The phase of the electromagnetic form factor of the pion

Pablo Sanchez-Puertas
pablosanchez@ugr.es
Departamento de Física Atómica, Molecular y Nuclear,
University of Granada, Spain

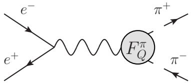
Based on E. Ruiz Arriola, PSP, arXiv:2403.07121

QNP 2024
11th July 2024, Barcelona, Spain

UNIVERSIDAD DE GRANADA

Motivation

• The $\pi^{\pm}$ electromagnetic form factor: hadronic structure in $\pi^+\pi^-\gamma^*$



• Its phase can be identified (modulo IB) with the $\pi\pi$ phase shift $\delta_1^1(s)$ below inelasticities.

• Such phase is a key ingredient in dispersive programs to describe a variety of phenomena ($\pi\pi$ scattering, πN scattering,... $F_{1,2}^N(s)$, $F_{P\gamma\gamma^*}(s)$,...).

• Above inelasticities, it probes heavy ρ-like resonances $\rightarrow$ spectroscopy.

This work

Can precise data of (the modulus) $F_Q^\pi(s)$ improve our knowledge on δ_Q?

Use long-known dispersion relations (DRs) to attack this problem

The approach proves useful to discuss other properites

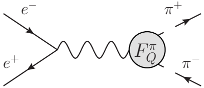
Section 1

Properties of the form factor and dispersion relations

The form factor and its phase: properties

- The pion electromagnetic form factor is defined as

$$\langle \pi^+\pi^- |\, J_Q^\mu(0)\,|0\rangle = (q_+ - q_-)^\mu F_Q^\pi(q^2), \quad J_a^\mu = \bar{q}\gamma^\mu \frac{\lambda^a}{2} q$$



- Recall that $J_Q^\mu = J_3^\mu + J_8^\mu/\sqrt{3}$, so if isospin breaking (IB), 2 form factors

$$F_Q^\pi(q^2) = F_3^\pi(q^2) + F_8^\pi(q^2)/\sqrt{3} \quad (F_8^\pi \xrightarrow{\not{IB}} 0)$$

- Generally unknown from first principles (QCD), but charge conservation...

$$F_Q^\pi(0) = F_3^\pi(0) = 1, \quad F_8^\pi(0) = 0$$

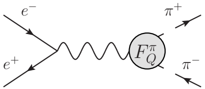
- ... and pQCD ($q^2 = -Q^2$)

$$\lim_{Q^2\to\infty} F_Q^\pi(-Q^2) = \frac{16\pi F_\pi^2 \alpha_s(\mu_R^2)}{Q^2}\left(1 + \frac{\alpha_s(\mu_R^2)}{\pi}\left[6.58 + \frac{9}{4}\ln\left(\frac{\mu_R^2}{Q^2}\right)\right]\right)$$

The form factor and its phase: properties

- The pion electromagnetic form factor is defined as

$$\langle \pi^+ \pi^- | \, J_Q^\mu(0) \, |0\rangle = (q_+ - q_-)^\mu F_Q^\pi(q^2), \quad J_a^\mu = \bar{q}\gamma^\mu \frac{\lambda^a}{2} q$$



- Recall that $J_Q^\mu = J_3^\mu + J_8^\mu/\sqrt{3}$, so if isospin breaking (IB), 2 form factors

$$F_Q^\pi(q^2) = F_3^\pi(q^2) + F_8^\pi(q^2)/\sqrt{3} \quad (F_8^\pi \xrightarrow{\not{IB}} 0)$$

- Generally unknown from first principles (QCD), but charge conservation...

$$F_Q^\pi(0) = F_3^\pi(0) = 1, \quad F_8^\pi(0) = 0$$

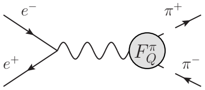
- Analytical continuation

$$\lim_{s \to \infty} F_Q^\pi(s) \to -\frac{16\pi F_\pi^2}{s} \frac{4\pi}{\beta_0} \frac{L + i\pi}{L^2 + \pi^2} \left[1 + \frac{6.58}{\pi} \frac{4\pi}{\beta_0} \frac{L + i\pi}{L^2 + \pi^2} \right],$$

The form factor and its phase: properties

- The pion electromagnetic form factor is defined as

$$\langle \pi^+\pi^- |\, J_Q^\mu(0)\, |0\rangle = (q_+ - q_-)^\mu F_Q^\pi(q^2), \quad J_a^\mu = \bar{q}\gamma^\mu \frac{\lambda^a}{2} q$$



- Recall that $J_Q^\mu = J_3^\mu + J_8^\mu/\sqrt{3}$, so if isospin breaking (IB), 2 form factors

$$F_Q^\pi(q^2) = F_3^\pi(q^2) + F_8^\pi(q^2)/\sqrt{3} \quad (F_8^\pi \xrightarrow{\not{IB}} 0)$$

- Generally unknown from first principles (QCD), but charge conservation...

$$F_Q^\pi(0) = F_3^\pi(0) = 1, \quad F_8^\pi(0) = 0$$

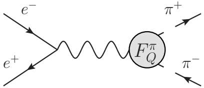
- Implying for the phase

$$\delta(s) = \pi \left[1 + L^{-1} + \frac{6.58}{\pi}\frac{4\pi}{\beta_0} L^{-2}\right] \bmod(2\pi) \qquad L = \ln(\frac{s}{\Lambda_{QCD}^2})$$

The form factor and its phase: properties

- The pion electromagnetic form factor is defined as

$$\langle \pi^+ \pi^- | \, J_Q^\mu(0) \, |0\rangle = (q_+ - q_-)^\mu F_Q^\pi(q^2), \quad J_a^\mu = \bar{q}\gamma^\mu \frac{\lambda^a}{2} q$$



