北京航空航天大學
BEIHANG UNIVERSITY

The Tomography of Nucleon:

Lattice QCD calculation of the unpolarized transverse-momentum-dependent parton distributions

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Jul. 10 @ QNP 2024

Lattice Parton Collaboration

Based on *PRD109, 114513 (2024)*

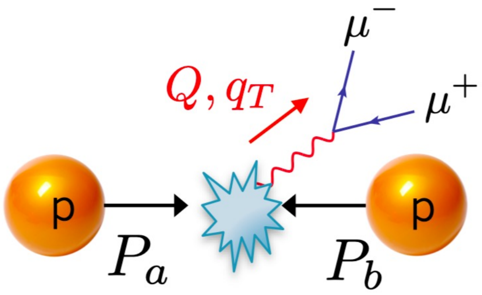
OUTLINE

- Motivation
- Lattice QCD calculation of TMDPDFs
  - Extract TMDPDFs from LaMET
  - Quasi TMDPDF matrix elements and their renormalization
  - From Quasi TMDPDF to physical TMDPDF
  - Numerical results
- Summary and Outlook

TMDPDFs: 3D tomography of the nucleon

TMD processes:

Drell-Yan

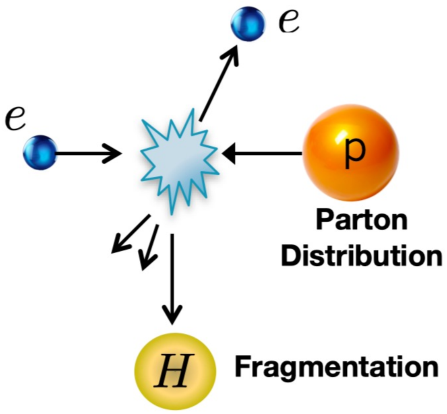


$q_T \ll Q$

LHC, FermiLab, RHIC, …

$$\sigma \sim f_{q/P}(x, k_T)\, f_{q/P}(x, k_T)$$

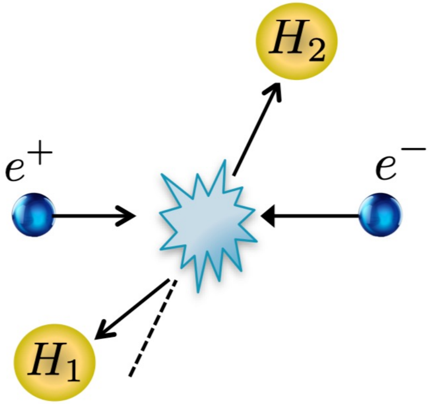
Semi-Inclusive DIS



HERMES, COMPASS, JLab, EIC, …

$$\sigma \sim f_{q/P}(x, k_T)\, D_{h/q}(x, k_T)$$

Dihadron in e+e-

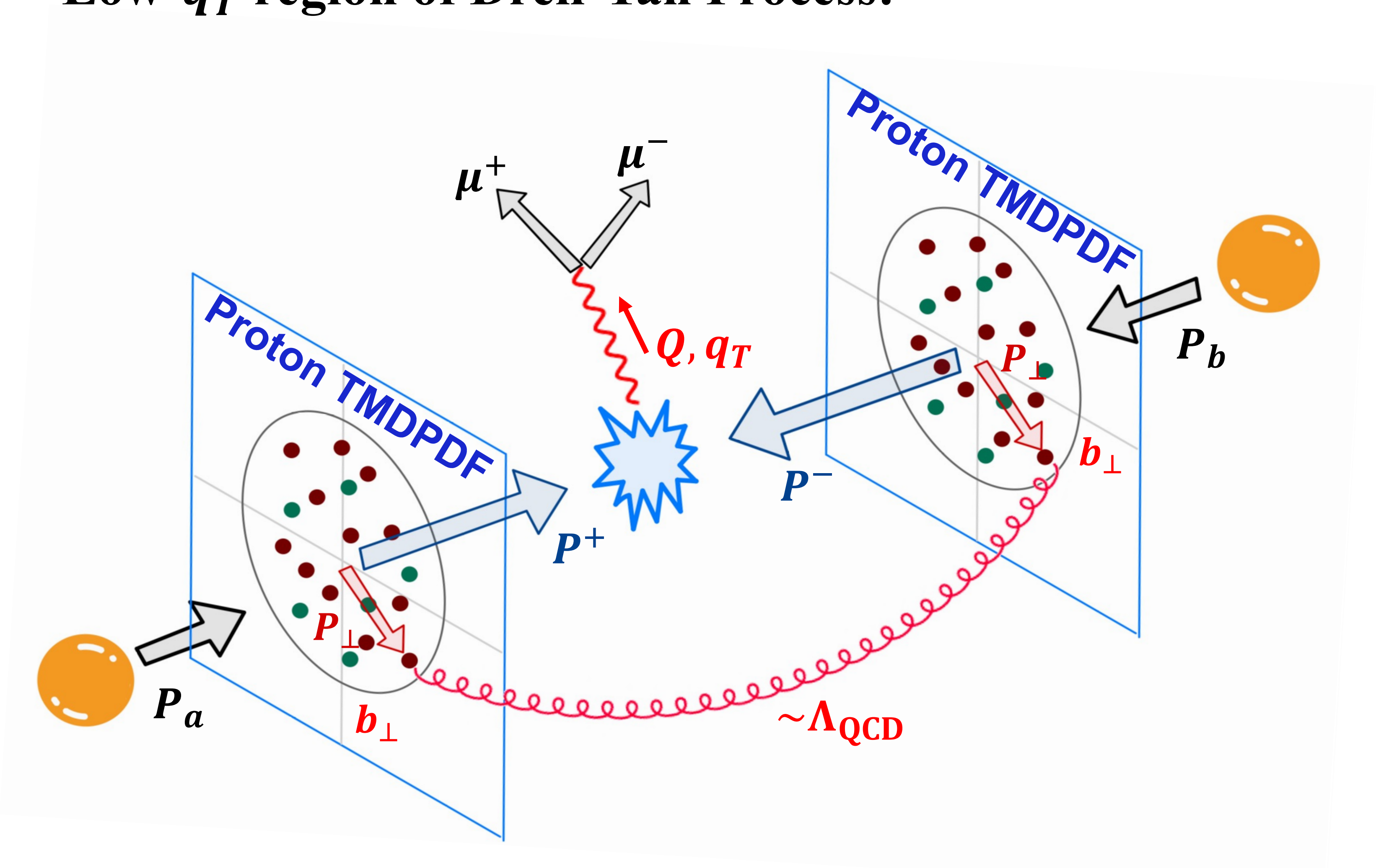


BESIII, Babar, Belle, …

$$\sigma \sim D_{h_1/q}(x, k_T)\, D_{h_2/q}(x, k_T)$$

