Eccentric and hyperbolic black hole binaries

Enhancing effective-one-body models for generic planar orbits using numerical information

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Workshop on Gravitational Wave Modelling

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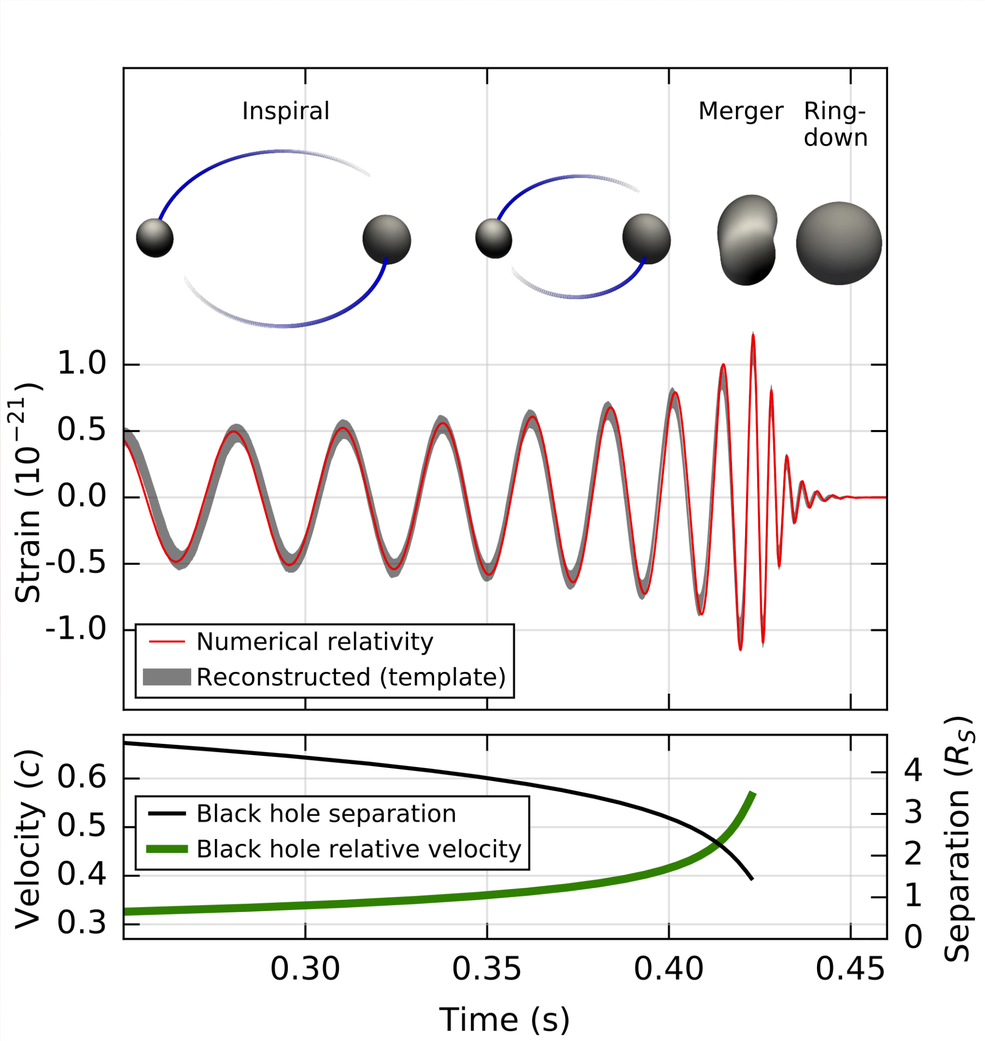
Outline

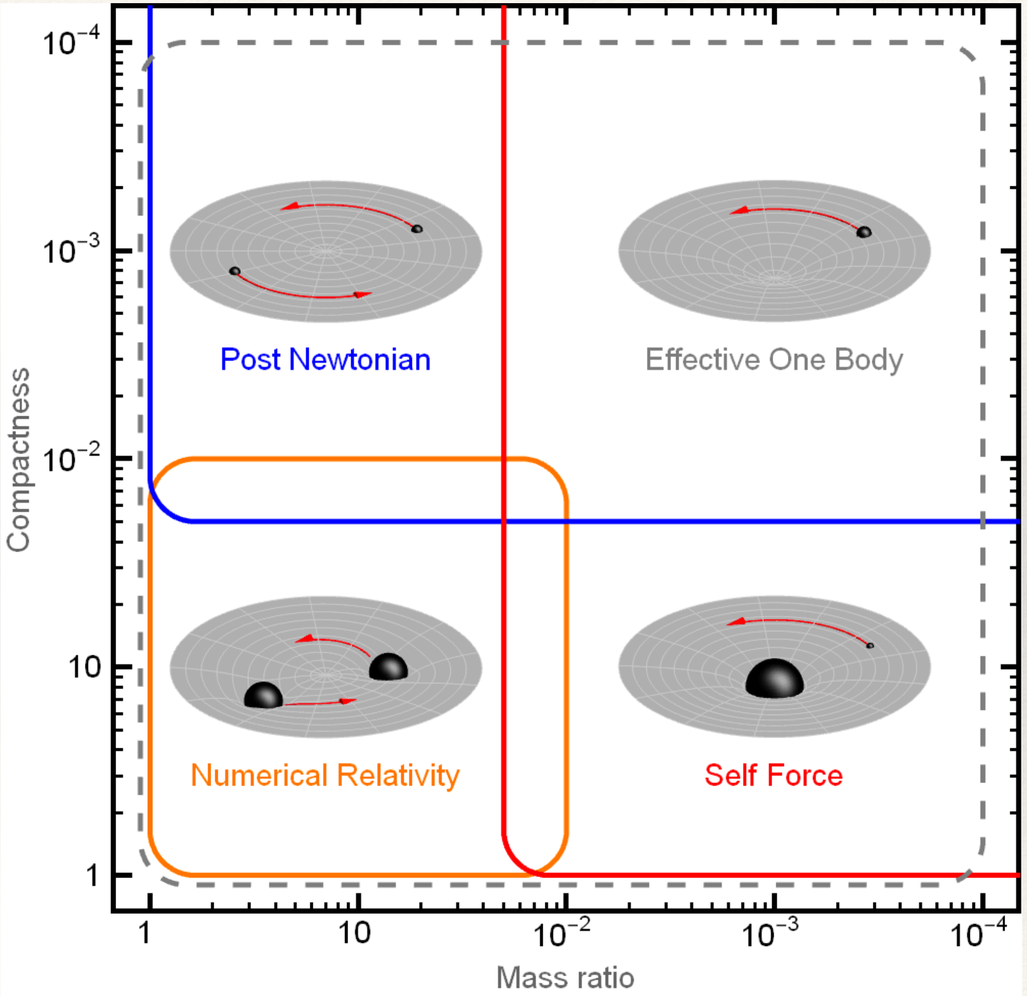
- Intro on Effective One-Body (EOB) models
- Using numerical data from black hole perturbation theory:
  - Ringdown model for eccentric EMRIs
  - NQC for eccentric EMRIs
- Dynamical captures: test-mass and $q = 1$ cases
- Future prospects

Gravitational waves

- Gravitational Waves (GWs) from compact binaries are the only sources, at the moment, for the LVK detectors
- No exact solutions for black-hole binaries: we must rely on analytical approximations or Numerical Relativity (NR) simulations
- Stages: inspiral, plunge, merger, ringdown



Effective one-body model



- (Semi)-analytical model to describe binaries in GR
- Basic idea: map the 2-body problem in a 1-body problem, where the body moves in an effective metric. This metric is a continuous ν-deformation of a black hole solution, where $\nu = m_1 m_2/(m_1 + m_2)^2 \in [0,1/4]$ is the symmetric mass-ratio
- The model is based on Post-Newtonian (PN) theory matched with Multipolar Post-Minkowskian (MPM) theory at large separations:
    - PN: expansion of Einstein Field Equations (EFE) for small velocities
    - MPM: expansion of EFE in powers of G
- Test-mass limit natively included, i.e. setting $\nu = 0$ everywhere gives geodesic motion in Kerr space-time (rotating black hole)
- Can be enhanced including information from NR and gravitational self-force (GSF) theory

TEOBResumS

- Model based on three building blocks (each of them known at a certain PN order):
  - EOB Hamiltonian: conservative part of the dynamics:

$$\hat{H}_{\rm EOB} = \frac{1}{\nu}\sqrt{1+2\nu\left(\hat{H}_\nu - 1\right)} \qquad \hat{H}_\nu = \sqrt{A_\nu\left(1+\frac{p_\varphi^2}{r_c^2}\right)+p_{r_*}^2+Q_\nu} + H_{\rm SO} \qquad A_\nu = A_\nu^{\rm orb}\frac{1+2/r_c}{1+2/r}$$

  - Radiation reaction: non-conservative part of the dynamics, i.e. back-reaction of GW-emission:

$$\hat{F}_\varphi = -\frac{32}{5}\nu r_\Omega^4\Omega^5\hat{f}_{{\rm nc}_{22}} \qquad F_r = \frac{32}{3}\nu\frac{p_{r_*}}{r^4}P_2^0[\hat{f}_r^{\rm 2PN}]$$

where $\hat{f}_{{\rm nc}_{22}}$ contains contributions from the higher modes, PN circular corrections, $\hat{h}_{\ell m}^{(N,\epsilon)_{\rm nc}}$