- Recall that $J_Q^\mu = J_3^\mu + J_8^\mu/\sqrt{3}$, so if isospin breaking (IB), 2 form factors

$$F_Q^\pi(q^2) = F_3^\pi(q^2) + F_8^\pi(q^2)/\sqrt{3} \quad (F_8^\pi \xrightarrow{\not{IB}} 0)$$

- Generally unknown from first principles (QCD), but charge conservation...

$$F_Q^\pi(0) = F_3^\pi(0) = 1, \quad F_8^\pi(0) = 0$$

- More precise statement via the principle of the argument

$$\delta(s) = \pi \left[1 + L^{-1} + \frac{6.58}{\pi} \frac{4\pi}{\beta_0} L^{-2} + N - P \right]$$

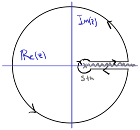
$N(P) =$ number of zeros(poles) in the first Riemann sheet (hence $P = 0$)

The form factor and its phase: dispersion relations I

- It satisfies Schwarz reflection principle, $F_Q^\pi(s^*) = [F_Q^\pi(s)]^*$
- Elastic unitarity demands $\delta_3 = \delta_1^1$ below inelasticities $[\omega\pi]$

$$\mathrm{Im}\, F_3^\pi(s) = \beta_\pi [t_1^1(s)]^* F_3^\pi(s) = \sin\delta_1^1(s) e^{-i\delta_1^1(s)} F_3^\pi(s) \Rightarrow F_3^\pi = |F_3^\pi| e^{i\delta_3 \simeq \delta_1^1}$$

- If no zeros (comments later), Cauchy's representation for $\ln F_Q^\pi(s)$ leads to (well-known) Omnès-like solution



$$F_Q^\pi(s) = \exp\left(\frac{s}{\pi} \int_{s_{th}}^{\infty} dz \frac{\delta_Q(z)}{z(z-s)} \right)$$

- Similarly, Cauchy's representation for $[\ln F_Q^\pi(s)]/\sqrt{s_{th} - s}$ leads to (the less known) modulus dispersion relation[1]

$$F_Q^\pi(s) = \exp\left(\frac{\sqrt{s_{th} - s}}{\pi} \int_{s_{th}}^{\infty} dz \frac{\ln |F_Q^\pi(z)|}{\sqrt{z - s_{th}}(z-s)} \right)$$

$\delta_Q(s)$ vs. $|F_Q^\pi(s)|$ (the latter accesible at experiment!)

---

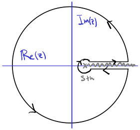
[1] T. N. Truong et al Phys.Rev.172, 1645 (1968)

The form factor and its phase: dispersion relations I

- It satisfies Schwarz reflection principle, $F_Q^\pi(s^*) = [F_Q^\pi(s)]^*$
- Elastic unitarity demands $\delta_3 = \delta_1^1$ below inelasticities $[\omega\pi]$

$$\mathrm{Im}\, F_3^\pi(s) = \beta_\pi [t_1^1(s)]^* F_3^\pi(s) = \sin\delta_1^1(s) e^{-i\delta_1^1(s)} F_3^\pi(s) \Rightarrow F_3^\pi = |F_3^\pi| e^{i\delta_3 \simeq \delta_1^1}$$

- If no zeros (comments later), Cauchy's representation for $\ln F_Q^\pi(s)$ leads to (well-known) Omnès-like solution



$$F_Q^\pi(s) = \exp\left(\frac{s}{\pi} \int_{s_{th}}^\infty dz \frac{\delta_Q(z)}{z(z-s)} \right)$$

- Similarly, Cauchy's representation for $[\ln F_Q^\pi(s)]/\sqrt{s_{th}-s}$ leads to (the less known) modulus dispersion relation[1]

$$\delta(s) = -\frac{\sqrt{s-s_{th}}}{\pi}\, \mathrm{PV} \int_{s_{th}}^\infty dz \frac{\ln|F_Q^\pi(z)|}{(z-s)\sqrt{z-s_{th}}}$$

$\delta_Q(s)$ vs. $|F_Q^\pi(s)|$ (the latter accesible at experiment!)

[1] T. N. Truong et al Phys.Rev.172, 1645 (1968)

The form factor and its phase: dispersion relations II

- Improve on convergence subtracting at 0 (DR1)

$$F_Q^\pi(s) = \exp\left(\frac{s\sqrt{s_{th}-s}}{\pi}\int_{s_{th}}^\infty dz\,\frac{\ln|F_Q^\pi(z)|}{z\sqrt{z-s_{th}}(z-s)}\right),$$
$$\delta(s) = -\frac{s\sqrt{s-s_{th}}}{\pi}\,\mathrm{PV}\int_{s_{th}}^\infty dz\,\frac{\ln|F_Q^\pi(z)|}{z\sqrt{z-s_{th}}(z-s)},$$

- Improve on convergence subtracting at 0 and threshold (P-wave) (DR2)

$$F_Q^\pi(s) = F_Q^\pi(s_{th})^{1-\left(\frac{s_{th}-s}{s_{th}}\right)^{3/2}}\exp\left[-\frac{s(s_{th}-s)^{3/2}}{\pi}\int_{s_{th}}^\infty dz\,\frac{\ln|F_Q^\pi(z)/F_Q^\pi(s_{th})|}{z(z-s_{th})^{3/2}(z-s)}\right],$$
$$\delta(s) = -\ln F_Q^\pi(s_{th})\left(\frac{s-s_{th}}{s_{th}}\right)^{3/2} - \frac{s(s-s_{th})^{3/2}}{\pi}\,\mathrm{PV}\int_{s_{th}}^\infty dz\,\frac{\ln|F_Q^\pi(z)/F_Q^\pi(s_{th})|}{z(z-s_{th})^{3/2}(z-s)},$$

- The following sum rule can be derived (important role!!)