TMDPDFs: 3D tomography of the nucleon

- **Low-q_T region of Drell-Yan Process:**



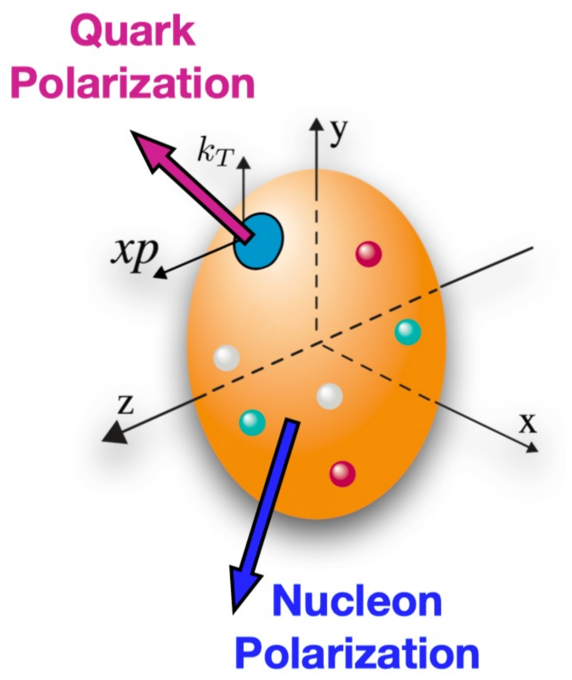
Revealing the

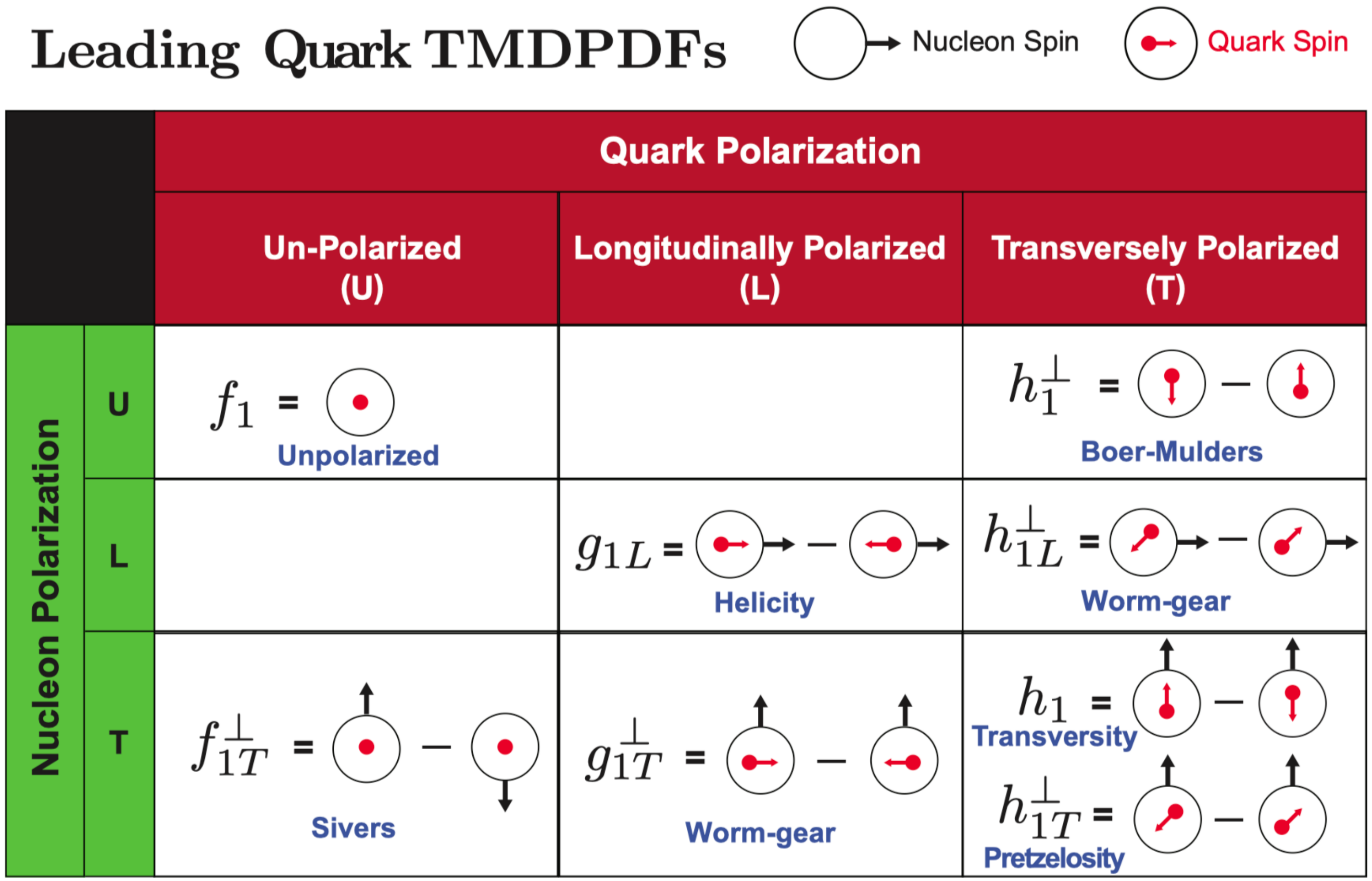
confined motion of

partons inside the nucleon

EPJA52, 268 (2016)

TMDPDFs: 3D tomography of the nucleon





Leading Quark TMDPDFs

⊙→ Nucleon Spin
⊙ Quark Spin

		Quark Polarization		
		Un-Polarized (U)	**Longitudinally Polarized (L)**	**Transversely Polarized (T)**
Nucleon Polarization	**U**	f_1 = ⊙ Unpolarized		$h_1^\perp$ = ⊙ − ⊙ Boer-Mulders
	L		g_{1L} = ⊙→ − ⊙→ Helicity	$h_{1L}^\perp$ = ⊙→ − ⊙→ Worm-gear
	T	$f_{1T}^\perp$ = ⊙ − ⊙ Sivers	$g_{1T}^\perp$ = ⊙ − ⊙ Worm-gear	h_1 = ⊙ − ⊙ Transversity $h_{1T}^\perp$ = ⊙ − ⊙ Pretzelosity

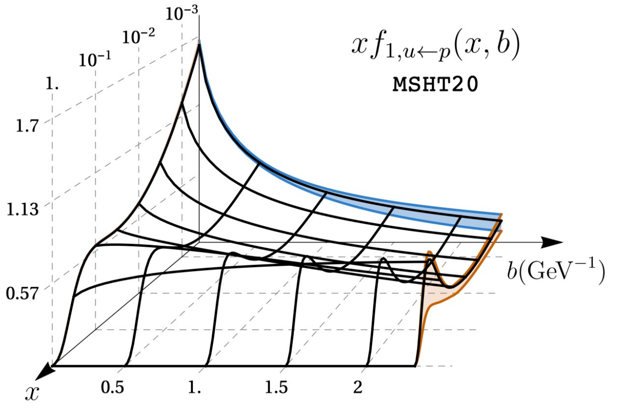
TMD Handbook, TMD Collaboration, 2304.03302

Progress in the study of TMDPDFs

- **Theoretical analysis**
  - **TMD factorization, evolution and resummation:**

    Boussarie et al., TMD handbook, 2304.03302;

    Collins, Foundations of perturbative QCD; ……



u-quark unpolarized TMDPDF, 2201.07114