  - Prescription to compute the waveform at infinity from the dynamics:

$$h_+ - ih_\times = D_L^{-1}\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} h_{\ell m}\,{}_{-2}Y_{\ell m}(\theta,\phi) \qquad h_{\ell m} = h_{\ell m}^{(N,\epsilon)_{\rm c}}\,\hat{h}_{\ell m}^{(N,\epsilon)_{\rm nc}}\,\hat{S}_{\rm eff}^{(\epsilon)}\,T_{\ell m}\,e^{i\delta_{\ell m}}\,(\rho_{\ell m})^\ell\,\hat{h}_{\ell m}^{(2{\rm PN},\epsilon)_{\rm nc}}$$

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noncircular terms

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Informing EOB models with NR

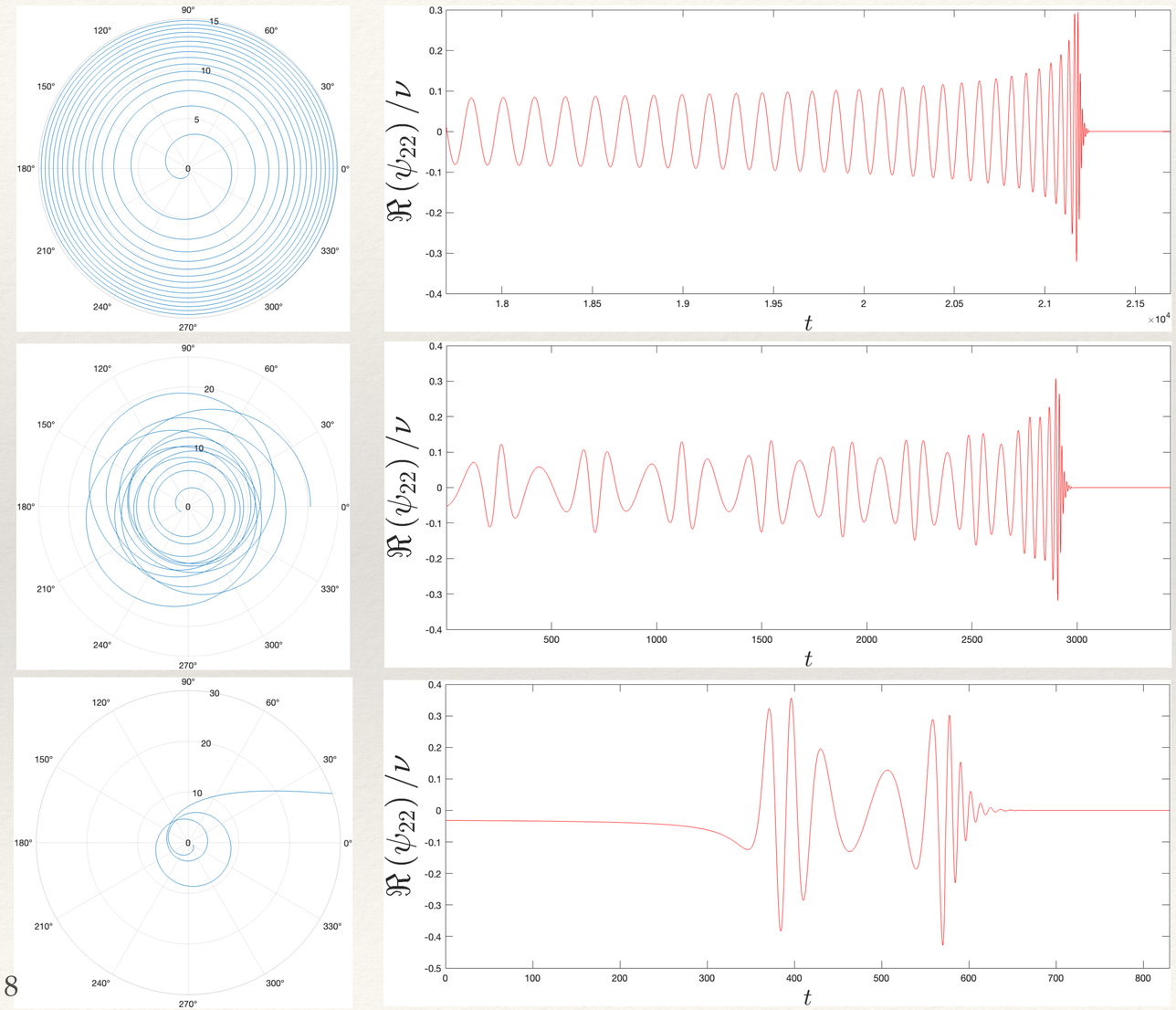
- Purely analytical EOB models cannot describe plunge accurately. Moreover, NR data and BH-perturbation theory results are needed to model the merger-ringdown.
- NR information included in EOB models:
  - Correct nonspinning sector adding free coefficients in PN expression that are then fitted on NR data
  - Next-to-Quasi-Circular (NQC) corrections for the plunge waveform: arbitrary basis of radius-derivative-like variables, coefficients extracted from NR → improves plunge
  - Ringdown model: based on Quasi-Normal-Modes (QNM) + fits on NR
- **EOB-NR models can describe the complete evolution of compact binaries**

$$h^{\rm EOB}_{\ell m} = \Theta(t_{\rm mrg} - t)\, h^{\rm inspiral}_{\ell m}\, \hat{h}^{\rm NQC}_{\ell m} \;+\; \Theta(t - t_{\rm mrg})\, h^{\rm rng}_{\ell m}$$

$t_{\rm mrg}$: A_{22}-peak time

Nonspinning equal-mass examples from TEOBResumS

- Quadrupolar waveform in the quasi-circular case
- Quadrupolar waveform in an eccentric case
- Quadrupolar waveform for a dynamical capture

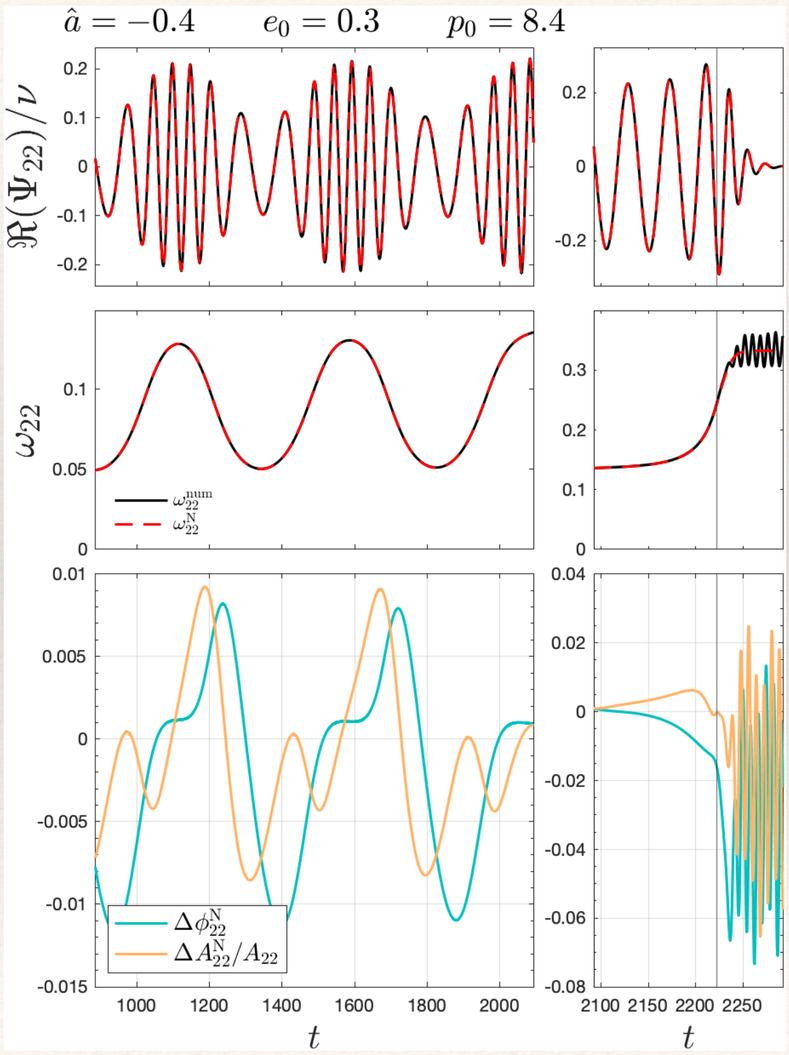


Test-mass limit

- Setting $\nu = 0$ in the Hamiltonian we recover motion in Kerr ($|\hat{a}| < 1$ is the dimensionless spin parameter)

$$\hat{H}^{\rm eq}_{\rm Kerr} = \sqrt{A\left(1+\frac{p_\varphi^2}{r_c^2}\right)+p_{r_*}^2} + \frac{2\hat{a}p_\varphi}{rr_c^2}$$

- Why should we bother with the test-mass limit?
  - Can be used to describe Extreme Mass Ratio Inspirals (EMRIs) that will be probably detected by LISA. However, for the accurate description the ν-terms in the conservative dynamics cannot be neglected, see e.g. Nagar-Albanesi:2207.14002
  - Useful to test EOB-prescriptions since the EOB dynamics can be used to compute numerical waveforms/fluxes. The dynamics is needed in the source term of the Teukolsky equation (linear perturbation of Kerr spacetime)



Ringdown model: basic idea

- $h^{\rm EOB}_{\ell m} = \Theta(t_{\rm mrg} - t)\, h^{\rm inspiral}_{\ell m}\, \hat{h}^{\rm NQC}_{\ell m} \;+\; \Theta(t - t_{\rm mrg}) \boxed{h^{\rm rng}_{\ell m}}$
- In principle described by Quasi-Normal-Mode frequencies

$$h^{\rm rng}_{\ell m} = \sum_n c_n e^{-\sigma^n_{\ell m}\tau}, \text{ where } \sigma^n_{\ell m} = \alpha^n_{\ell m} + i\omega^n_{\ell m} \text{ and } \tau = (t - t_{\rm mrg})/\alpha^1_{22}$$