$$\frac{s_{th}^{3/2}}{\pi\ln F_Q^\pi(s_{th})}\int_{s_{th}}^\infty dz\,\frac{\ln|F_Q^\pi(z)/F_Q^\pi(s_{th})|}{z(z-s_{th})^{3/2}} = 1.$$

The form factor and its phase: dispersion relations II

- Improve on convergence subtracting at 0 (DR1)

$$F_Q^\pi(s) = \exp\left(\frac{s\sqrt{s_{th}-s}}{\pi}\int_{s_{th}}^{\Lambda^2} dz \frac{\ln|F_Q^\pi(z)|}{z\sqrt{z-s_{th}}(z-s)}\right),$$

$$\delta(s) = -\frac{s\sqrt{s-s_{th}}}{\pi}\,\mathrm{PV}\int_{s_{th}}^{\Lambda^2} dz \frac{\ln|F_Q^\pi(z)|}{z\sqrt{z-s_{th}}(z-s)},$$

- Improve on convergence subtracting at 0 and threshold (P-wave) (DR2)

$$F_Q^\pi(s) = F_Q^\pi(s_{th})^{1-\left(\frac{s_{th}-s}{s_{th}}\right)^{3/2}} \exp\left[-\frac{s(s_{th}-s)^{3/2}}{\pi}\int_{s_{th}}^{\Lambda^2} dz \frac{\ln|F_Q^\pi(z)/F_Q^\pi(s_{th})|}{z(z-s_{th})^{3/2}(z-s)}\right],$$

$$\delta(s) = -\ln F_Q^\pi(s_{th})\left(\frac{s-s_{th}}{s_{th}}\right)^{3/2} - \frac{s(s-s_{th})^{3/2}}{\pi}\,\mathrm{PV}\int_{s_{th}}^{\Lambda^2} dz \frac{\ln|F_Q^\pi(z)/F_Q^\pi(s_{th})|}{z(z-s_{th})^{3/2}(z-s)},$$

- The following sum rule can be derived (important role!!)

$$\frac{s_{th}^{3/2}}{\pi\ln F_Q^\pi(s_{th})}\int_{s_{th}}^{\Lambda^2} dz \frac{\ln|F_Q^\pi(z)/F_Q^\pi(s_{th})|}{z(z-s_{th})^{3/2}} = 1.$$

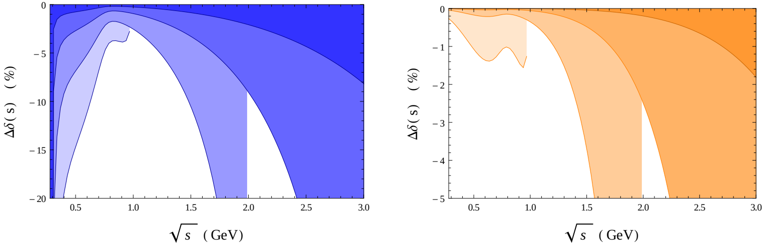
- $|F_Q^\pi(s)|$ measured up to finite Λ; explore performance

Modulus DR: performance

- Data available in $s \in (s_{th}, 3\ \mathrm{GeV})$: assess performance for finite cutoff
- Employ the following (*analytic*) toy model[2] (ρ with $\pi\pi$ resummation)

$$F_Q^{\pi}(s) = \frac{m_\rho^2}{m_\rho^2 - s - \frac{192\pi}{\beta_\rho^3}\frac{\Gamma_\rho}{m_\rho} H_{\pi\pi}(s)}, \qquad H_{\pi\pi}(s) = s\beta^2 \bar{B}_0(s; m_\pi, m_\pi) + \ldots$$

- Rel. uncertainty if DR cutted-off at $\{1, 2, 3, 5\}$ GeV (DR1/DR2 left/right)



- DR2 much better performance → our favorite choice!
- For DR2 sum rule matters! SR=$\{1.02, 1.004, 1.0002\}$ vs. 1 for $\{1, 3, 10\}$ GeV

[2] Dumm, Pich, Portoles, PRD62, 054014 (2000)

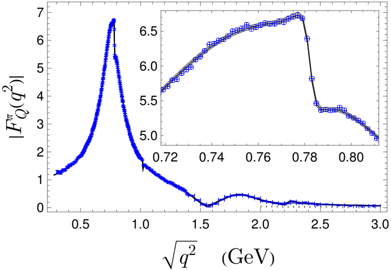
Section 2

Real data analysis

Real data analysis

Data from $e^+e^- \to \pi^+\pi^-$

- Among precise exp. only BaBar beyond 1 GeV
- Combining? problems due to tensions
- Select BaBar[3] for this analysis!



- Integrating $\ln|F_Q^\pi(s)|$ in DR: interpolation $\Rightarrow$ use a well-motivated model:

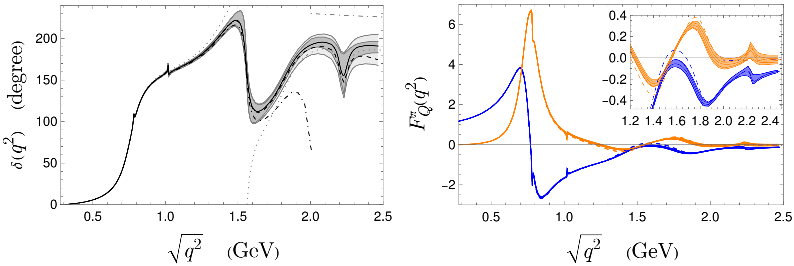
$$F_Q^\pi(s) = [c_\rho D_\rho(1+s[c_\omega BW_\omega + c_\phi BW_\phi]) + c_{\rho'} D_{\rho'} + c_{\rho''} D_{\rho''} + c_{\rho'''} D_{\rho'''}]/[\sum_\rho c_\rho]$$