- **Phenomenological parametrizations and extractions**
  - **Unpolarized:**

    Moos, JHEP05 (2024); Bacchetta, JHEP10 (2022); Bury, JHEP10 (2022);

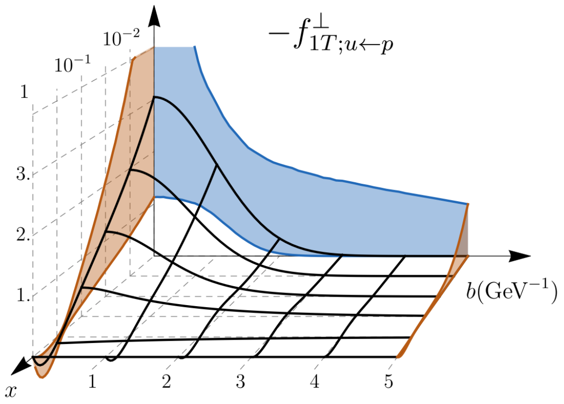
    Scimem, JHEP06 (2020); Bacchetta, JHEP06 (2017); ……

  - **Sivers, Boer-Mulders:**

    Bury, PRL126 (2021), JHEP05 (2021) ; Cammarota, PRD102(2020);

    Zhang, PRD77 (2008), Lu, PRD81 (2010) ; ……

  - **Others: worm-gear, gluon TMDs, ……**



u-quark Sivers function, PRL126 (2021)

➢ Lattice calculations

- **Lorentz-invariant approach:** ratios of Mellin moments

  Hagler, EPL88(2009); Musch, PRD85(2012); Engelhardt, PRD93(2016); Yoon, 1601.05717, PRD96(2017); ……

- **LaMET formalism:**

  ✓ **I: theoretical analysis of matching kernel, soft function, Collins-Soper kernel, ……**

  Rio, PRD108(2023); Ji, JHEP08(2023), RMP93(2021), NPB955(2020), PLB811(2020);

  Ebert, JHEP04(2022); Deng, JHEP09(2022)……

  ✓ **II: lattice calculation of intrinsic soft function, Collins-Soper kernel, beam function, ...**

  LPC, JHEP08(2023), PRL125(2020); Li, PRL128(2022); LPC, PRD106(2022);