- Problem: time-dependent coefficients $c_n \equiv c_n(\tau)$ are needed to accurately describe the ringdown
- We rescale the waveform with the fundamental QNM-frequency and write the rescaled-signal using amplitude and phase templates

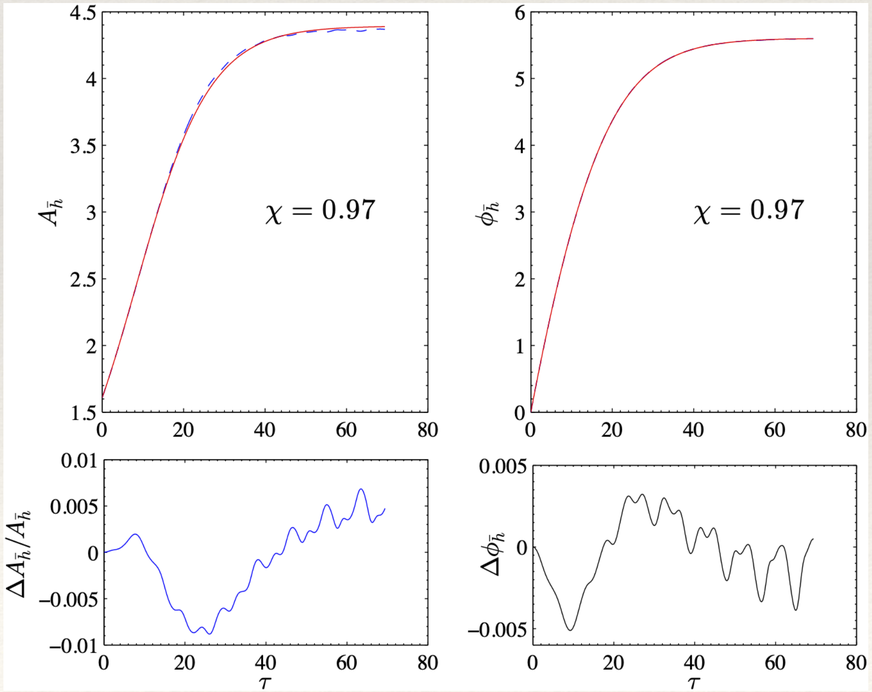
$$\bar{h}(\tau) = e^{\omega_1\tau + i\phi^{\rm mrg}_{\ell m}} \boxed{h^{\rm rng}_{\ell m}(\tau)} = A_{\bar{h}}(\tau) e^{\phi_{\bar{h}}(\tau)}$$

- The templates $A_{\bar{h}}$ and $\phi_{\bar{h}}$ depend on some parameters extracted fitting numerical waveform: **primary fits**
- We repeat the procedure for many waveforms trying to cover the parameter space (e.g. for quasi-circular binaries the parameter space could be the effective spin and the symmetric mass ratio)
- We perform **global fits** of the quantities that we need on the parameter space so that we have an analytical representation of the ringdown for all the possible configurations

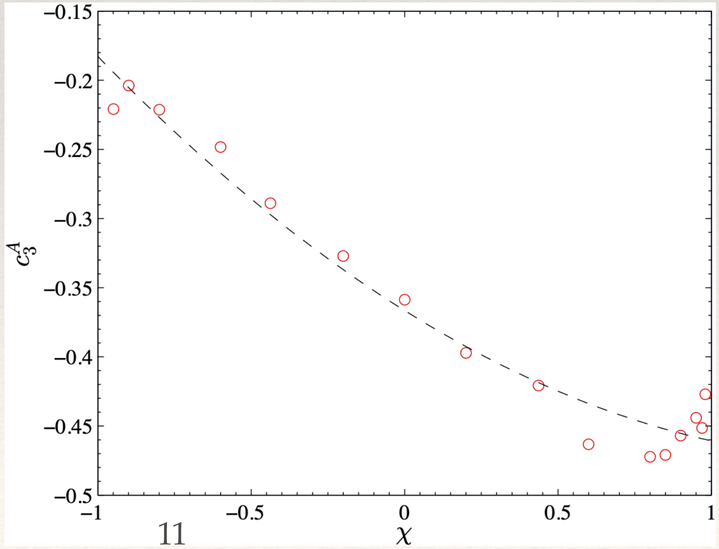
Ringdown model: QC example

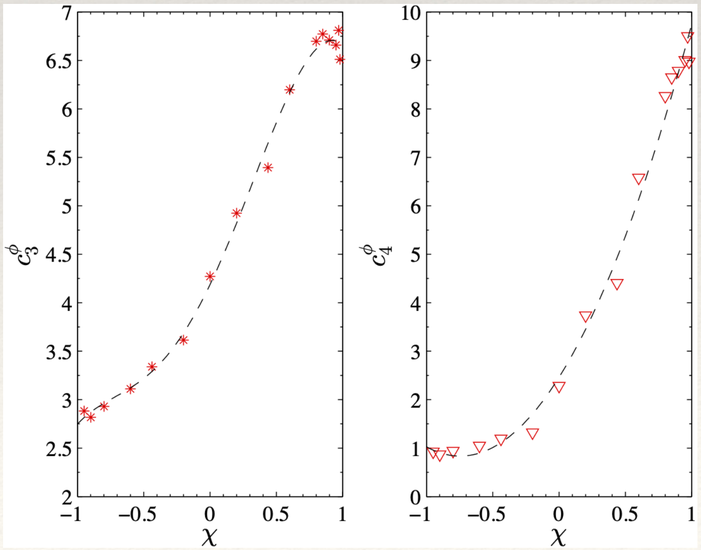
- Quasi-circular equal mass binaries, results taken from Damour-Nagar:1406.0401
- Templates: $A_{\bar{h}}(\tau) = c_1^A \tanh\left(c_2^A \tau + c_3^A\right) + c_4^A$ and $\phi_{\bar{h}}(\tau) = -\, c_1^\phi \ln\left[(1 + c_3^\phi e^{-c_2^\phi \tau} + c_4^\phi e^{-2c_2^\phi \tau})/(1 + c_3^\phi + c_4^\phi)\right]$
- Some parameters constrained by continuity: $c_2^A, c_1^A, c_4^A, c_2^\phi, c_1^\phi$ written in terms of QNM-frequencies (fundamental and first overtone), M_{BH}, $A_{22}^{\rm mrg}$, $\omega_{22}^{\rm mrg}$
- 1D parameter space: effective-spin

here τ is $(t - t_{\rm mrg})/M_{BH}$



- We need global fits for the coefficients $c_3^A, c_3^\phi, c_4^\phi$ and the NR quantities $A_{22}^{\rm mrg}, \omega_{22}^{\rm mrg}$





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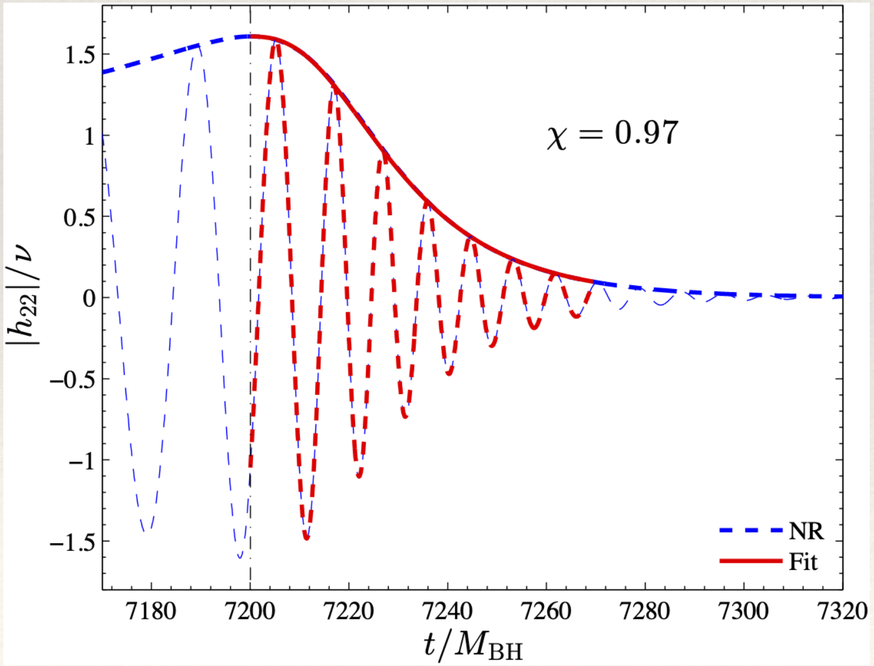
- 1D parameter space: effective-spin