- D_ρ= Gounaris Sakurai; BW Breit-Wigner; c_V's complex coefficient
- Model violates Schwarz; just to interpolate; DR unitarize it; systematics later
- $\chi^2 = \text{res}^T(Cov)^{-1}$ res: d'Agostini bias; MC analysis keeps correlations
- Sum rule is verified form most fits (discard if not)

[3] J. P. Lees et al (BaBar Coll), PRD86, 032013 (2012)

Real data analysis: δ_Q

- Our results for the and Re/Im parts



- Inner/outer band= stat(stat+syst)
- Phase differs wrt original model (expected, comments later)
- This is not $\delta_1^1(s)$ yet.[4] For that, $F_8^\pi/\sqrt{3}$ must be turned off, leaving F_3^π

$$F_Q^\pi(s) = [c_\rho(D_\rho + sD_\rho[c_\omega BW_\omega + c_\phi BW_\phi]) + c_{\rho'} D_{\rho'} + c_{\rho''} D_{\rho''} + c_{\rho'''} D_{\rho'''}]/[\sum c_\rho]$$

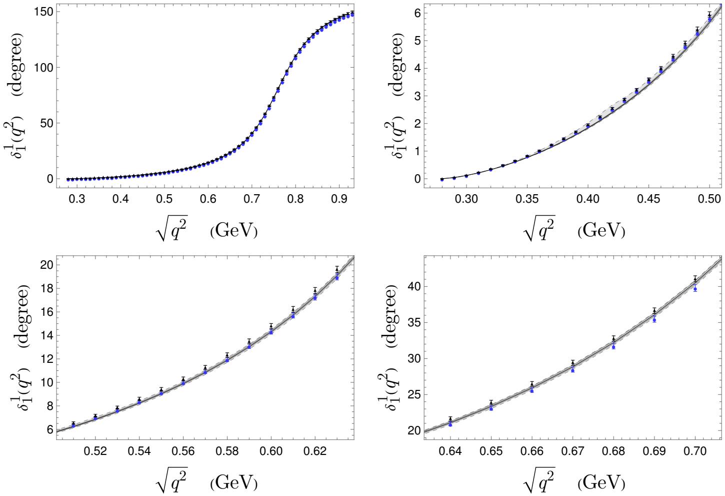
- Process can be repeated for F_3^π to extract $\delta_3 (= \delta_1^1$ elastic) and δ_8

[4] Arriola, Broniowski, PSP, PLB822, 136680 (2021)

Real data analysis: δ_1^1

- Our results for δ_1^1 ($= \delta_3$) in the elastic region. Compared to Roy analysis[5].

Systematics: reparametrization of $F_Q^\pi(s)$ at threshold + Sum Rule

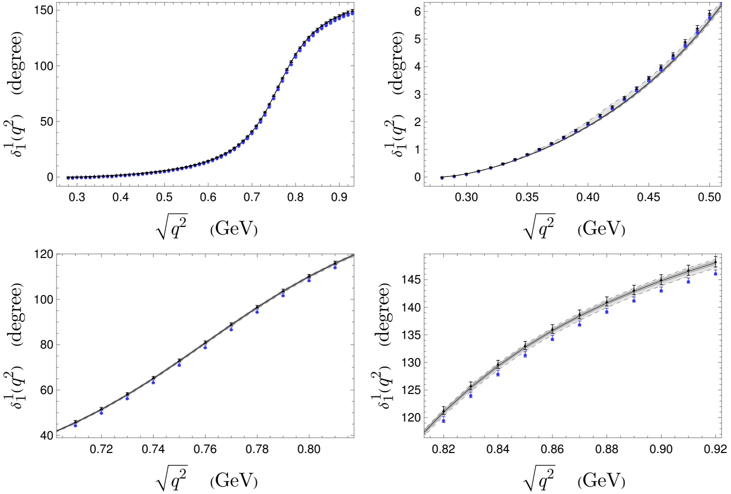


[5] Madrid phase: PRD83, 074004 (2011), Bern phase: JHEP 02, 006 (2019)

Real data analysis: δ_1^1

- Our results for δ_1^1 $(= \delta_3)$ in the elastic region. Compared to Roy analysis[5].

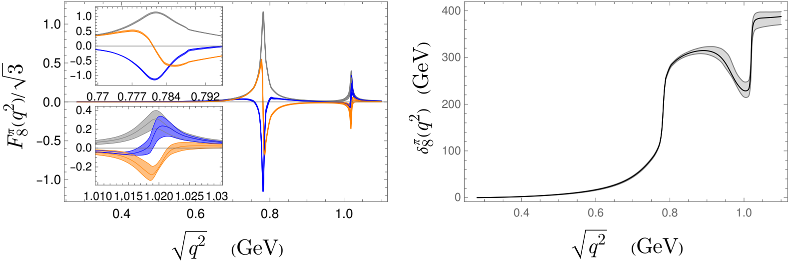
Systematics: reparametrization of $F_Q^\pi(s)$ at threshold + Sum Rule



[5] Madrid phase: PRD83, 074004 (2011), Bern phase: JHEP 02, 006 (2019)

Real data analysis: F_8^π

- From results on $F^\pi_{Q,3} \to F_8^\pi/\sqrt{3}$



- Note the phase approaches 2π (not π) (argument principle + $F_8^\pi(0) = 0$)