  Shanahan, PRD104(2021); Schlemmer, JHEP08(2021); ……

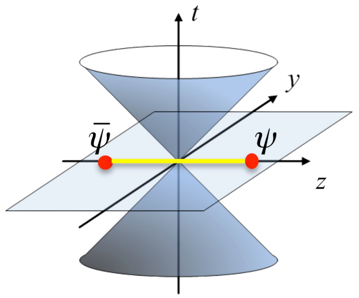
  ✓ **III: Nonperturbative renormalization, resummation, ……**

  Zhang, PLB884(2023); Ji, JHEP08(2023); Su, NPB991(2023); LPC, PRL129(2022); NPB991(2023)……

  ➢ **IV: A real lattice calculation of TMD observable?**

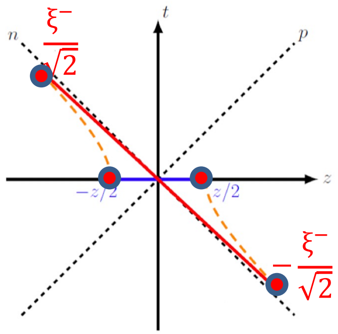
Extracting TMDs in LaMET formalism

- **Large-momentum effective theory: connecting Euclidean lattice and physical observables**

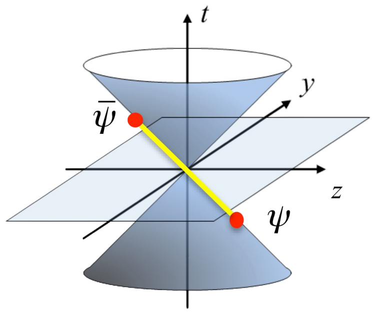


Lorentz boost

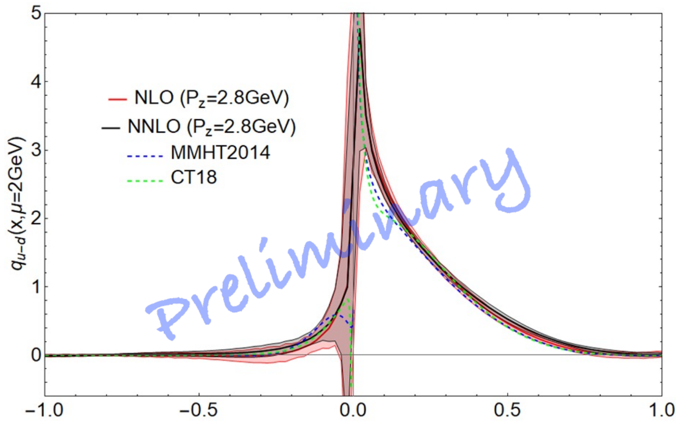
Ji, PRL110(2013), RMP93(2021), …



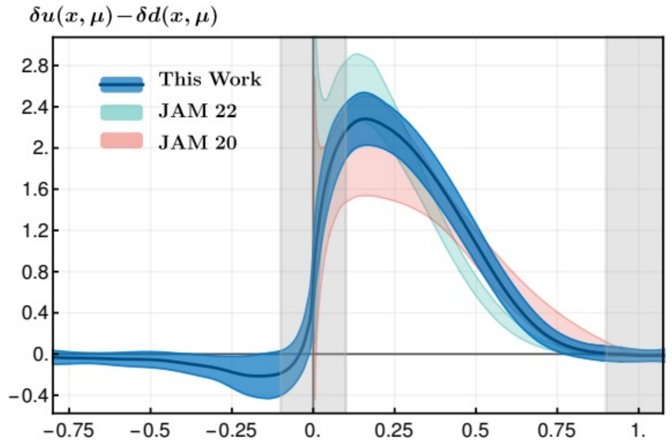
EFT



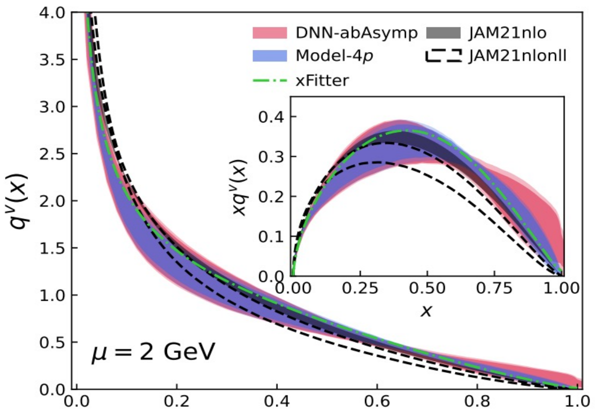
- **Achieved great success in the studies of PDF:**



Proton unpolarized PDF, in preparation

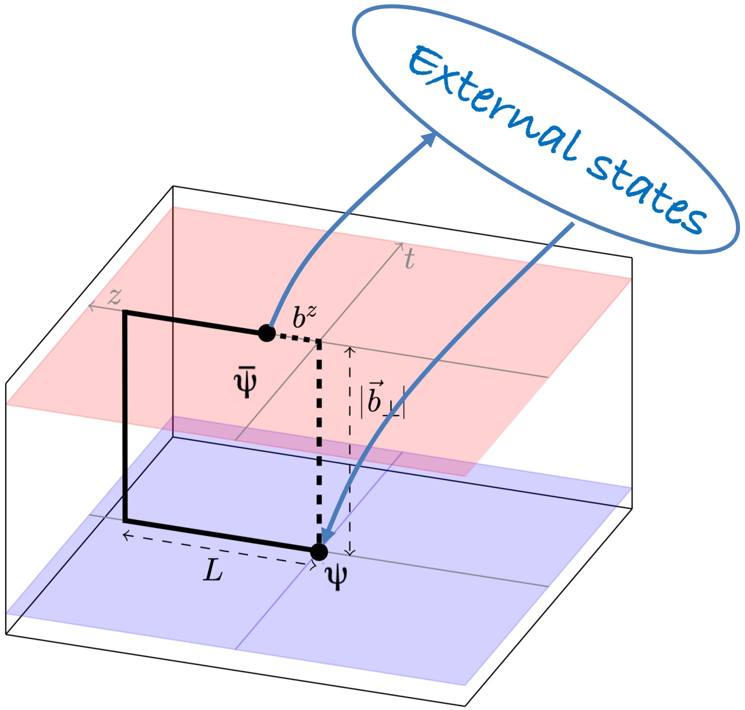


Proton transversity PDF, PRL131(2023)



