathcal{L}_{\Psi} + \mathcal{L}_{\Phi} + \mathcal{L}_{\mathsf{Yuk}} + \mathcal{L}_{\mathsf{gh}} + \mathcal{L}_{\mathsf{g\text{-}fix}} + \mathcal{L}_{\mathsf{ext}}\right),$$

with:

$$\mathcal{L}_{\mathsf{YM}} = \frac{-1}{4} F^a_{\mu\nu} F^{a\,\mu\nu}\,,\ \mathcal{L}_{\Phi} = \frac{1}{2}(D_\mu \Phi^m)^2 - \frac{\lambda_H^{mnop}}{4!}\Phi_m\Phi_n\Phi_o\Phi_p\,,$$

$$\mathcal{L}_{\Psi} \Rightarrow i\overline{\Psi}_i \not{D}_R^{ij}\Psi_j = i\overline{\Psi}_i \not{\partial}\Psi_i + g_S T_{R\,ij}^{\;a}\overline{\Psi_{R\,i}}\mathbb{P}_{\mathsf{L}}\not{G}^a\mathbb{P}_{\mathsf{R}}\Psi_{R\,j}\,,$$

$$\mathcal{L}_{\mathsf{Yuk}} = -(Y_R)^m_{ij}/2\,\Phi_m \overline{\Psi_{R\,i}}^C \mathbb{P}_{\mathsf{R}}\Psi_{R\,j} + \text{h.c.}\,,$$

$$\mathcal{L}_{\mathsf{gh}} = \partial_\mu \bar{c}_a \cdot D^{ab\,\mu} c_b\,,\ \mathcal{L}_{\mathsf{g\text{-}fix}} = \frac{\xi}{2}B^a B_a + B^a\partial^\mu G^a_\mu\,,$$

$$\mathcal{L}_{\mathsf{ext}} = \rho^\mu_a s_d G^a_\mu + \zeta_a s_d c^a + \bar{R}^i s_d \Psi_{R\,i} + s_d\overline{\Psi_{R\,i}}R^i + \mathcal{Y}^m s_d \Phi_m\,.$$

Quantum numbers (mass dimension, ghost number and (anti)commutativity):

	G^a_μ	$\overline{\Psi}_i$, Ψ_i	Φ_m	c^a	$\bar{c}^a$	B^a	ρ^μ_a	ζ_a	R^i, $\bar{R}^i$	$\mathcal{Y}^m$	∂_μ	s
mass dim.	1	3/2	1	0	2	2	3	4	5/2	3	1	0
ghost #	0	0	0	1	-1	0	-1	-2	-1	-1	0	1
comm.	+	−	+	−	−	+	−	+	+	−	+	−

What about a L-Model?

How do the results modify for left-handed (LH) fermions? Two approaches:

1. Either note that $\mathbb{P}_{\mathsf{R}} \leftrightarrow \mathbb{P}_{\mathsf{L}}$, corresponding to the change $\gamma_5 \leftrightarrow -\gamma_5$, and related change $\epsilon^{\mu\nu\rho\sigma} \leftrightarrow -\epsilon^{\mu\nu\rho\sigma}$.

2. Or, view LH fermions in a "left" (L) representation of $\mathcal{G}$, as being the charge-conjugate of corresponding RH fermions that would belong to the conjugate representation of the "left" ones: $\mathbb{P}_{\mathsf{L}} \Psi_L \equiv (\mathbb{P}_{\mathsf{R}} \Psi_R)^C$, and $T_L \leftrightarrow T_R \equiv T_{\overline{L}}$.

NOTE: Possible mixings between RH and LH fermions (in the Yukawa sector...)!

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