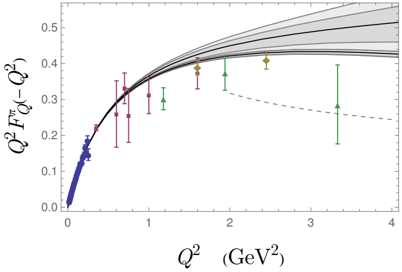
Other results

- We can use DR1/DR2 to go to the spacelike

$$F_Q^\pi(-Q^2) = \exp\left(\frac{-Q^2\sqrt{s_{th}+Q^2}}{\pi}\int_{s_{th}}^{\Lambda^2} dz\,\frac{\ln|F_Q^\pi(z)|}{z\sqrt{z-s_{th}}(z+Q^2)}\right)$$

$$F_Q^\pi(-Q^2) = F_Q^\pi(s_{th})^{1-\left(\frac{s_{th}+Q^2}{s_{th}}\right)^{3/2}} \exp\left[\frac{Q^2(s_{th}+Q^2)^{3/2}}{\pi}\int_{s_{th}}^{\Lambda^2} dz\,\frac{\ln|F_Q^\pi(z)/F_Q^\pi(s_{th})|}{z(z-s_{th})^{3/2}(z+Q^2)}\right]$$

- For large s, $F_Q^\pi(s) \ll F_Q^\pi(s_{th}) \Rightarrow$ DR1/DR2 = lower/upper bound for finite Λ^2



- DR1 less systematics but DR2 faster convergence
- Expansion at $q^2 = 0$ and radius

$$\langle r_\pi^2\rangle = 11.01(7)_{\rm st}(^{+10}_{-4})_{\rm sys}{\rm GeV}^{-2}$$
$$= 0.429(2)_{\rm st}(^{+3}_{-1})_{\rm sys}\ {\rm fm}^2$$

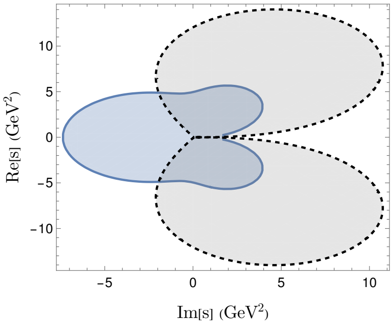
Bern analysis: 0.429(4) fm^2
Further terms in series in our work

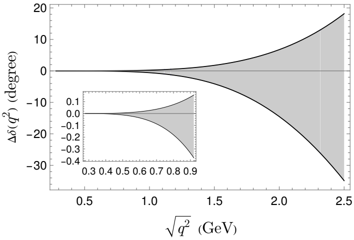
Section 3

Final remarks

Comments on zeros

- What if zeros are present? ⇒ Conformal polynomial containing such zeros[6]
- Zeros are excluded in the low-energy region[6]
- The sum rule is modified and helps to constrain zeros





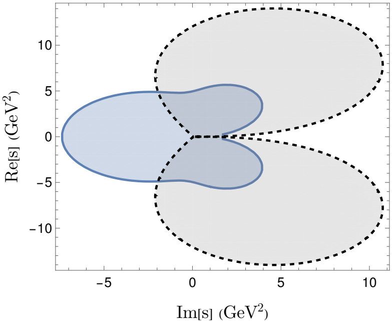
Impact for "deep" complex zeros

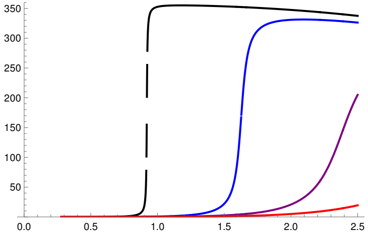
- Zeros also relevant for Omnès-like DR, worth attention, but hard to reach conclusive answer. If established, can be incorporated, so welcome.

[6] Cronstrom, PLB49, 283 (1974); Ananthanaryan et al PRD93, 096002 (2011)

Comments on zeros

- What if zeros are present? ⇒ Conformal polynomial containing such zeros[6]
- Zeros are excluded in the low-energy region[6]
- The sum rule is modified and helps to constrain zeros





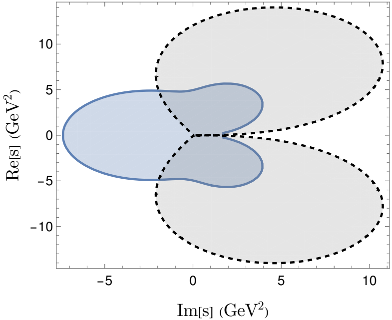
Impact for near-axis complex zeros: sudden jump (argument principle)

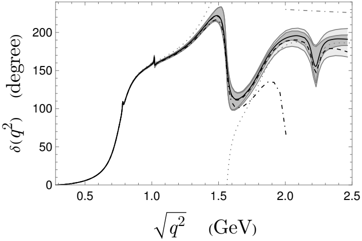
- Zeros also relevant for Omnès-like DR, worth attention, but hard to reach conclusive answer. If established, can be incorporated, so welcome.

[6] Cronstrom, PLB49, 283 (1974); Ananthanaryan et al PRD93, 096002 (2011)

Comments on zeros

- What if zeros are present? ⇒ Conformal polynomial containing such zeros[6]
- Zeros are excluded in the low-energy region[6]
- The sum rule is modified and helps to constrain zeros





Impact for near-axis complex zeros: sudden jump (argument principle)

- Zeros also relevant for Omnès-like DR, worth attention, but hard to reach conclusive answer. If established, can be incorporated, so welcome.

[6] Cronstrom, PLB49, 283 (1974); Ananthanaryan et al PRD93, 096002 (2011)

Outlook and summary

- Modulus DR to study the phase of the form factor
- Data-based, unitarization method
- Extracted δ_1^1, in agreement with precise determinations
- Also determined spacelike and radius, F_8^π
- Discussed impact of zeros: worth exploring (often ignored)
- Useful to compare τ data at high energies?
- Form factor on the 1RS: useful for resonance hunting
- Ideas on how to explore other form factors

Section 4

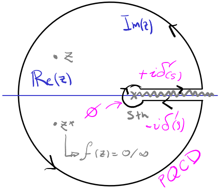
Backup

Principle of the argument

- Given $f(z)$ with poles and zeros at z_i, $\frac{d\ln f(z)}{dz} = f'(z)/f(z)$ has poles with residue ± 1 located at original zeros/poles.