Pion valance PDF, PRD106(2022)

- **Matching from quasi TMDs to TMDs**



- Hadronic matrix element reduced from equal-time correlators:

$$\tilde{h}^0_\Gamma(z, b_\perp, P^z) = \lim_{L\to\infty} \Big\langle P^z \Big| \bar{\psi}(b_\perp \hat{n}_\perp)\Gamma$$

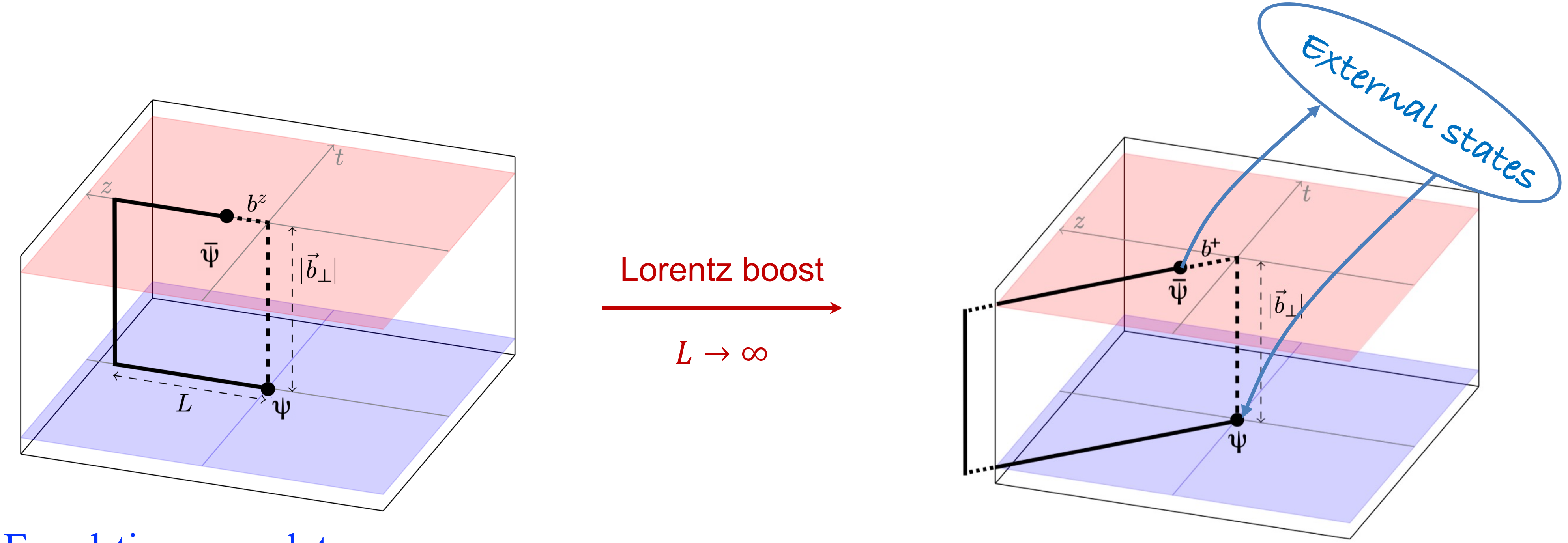
$$\times\, U_\sqsubset(b_\perp \hat{n}_\perp \leftarrow b_\perp \hat{n}_\perp + L\hat{n}_z; b_\perp \hat{n}_\perp + L\hat{n}_z \leftarrow L\hat{n}_z; L\hat{n}_z \leftarrow z\hat{n}_z)$$

$$\times\ \psi(z\hat{n}_z) \Big| P^z \Big\rangle$$

- Subtracted quasi TMDPDFs:

$$\tilde{f}_\Gamma(x, b_\perp, P^z, \mu) \equiv \lim_{\substack{a\to 0\\ L\to\infty}} \int \frac{dz}{2\pi} e^{-iz(xP^z)} \frac{\tilde{h}^0_\Gamma(z, b_\perp, P^z, a, L)}{\sqrt{Z_E(2L+z, b_\perp, a)}\, Z_O(1/a, \mu, \Gamma)}$$

Equal-time correlators
with staple-shaped Wilson link,
directly calculable on lattice



Equal-time correlators,
directly calculable on lattice

Space-like correlators,
NO effective method for directly calculation

Connected at large-momentum limit

Ji, PLB811(2020); Ebert, JHEP04(2022)