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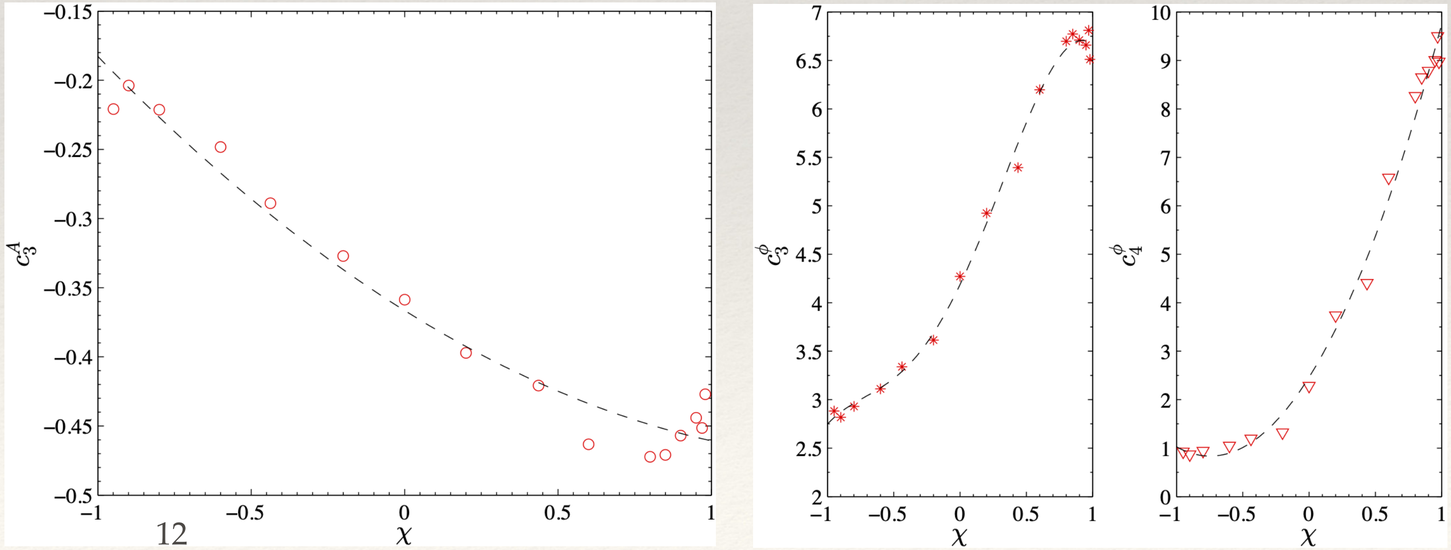


Ringdown model: eccentric, $q \to \infty$ case

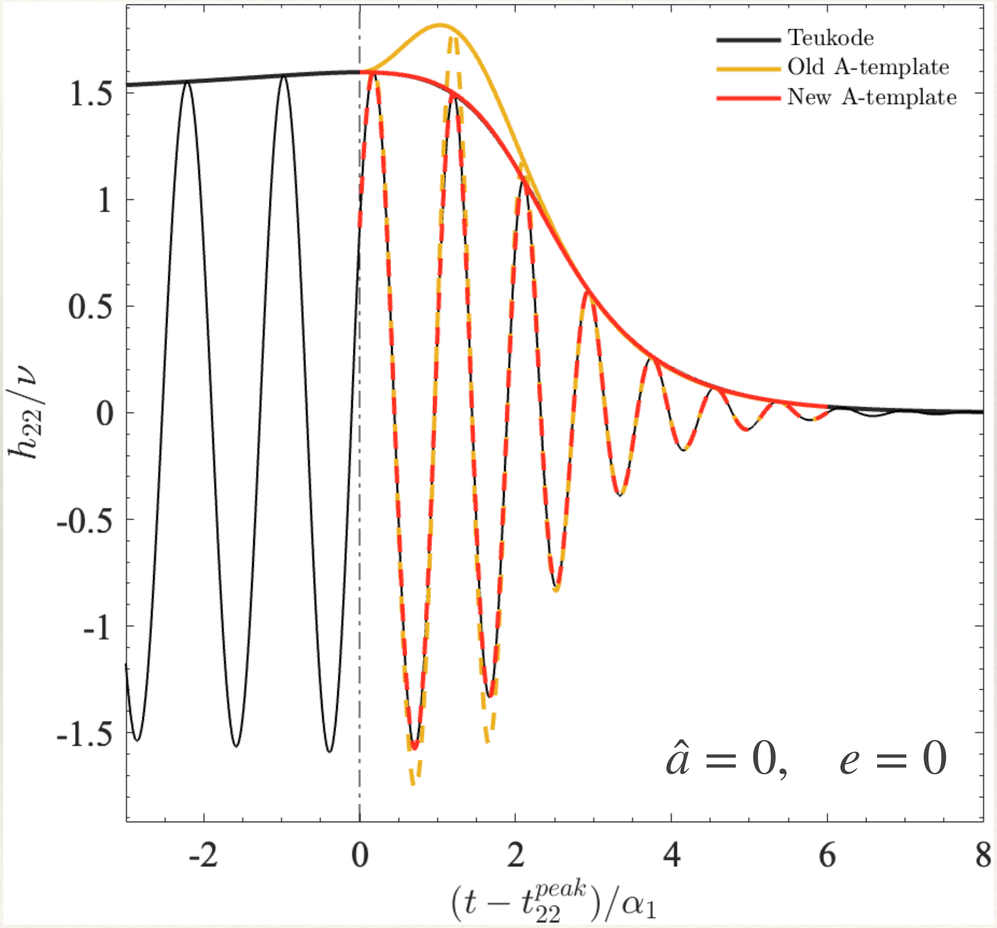
- Problem: extend the model to noncircularized binaries
- Start from the planar test-mass case, i.e. plunges in the equatorial plane of Kerr black holes.
- New amplitude template:

$$A_{\bar{h}}(\tau) = \left[c_1^A / \left(1 + e^{-c_2^A \tau + c_3^A}\right) + c_4^A\right]^{1/c_5^A}$$

- Same phase template:

$$\phi_{\bar{h}}(\tau) = -\, c_1^\phi \ln\left[(1 + c_3^\phi e^{-c_2^\phi \tau} + c_4^\phi e^{-2c_2^\phi \tau})/(1 + c_3^\phi + c_4^\phi)\right]$$

- We need global fits of 7 quantities (+ QNMs): c_2^A, c_3^A, c_1^ϕ, c_2^ϕ, $A_{\rm mrg}$, $\ddot{A}_{\rm mrg}$, $\omega_{\rm mrg}$
- 2D parameter space: Kerr-spin, a parameter that characterize non-circularity at the end of the evolution



Albanesi+:2104.10559

Eccentricity

- Not gauge-invariant: choose the definition you like most!
- Our definition (used in TEOBResumS):

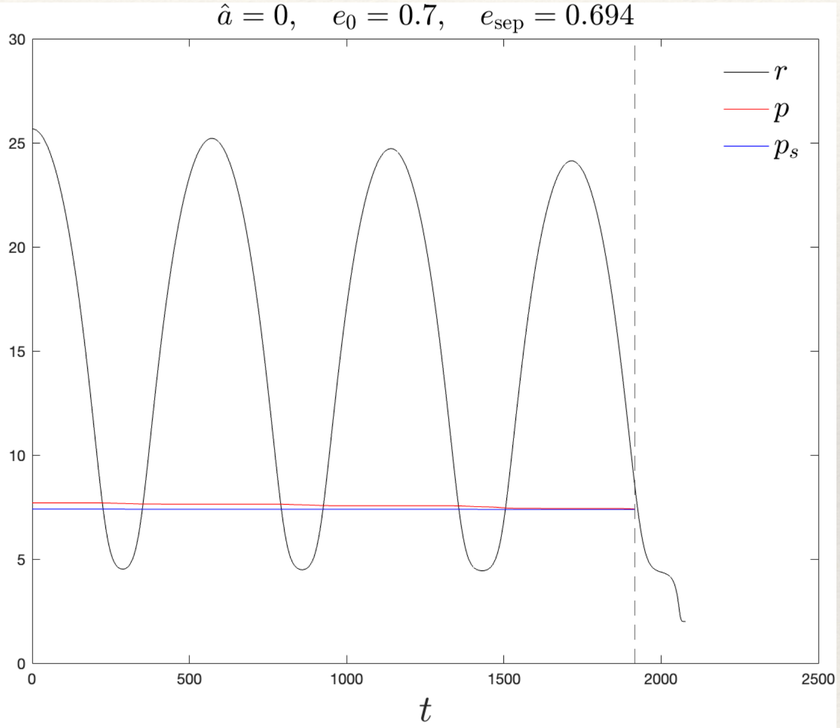
$$e = \frac{r_+ - r_-}{r_+ + r_-}, \quad p = \frac{2r_+ r_-}{r_+ + r_-}$$

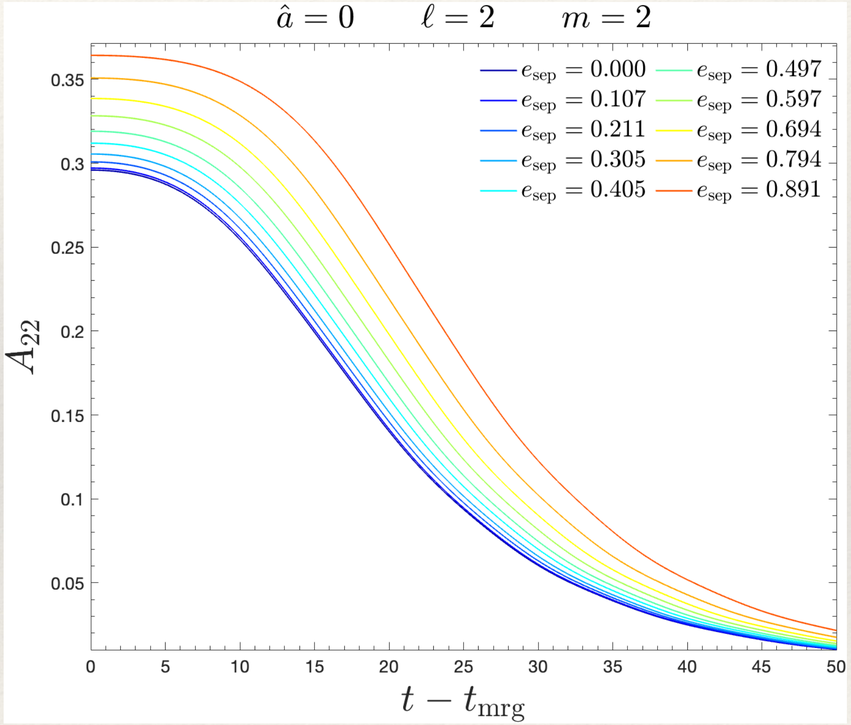
where r_+ and r_- are the apastron and the periastron. We found them from the energy equation $\hat{E} = \hat{H}_{\mathrm{EOB}}\,|_{\mathrm{p_{r*}}=0}$