- Meaning that, with an appropriate contour

$$\oint_C \frac{F_Q^{\pi\prime}(z)}{F_Q^{\pi}(z)} dz = 2\pi i(N - P)$$



- Evaluating the lhs with contour (radius s):

$$2i\delta(s)|_{\rm disc} - 2\pi i[1 + L^{-1} + \ldots]_{\rm pQCD} = 2\pi i(N - P)$$

- Thus recovering the result

$$\delta(s) = \pi\left[1 + L^{-1} + \frac{6.58}{\pi}\frac{4\pi}{\beta_0}L^{-2} + N - P\right]$$

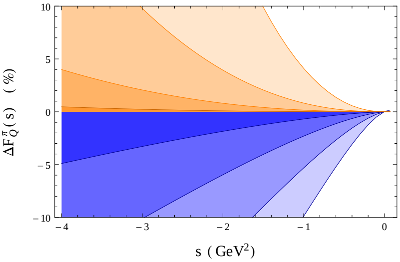
- Note that no poles in RS1 and Schwarz demands complex-conjugated zeros

Spacelike and radius

- Convergence for DR1/DR2 (blue/orange) for $\Lambda = \{1, 2, 3, 5\}$ GeV
- Results for radius etc:

$$F_Q^\pi(s) = 1 + b_\pi s + c_\pi s^2 + d_\pi s^3 + \ldots$$

- Note that $\langle r_\pi^2 \rangle = 6 b_\pi$



$$\langle r_\pi^2 \rangle = 11.01(7)_{\rm st}(^{+10}_{-4})_{\rm sys}{\rm GeV}^{-2} = (0.655(2)_{\rm st}(^{+4}_{-2})_{\rm sys}\ {\rm fm})^2 = 0.429(2)_{\rm st}(^{+3}_{-1})_{\rm sys}\ {\rm fm}^2,$$

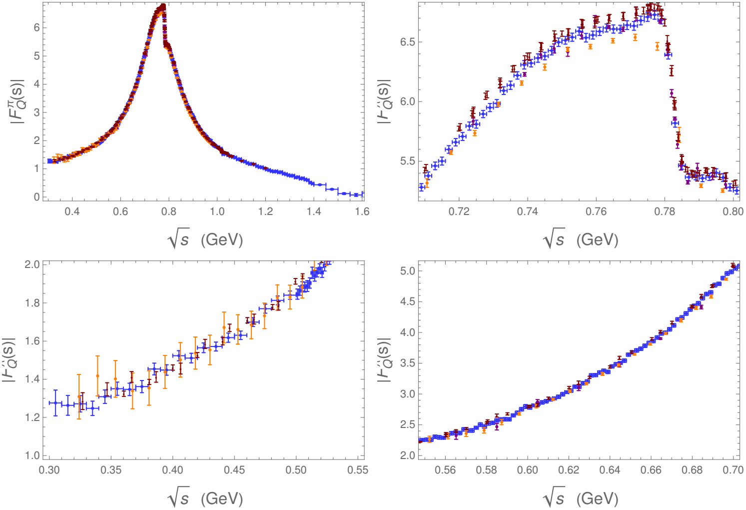
$$c_\pi = 3.84(3)_{\rm st}(^{+5}_{-2})_{\rm sys}\ {\rm GeV}^{-4}, \qquad d_\pi = 10.1(1)_{\rm st}(^{+3}_{-1})_{\rm sys}\ {\rm GeV}^{-6}.$$

- Analytic[7] $[0.657(3)\ {\rm fm}]^2$, $c_\pi \in (3.79, 4)\ {\rm GeV}^{-4}$, $d_\pi \in (10.14, 10.56)\ {\rm GeV}^{-6}$
- τ-based[7] $11.28(8)\ {\rm GeV}^{-2}$, $c_\pi = 3.94(4)\ {\rm GeV}^{-4}$, $d_\pi = 10.54(5)\ {\rm GeV}^{-6}$

[7] Ananthanaran et al, PRL119, 132002 (2017), [7] S. González-Solis, P. Roig, EPJC79, 436 (2019)

Some data sets

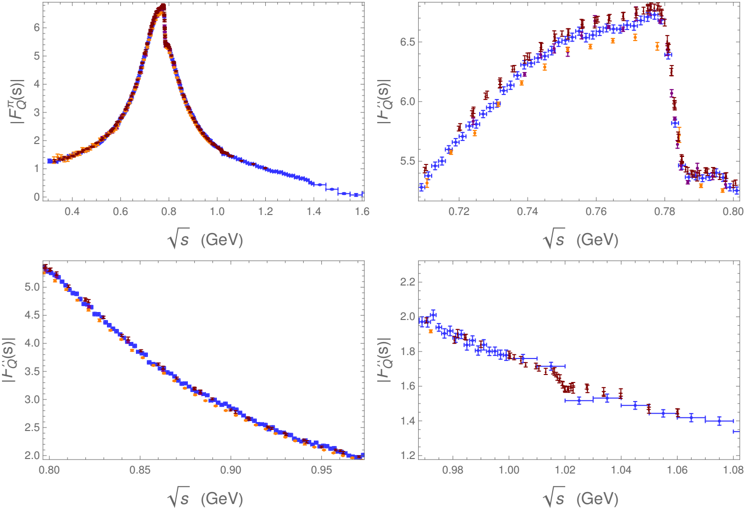
- We collect the most conflicitng datasets



- Color code: CMD-3, BaBar, KLOE

Some data sets

- We collect the most conflicitng datasets



- Color code: CMD-3, BaBar, KLOE

DR in the presence of zeros

- We introduce a polynomial with z_i the zeros[8]

$$B(s) = \prod_i \frac{\sqrt{s_{th} - z_i} - \sqrt{s_{th} - s}}{\sqrt{s_{th} - z_i} + \sqrt{s_{th} - s}}, \qquad |B(s \geq s_{th})| = 1$$