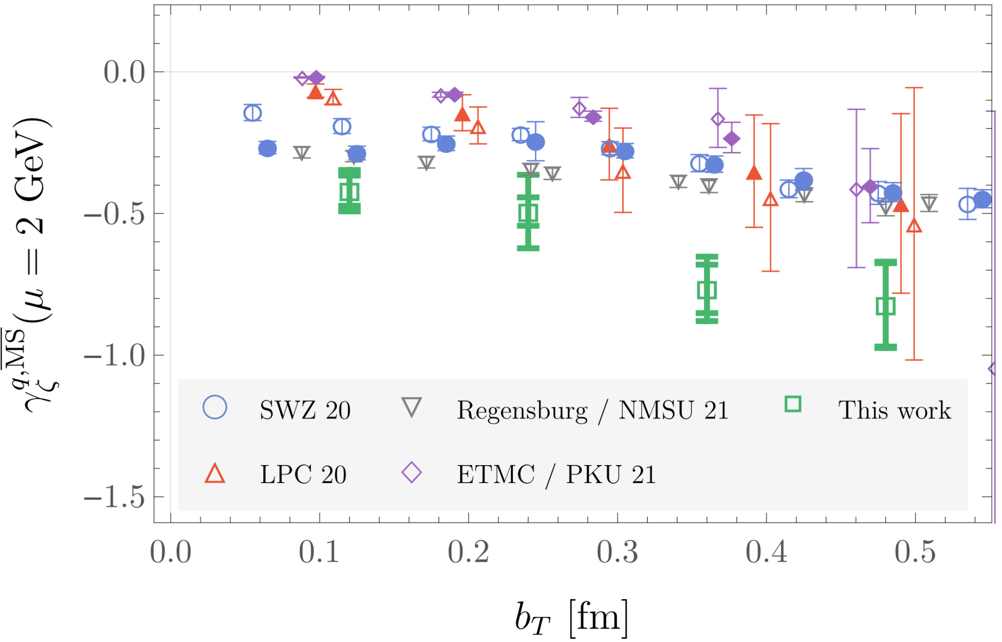
$$\underset{\text{Quasi TMDPDF}}{\boxed{\tilde f_\Gamma\left(x,b_\perp,\zeta_z,\mu\right)}}\sqrt{\underset{\text{Intrinsic soft function}}{\boxed{S_I\left(b_\perp,\mu\right)}}}=\overset{\text{Matching kernel}}{\boxed{H_\Gamma\left(\frac{\zeta_z}{\mu^2}\right)}}\,e^{\frac{1}{2}\ln\left(\frac{\zeta_z}{\zeta}\right)\underset{\text{Collins-Soper kernel}}{\boxed{K(b_\perp,\mu)}}}\underset{\text{Light-cone TMDPDF}}{\boxed{f\left(x,b_\perp,\mu,\zeta\right)}}+\mathcal{O}\left(\frac{\Lambda^2_{\mathrm{QCD}}}{\zeta_z},\frac{M^2}{(P^z)^2},\frac{1}{b_\perp^2\zeta_z}\right)$$

Collins–Soper kernel and intrinsic soft function

- **Collins-Soper kernel**

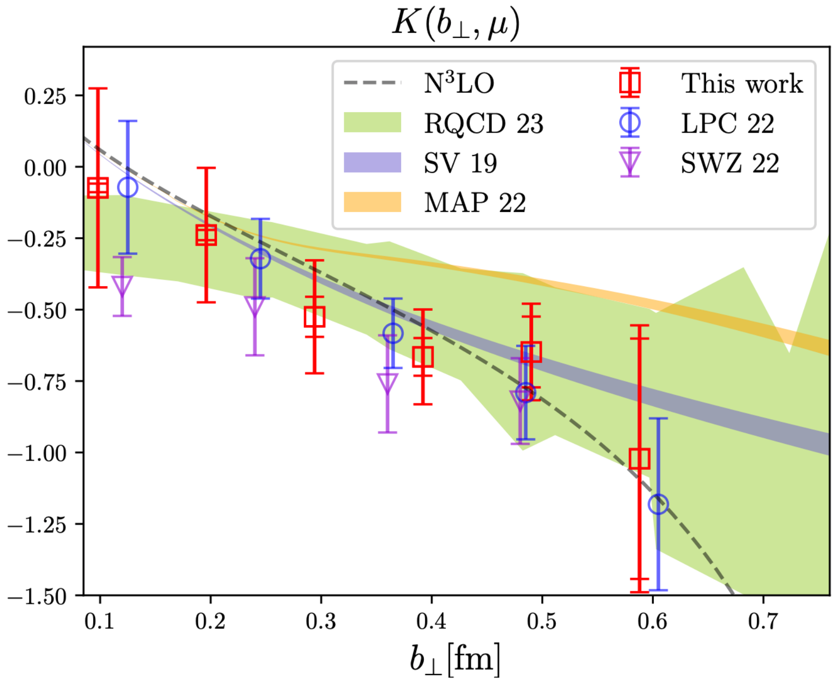
From quasi beam function:

Shanahan, PRD104(2021), PRD102(2020);
Schlemmer, JHEP08(2021); ……



From quasi TMDWF:

Chu, JHEP08(2023), PRD106(2022);
Zhang, PRL125(2020); Li, PRL128(2022); ……



Collins–Soper kernel and intrinsic soft function

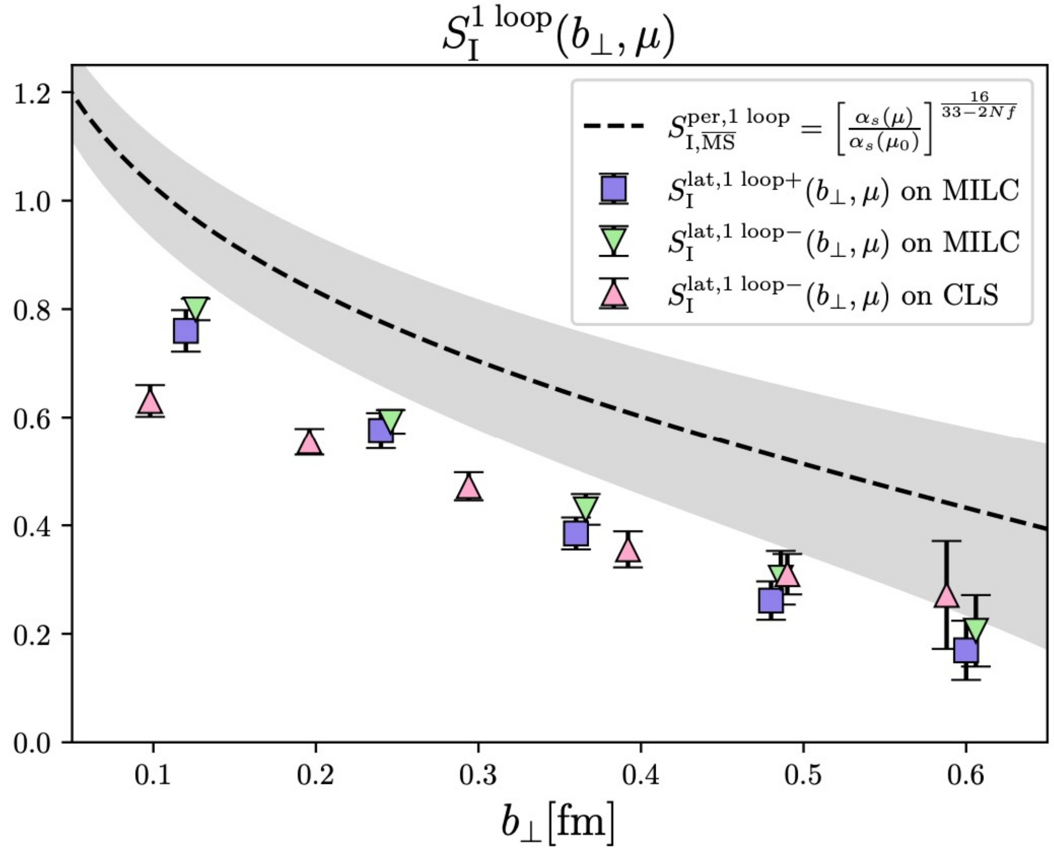
- **Intrinsic/reduced soft function**

From quasi TMDWF + 4-quark matrix element:

Chu, PRD109(2024); Ji, NPB955(2020);

Zhang, PRL125(2020); Li, PRL128(2022);

……

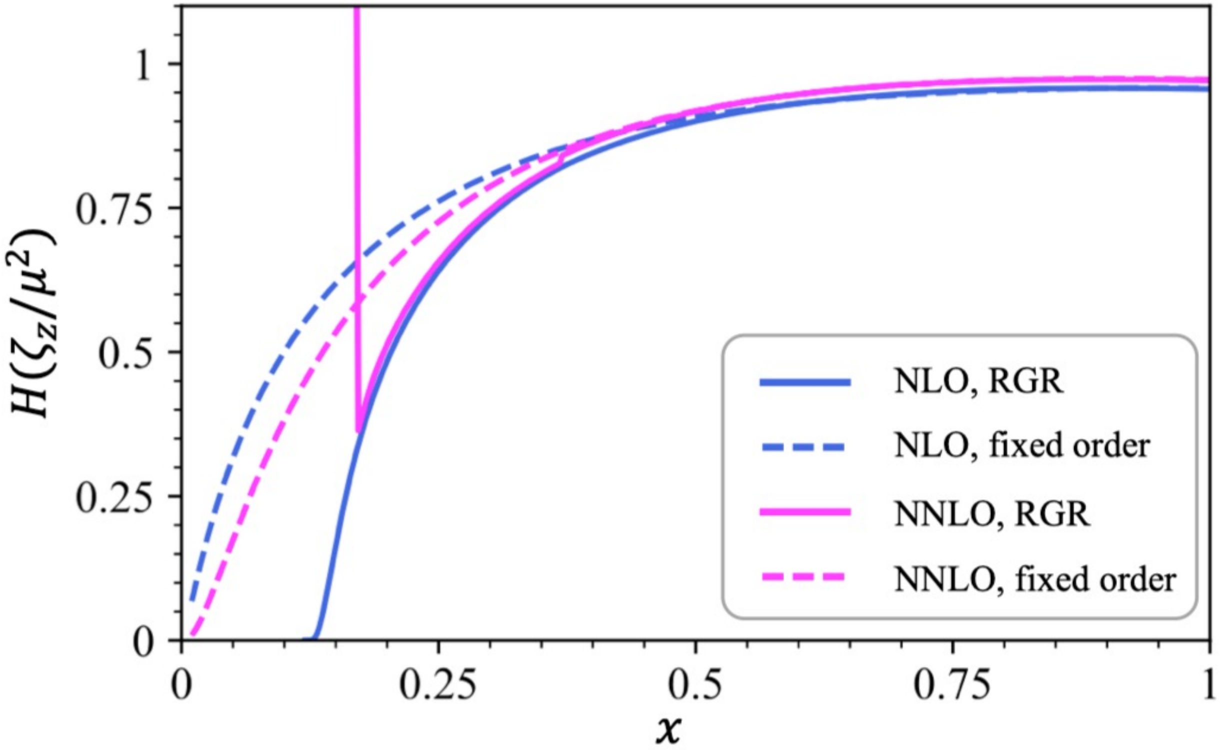


Matching kernel and RG resummation

$$\boxed{\tilde{f}_\Gamma\left(x, b_\perp, \zeta_z, \mu\right)}\sqrt{\boxed{S_I\left(b_\perp, \mu\right)}} = \boxed{H_\Gamma\left(\frac{\zeta_z}{\mu^2}\right)} e^{\frac{1}{2}\ln\left(\frac{\zeta_z}{\zeta}\right)\boxed{K(b_\perp,\mu)}} f\left(x, b_\perp, \mu, \zeta\right) + \mathcal{O}\left(\frac{\Lambda^2_{\mathrm{QCD}}}{\zeta_z}, \frac{M^2}{\left(P^z\right)^2}, \frac{1}{b_\perp^2 \zeta_z}\right)$$