- Not defined through the whole evolution because at some point the radial turning points are no longer defined
- Cannot be used for unbound orbits and dynamical captures

Ringdown model: eccentric, $q \to \infty$ case

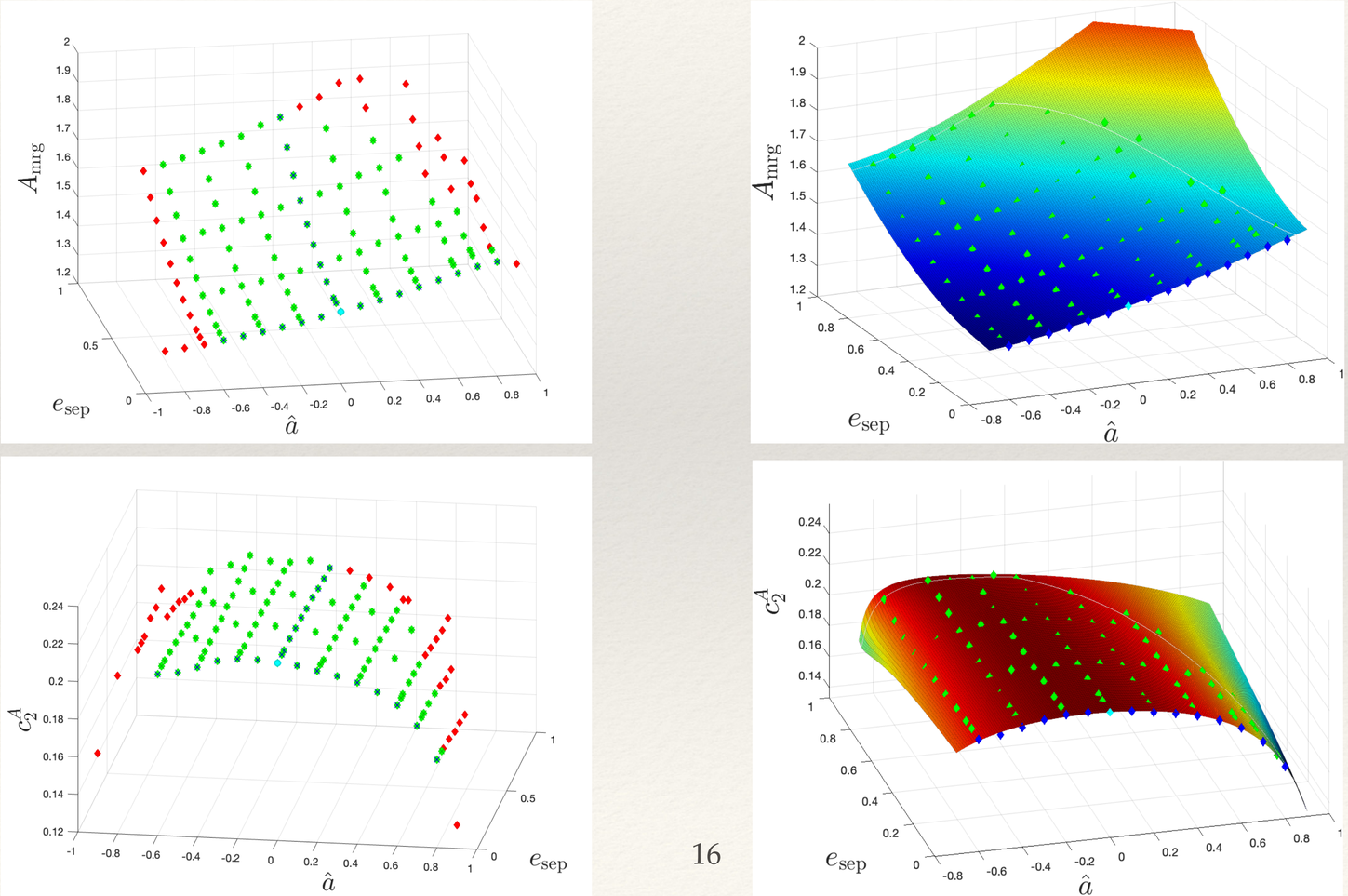
- Parameter to characterize non-circularity 'near' merger: eccentricity $e_{\rm sep}$ when the semilatus rectum p crosses the separatrix p_s (eccentric generalisation of last-stable-orbit), i.e. when the motion stops being bound and eccentricity is no longer defined
- Higher values of eccentricity produce higher amplitude at merger





Ringdown model: eccentric, $q \to \infty$ case

- $e_{\rm sep}$ is not optimal, but works. Example of 2D fits in the $(\hat{a}, e_{\rm sep})$-plane. Note that **not** all the points are considered in the global fits (red points are excluded), i.e. model not reliable if both high Kerr-spin and high eccentricity



Ringdown model: eccentric, $q \to \infty$ case

- How do we know the merger time when computing the EOB waveform from the I dynamics?

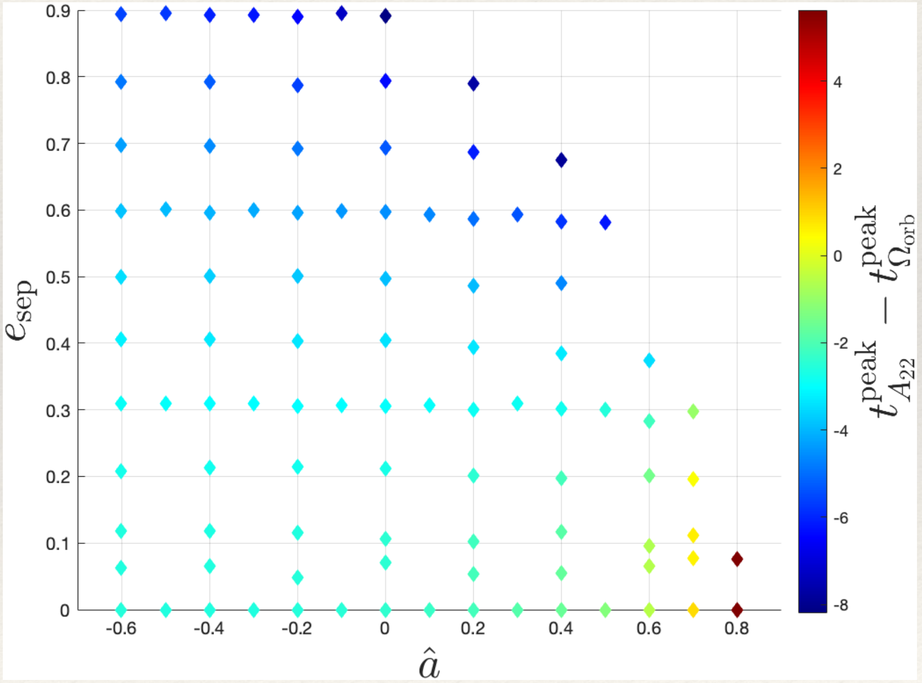
- In the quasi-circolar case a rule of thumb that works is

$$t_{\rm mrg} = t^{\rm peak}_{\Omega_{\rm orb}} - 3$$

where on the RHS there is the peak of the orbital frequency

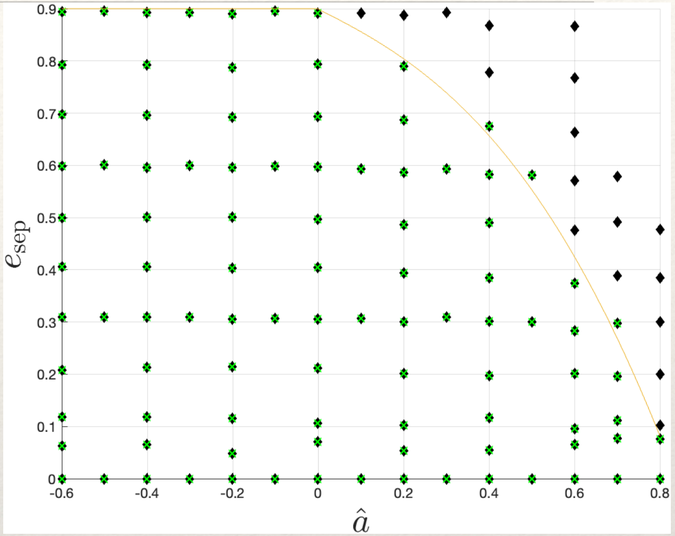
- In the noncircular case this is not very accurate