- In the presence of zeros DR, DR1

$$F_Q^\pi(s) = B(s) \exp\left(\frac{\sqrt{s_{th} - s}}{\pi} \int_{s_{th}}^{\infty} \frac{\ln |F_Q^\pi(z)| \, dz}{\sqrt{z - s_{th}}(z - s)} \right),$$

$$\delta(s) = \phi(s) - \frac{\sqrt{s - s_{th}}}{\pi} \, \mathrm{PV} \int_{s_{th}}^{\infty} \frac{\ln |F_Q^\pi(z)| \, dz}{(z - s)\sqrt{z - s_{th}}},$$

$$F_Q^\pi(s) = \frac{B(s)}{B(0)^{\sqrt{\frac{s_{th} - s}{s_{th}}}}} \exp\left(\frac{s\sqrt{s_{th} - s}}{\pi} \int_{s_{th}}^{\infty} \frac{\ln |F_Q^\pi(z)| \, dz}{z\sqrt{z - s_{th}}(z - s)} \right),$$

$$\delta(s) = \phi(s) + \sqrt{\frac{s - s_{th}}{s_{th}}} \ln B(0) - \frac{s\sqrt{s - s_{th}}}{\pi} \, \mathrm{PV} \int_{s_{th}}^{\infty} dz \frac{\ln |F_Q^\pi(z)|}{z\sqrt{z - s_{th}}(z - s)},$$

[8] Cronstrom, PLB49, 283 (1974); Ananthanaryan et al PRD93, 096002 (2011)

DR in the presence of zeros

- We introduce a polynomial with z_i the zeros[8]

$$B(s) = \prod_i \frac{\sqrt{s_{th} - z_i} - \sqrt{s_{th} - s}}{\sqrt{s_{th} - z_i} + \sqrt{s_{th} - s}}, \qquad |B(s \geq s_{th})| = 1$$

- For DR2, Cauchy's representation with care:

$$g(s) \equiv \frac{1}{(s_{th} - s)^{3/2}} \ln \frac{F_Q^\pi(s)}{F(s_{th})B(s)} \simeq \frac{-2\sum_i \frac{1}{\sqrt{s_{th} - z_i}}}{s - s_{th}} \equiv \frac{\text{Res}\, g(s_{th})}{s - s_{th}}$$

$$F_Q^\pi(s) = \frac{F_Q^\pi(s_{th})B(s)}{\left[F_Q^\pi(s_{th})B(0)\right]^{\left(\frac{s_{th}-s}{s_{th}}\right)^{3/2}}} \exp\left[-\frac{s}{s_{th}}\sqrt{s_{th} - s}\, \text{Res}\, g(s_{th}) \right.$$
$$\left. - \frac{s(s_{th} - s)^{3/2}}{\pi} \int_{s_{th}}^{\infty} \frac{\ln[|F_Q^\pi(z)|/F_Q^\pi(s_{th})]\, dz}{z(z - s_{th})^{3/2}(z - s)} \right],$$

$$\delta(s) = \phi(s) + \frac{s}{s_{th}}\sqrt{s - s_{th}}\, \text{Res}\, g(s_{th}) - ([s - s_{th}]/s_{th})^{3/2} \ln[F_Q^\pi(s_{th})B(0)]$$
$$- \frac{(s - s_{th})^{3/2}}{\pi} \text{PV} \int_{s_{th}}^{\infty} \frac{\ln[|F_Q^\pi(z)|/F_Q^\pi(s_{th})]\, dz}{(z - s_{th})^{3/2} z(z - s)}.$$

[8] Cronstrom, PLB49, 283 (1974); Ananthanaryan et al PRD93, 096002 (2011)

DR in the presence of zeros

- We introduce a polynomial with z_i the zeros[8]

$$B(s)=\prod_i \frac{\sqrt{s_{th}-z_i}-\sqrt{s_{th}-s}}{\sqrt{s_{th}-z_i}+\sqrt{s_{th}-s}}, \qquad |B(s\geq s_{th})|=1$$

- For DR2, Cauchy's representation with care:

$$g(s)\equiv \frac{1}{(s_{th}-s)^{3/2}}\ln\frac{F_Q^\pi(s)}{F(s_{th})B(s)} \simeq \frac{-2\sum_i \frac{1}{\sqrt{s_{th}-z_i}}}{s-s_{th}} \equiv \frac{\mathrm{Res}\, g(s_{th})}{s-s_{th}}$$

- As well as the modified sum rule

$$\frac{\sqrt{s_{th}}\,\mathrm{Res}\, g(s_{th})+\frac{s_{th}^{3/2}}{\pi}\int_{s_{th}}^{\infty}\frac{\ln\frac{|F_Q^\pi(z)|}{F_Q^\pi(s_{th})}\, dz}{z(z-s_{th})^{3/2}}}{\ln[F_Q^\pi(s_{th})B(0)]}=1.$$

- Evaluating the integral can set bounds on zeros

[8] Cronstrom, PLB49, 283 (1974); Ananthanaryan et al PRD93, 096002 (2011)

Systematics

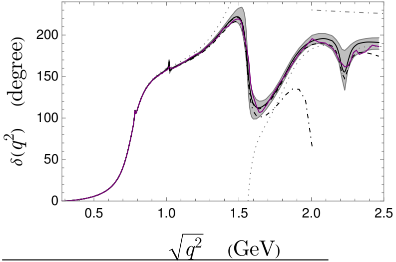
- First, systematics close to threshold (integrand sentitive to them)
  Vary $F_Q^\pi(s_{th})$ (most relevant) and expansion at s_{th} (up to $\Lambda = 0.42$ GeV)

$$\frac{F_Q^\pi(s)}{F_Q^\pi(s_{th})} = 1 + \frac{\alpha_1 s}{1-\alpha_2 s} + \frac{i\beta \frac{s^{3/2}}{M_\rho^2\sqrt{s_{th}}}}{1-\frac{s}{M_\rho^2}},$$

- Rapid variations, but sum rule must be fulfilled → strong constraint!