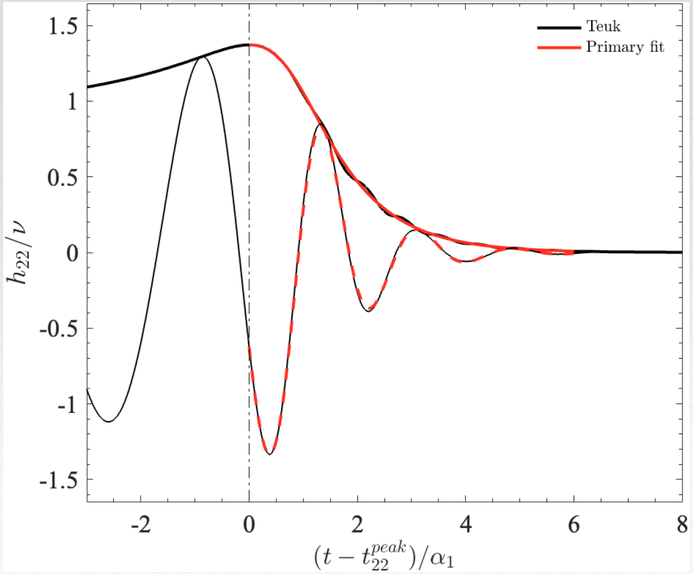
- Solution: global fit of $\Delta t = t^{\rm peak}_{A_{22}} - t^{\rm peak}_{\Omega_{\rm orb}}$



Ringdown model: eccentric, $q \to \infty$ case

- Ringdown model for eccentric plunges in Kerr
- Higher-modes not discussed but similar (important feature: the peak is delayed)
- Not reliable for both high spin and eccentricity
- Mode-mixing not included in the model
- A better parameter than $e_{\rm sep}$ could be useful for extending the model to dynamical capture. One possibility is the impact parameter





Next-to-Quasi-Circular corrections

- $h^{\rm EOB}_{\ell m} = \Theta(t_{\rm mrg} - t)\, h^{\rm inspiral}_{\ell m} \boxed{\hat{h}^{\rm NQC}_{\ell m}} + \Theta(t - t_{\rm mrg})\, h^{\rm rng}_{\ell m}$
- Named this way for historical reasons
- NQC corrections provide a smooth connection between the analytical-inspiral solution and the ringdown model
- Numerical-informed corrections for the waveform during the plunge

$$\boxed{\hat{h}^{\rm NQC}_{\ell m}} = \left(1 + \sum_{j=1}^{3} \boxed{a_j^{\ell m}}\boxed{n_j}\right) \exp\left(\mathrm{i} \sum_{j=4}^{6} \boxed{b_{j-3}^{\ell m}}\boxed{n_j}\right) \boxed{\sigma(t;\alpha,t_0)}$$

$$n_1 = \frac{p_{r_*}^2}{(r\Omega)^2}, \quad n_2 = \frac{\ddot{r}}{r\Omega^2}, \quad n_3 = n_1\, p_{r_*}^2, \qquad n_4 = \frac{p_{r_*}}{r\Omega}, \quad n_5 = n_4\, \Omega^{2/3}, \quad n_6 = n_5\, p_{r_*}^2\,.$$

- Coefficients $a_j^{\ell m}$ and $b_j^{\ell m}$ extracted from numerical data
- The sigmoid $\sigma(t;\alpha,t_0) = 1/[1 + e^{-\alpha(t-t_0)}]$ is used to switch-off the noncircular analytical corrections corrections and to switch-on the NQC corrections during the plunge

Next-to-Quasi-Circular corrections

- How to extract $a_j^{\ell m}$ and $b_j^{\ell m}$:

$$A_{\ell m}^{\rm EOB}(t_{\rm NQC}) = A_{\ell m}^{\rm num}(t_{\rm NQC}) \qquad \omega_{\ell m}^{\rm EOB}(t_{\rm NQC}) = \omega_{\ell m}^{\rm num}(t_{\rm NQC})$$

$$\dot{A}_{\ell m}^{\rm EOB}(t_{\rm NQC}) = \dot{A}_{\ell m}^{\rm num}(t_{\rm NQC}) \qquad \dot{\omega}_{\ell m}^{\rm EOB}(t_{\rm NQC}) = \dot{\omega}_{\ell m}^{\rm num}(t_{\rm NQC})$$

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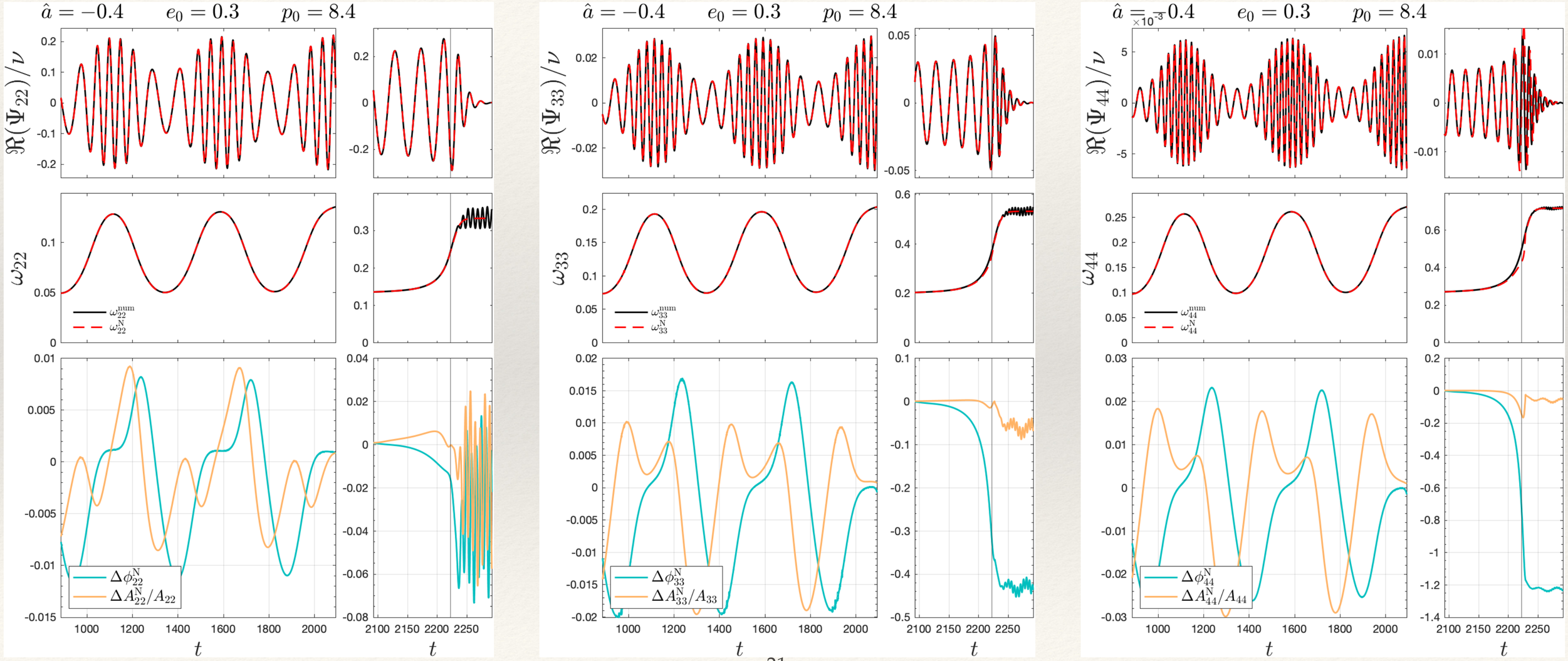
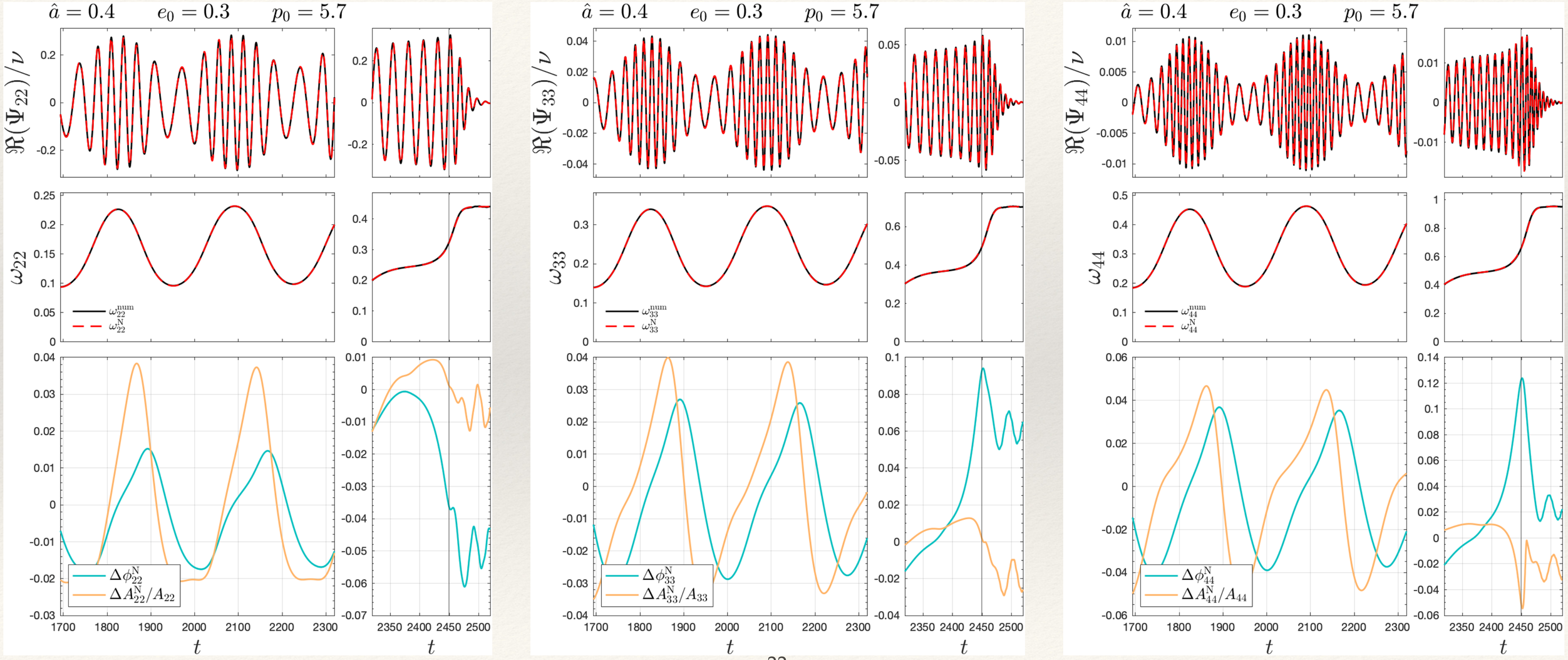
- $t_{\rm NQC} \geq t_{\rm peak} \;\rightarrow\; h^{\rm num}(t_{\rm NQC}) \equiv h^{\rm rng}(t_{\rm NQC})$, otherwise $h(t_{\rm NQC})^{\rm num}$ have to be fitted on the parameter space
- At the moment we are using $t_{\rm NQC} = t_{\rm peak} + 2$
- With these corrections there is a smooth connection between the EOB inspiral and the merger model