Account for[9] $F_Q^\pi(s_{th}) = 1.176(2)$ [we find $F_Q^\pi(s_{th}) = 1.174(1)$]

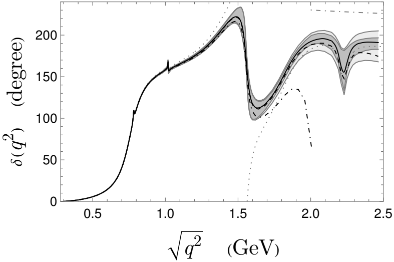
- Second, above 600 MeV, linear interpolation for data: irrelevant!

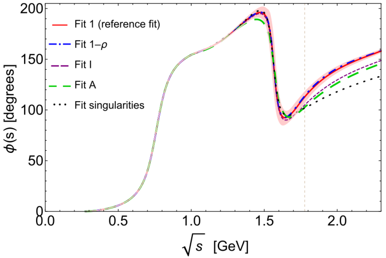


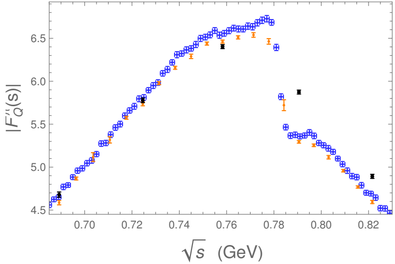
[9] Ananthanarayan et al, PRD98, 114015 (2018)

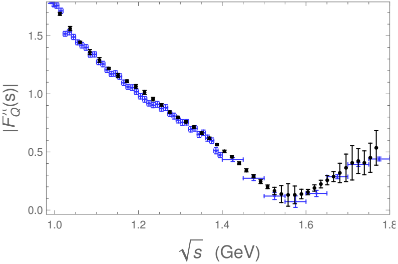
Systematics

- Comparison to τ data analysis[10]









Colors

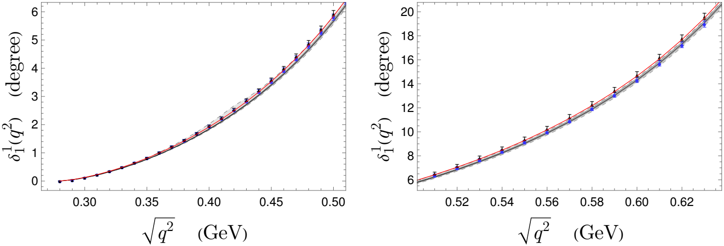
[10] S. González-Solís, P. Roig, EPJC79, 436 (2019)

Systematics

- An alternative approach independent of zeros: model imaginary part

$$F_Q^\pi(z) = 1 + \frac{s}{\pi}\int_{s_{th}}^{\infty} \frac{\text{Im}\, F_Q^\pi(z)}{z(z-s)} dz$$

- Could take the one from GS, remove by hand the residual phase at threshold



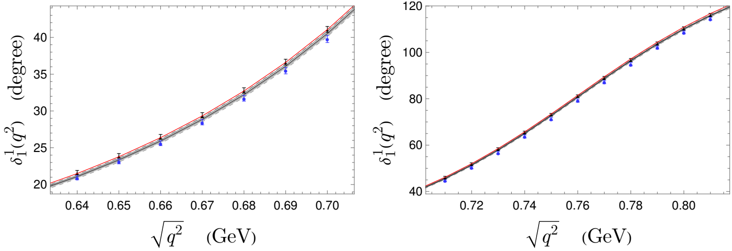
- But much more model dependence in solving the inverse problem
- E.g.: Imaginary part from DR also valid in Cauchy
- Other resonant models significantly different phases
- GS misses $\omega\pi, KK, a_1\pi$ effects and violates Schwarz
- On turn, DR stable (see discretizing HE) but systematics from zeros

Systematics

- An alternative approach independent of zeros: model imaginary part

$$F_Q^\pi(z) = 1 + \frac{s}{\pi}\int_{s_{th}}^{\infty} \frac{\text{Im}\, F_Q^\pi(z)}{z(z-s)}\,dz$$

- Could take the one from GS, remove by hand the residual phase at threshold



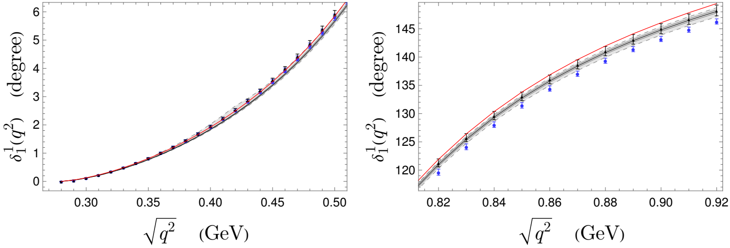
- But much more model dependence in solving the inverse problem
- E.g.: Imaginary part from DR also valid in Cauchy
- Other resonant models significantly different phases
- GS misses $\omega\pi, KK, a_1\pi$ effects and violates Schwarz
- On turn, DR stable (see discretizing HE) but systematics from zeros

Systematics

- An alternative approach independent of zeros: model imaginary part

$$F_Q^\pi(z) = 1 + \frac{s}{\pi}\int_{s_{th}}^{\infty} \frac{\text{Im}\, F_Q^\pi(z)}{z(z-s)}dz$$

- Could take the one from GS, remove by hand the residual phase at threshold



- But much more model dependence in solving the inverse problem
- E.g.: Imaginary part from DR also valid in Cauchy
- Other resonant models significantly different phases
- GS misses $\omega\pi$, KK, $a_1\pi$ effects and violates Schwarz
- On turn, DR stable (see discretizing HE) but systematics from zeros