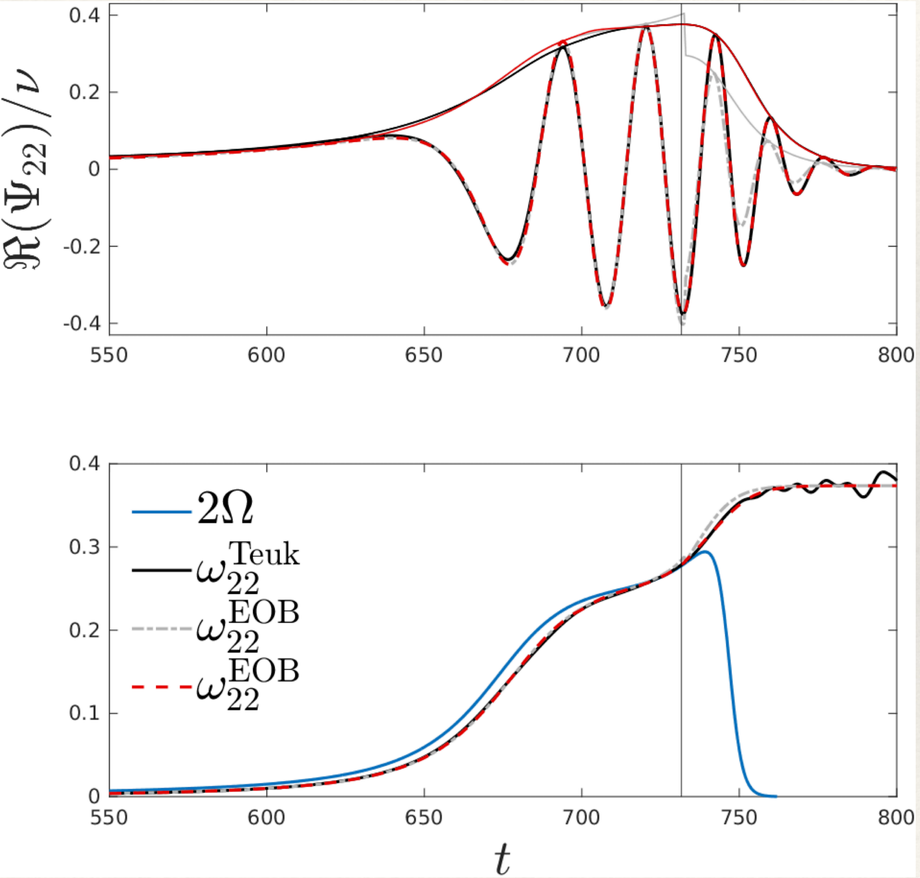
Anti-aligned spins



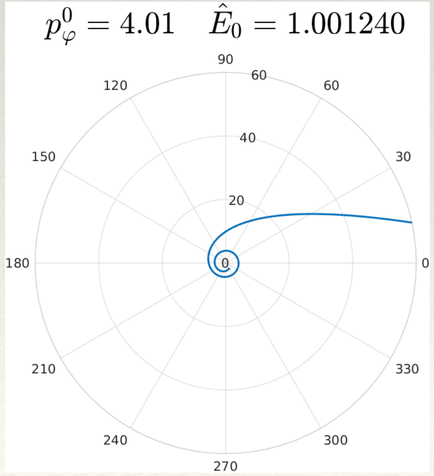
Aligned spins



Dynamical captures



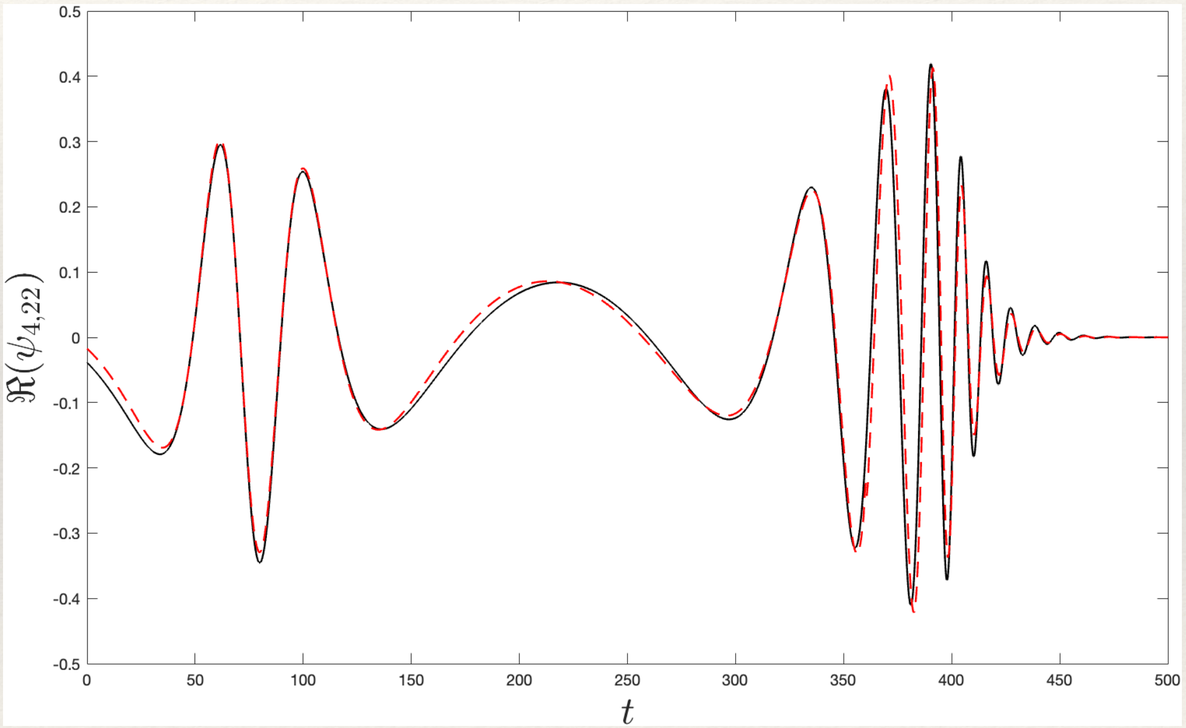
- Test-mass limit
- We currently have a few configurations, so we do not have global fits.
- Merger-time taken from numerical waveform
- Grey waveform: no NQC, quasi-circular ringdown model
- Red waveform: ringdown parameter taken from primary fits, used NQC
- Proof of principle that the templates and the general idea work also in the hyperbolic case (at least for the non-spinning configuration checked)



Dynamical captures

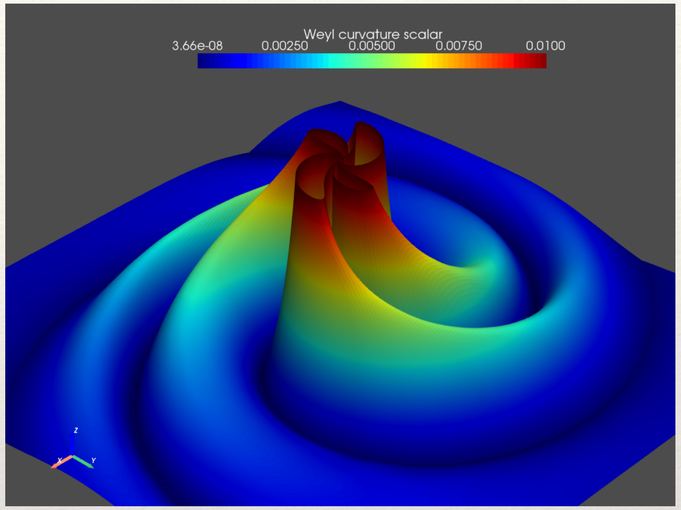
- Model used to analyze GW190521 (Gamba+:2106.05575)
- Equal mass case: work in progress
- Some key points from EOB/NR comparisons:
  - Amplitude from QC-ringdown model underestimates the real one
  - The frequency is mostly ok

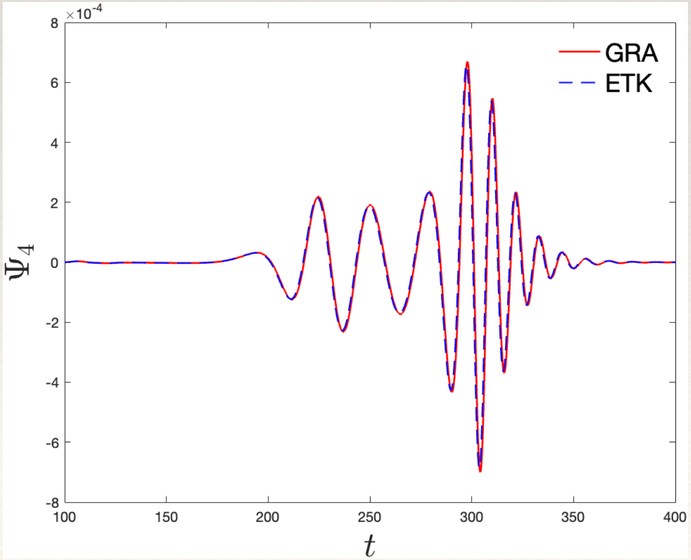
Courtesy of Tomás and Juan (QC-ringdown model)



NR simulations of dynamical captures

- Performed mostly without spins
- The integration of the Weyl scalar is tricky. Cannot use FFI methods and TD integrations are not really accurate
- Compared dynamical captures from ETK and GR-Athena++ (work with Tomás and Juan)
- Work in progress!





Summary

- Eccentric ringdown: model based on primary and global fits of numerical waveform in the test-mass limit. The same logic can be applied to the comparable mass case using NR waveforms
- NQC determined using the ringdown model
- Still many thing to do in the dynamical capture scenario, even in test-mass case
- Noncircular inspiral not discussed here, but for more info see e.g. Chiaramello+:2001.11736, Albanesi+:2104.10559, Placidi+:2112.05448, Albanesi+:2202.10063, Albanesi+:2203.16286

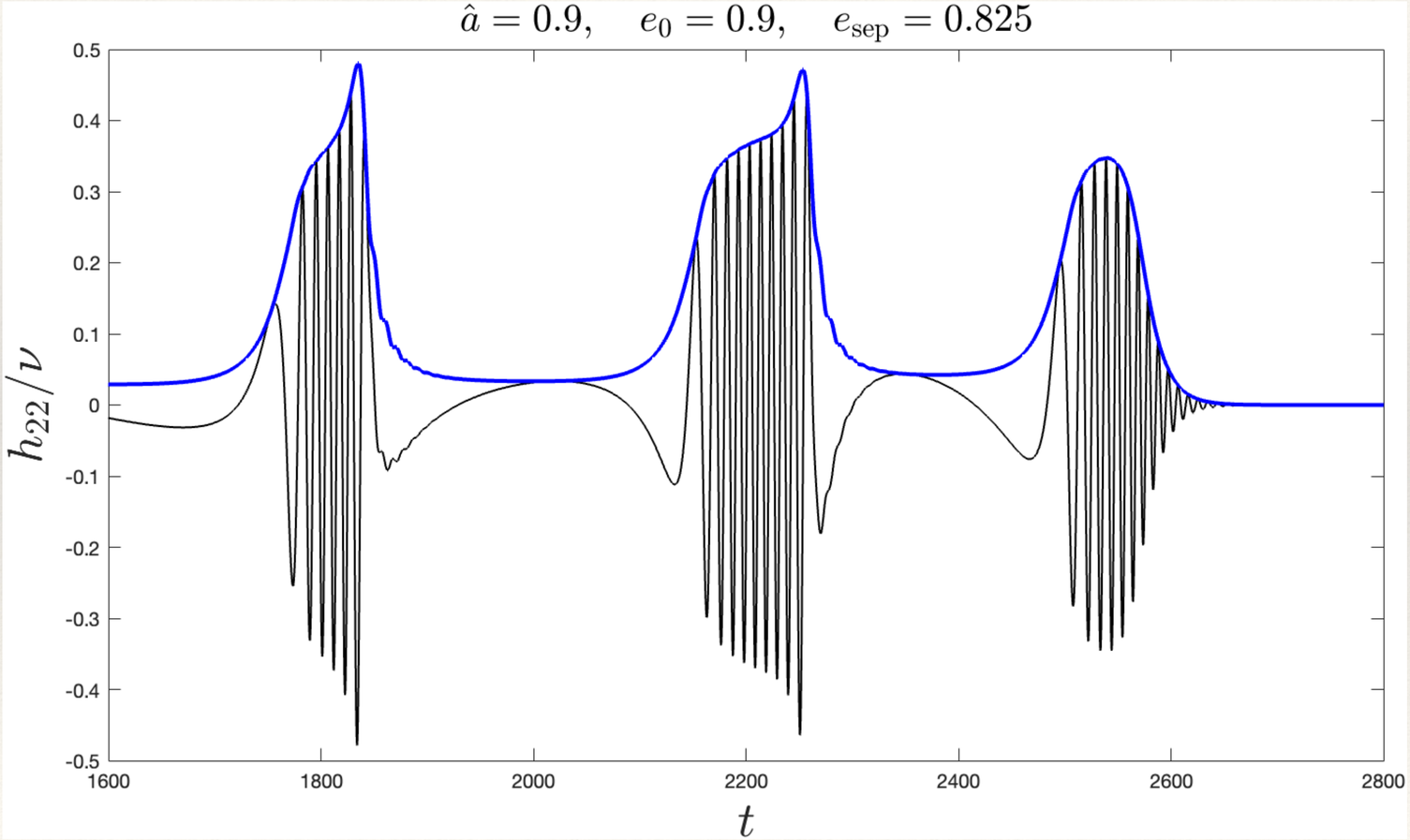
Next steps

- Including the mass-ratio in the parameter space for eccentric binaries. Probably extension to higher-dimensional parameter space could be more difficult
- Use impact parameter at merger instead of eccentricity at separatrix-crossing?
- Produce more numerical waveforms for dynamical captures in Kerr
- Other approaches? Surrogate model for ringdown could be an option

Thank you for your attention!

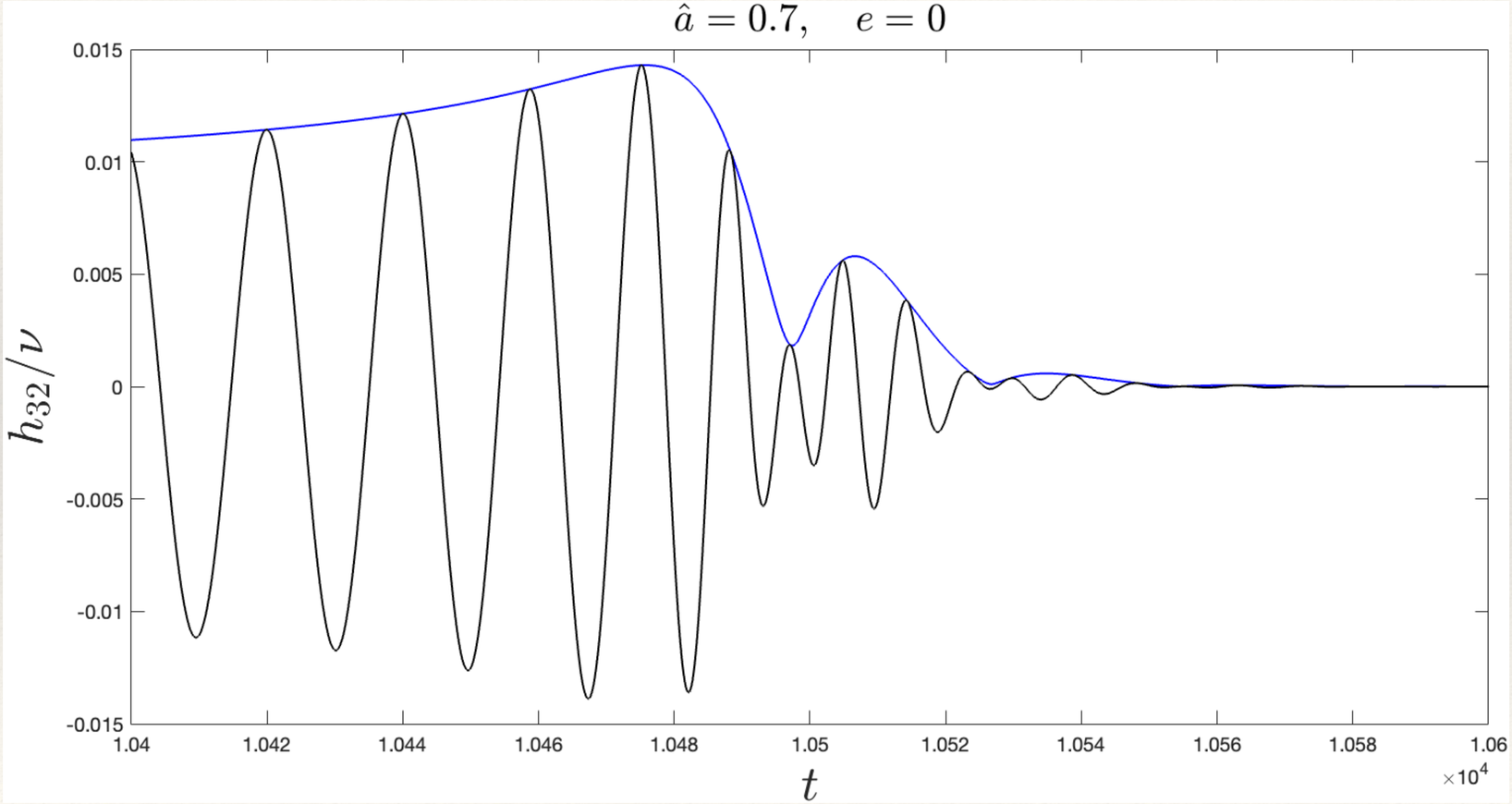
Supplemental material

- Extreme-configuration: high spin and high eccentricity also lead to QNM-burst during the inspiral



Supplemental material

- Higher modes can show mode-mixing effects also at high positive spins



Supplemental material

- Amplitude-hierarchy less clear at high positive spins