Matching kernel

- **NLO:** *Ji, PLB811(2020); RMP93(2021)*
- **NNLO:** *Río, PRD108(2023); Ji, JHEP08(2023)*



- ➢ **Fixed order: $\mu = 2$GeV;**
- ➢ **RGR: RG evolution from lattice scale $\zeta_z = 2xP^z$ to $\overline{\text{MS}}$ scale $\mu = 2$GeV.**

Lattice calculation of physical TMDPDF?

Matching kernel ✓

$$\boxed{\tilde{f}_{\Gamma}\left(x,b_{\perp},\zeta_{z},\mu\right)}\sqrt{\boxed{S_{I}\left(b_{\perp},\mu\right)}}=\boxed{H_{\Gamma}\left(\frac{\zeta_{z}}{\mu^{2}}\right)}e^{\frac{1}{2}\ln\left(\frac{\zeta_{z}}{\zeta}\right)\boxed{K(b_{\perp},\mu)}}f\left(x,b_{\perp},\mu,\zeta\right)+\mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}^{2}}{\zeta_{z}},\frac{M^{2}}{(P^{z})^{2}},\frac{1}{b_{\perp}^{2}\zeta_{z}}\right)$$

Quasi TMDPDF

Intrinsic soft function ✓

Collins-Soper kernel ✓

⇩

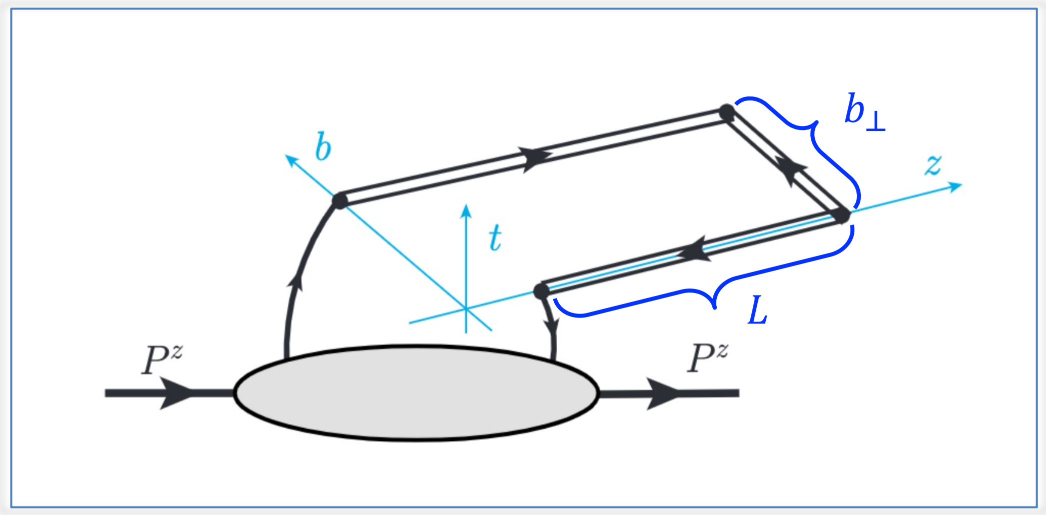
Simulating quasi TMDPDF on a Euclidean lattice:

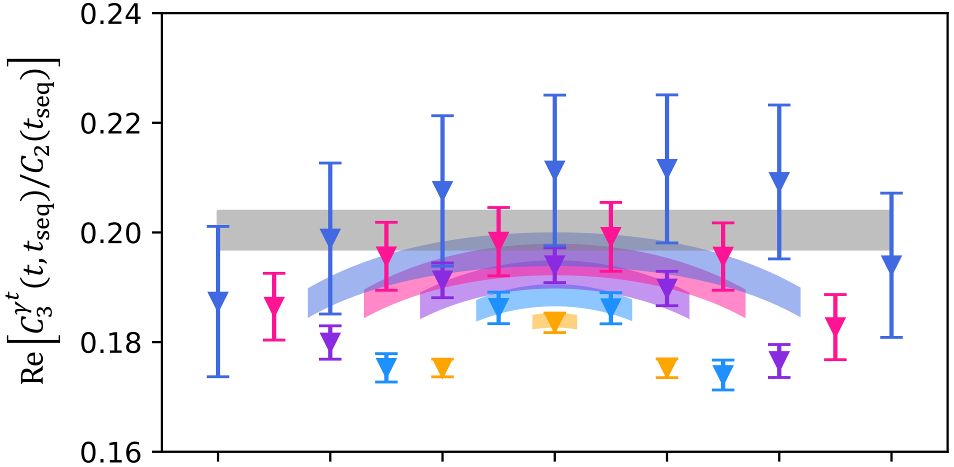
- MILC configuration: $48^3 \times 64$, $a = 0.12\text{fm}$;
- Pion mass: $\mathrm{m}_\pi^{\text{sea}} = 130\text{MeV}$, $m_\pi^{\text{val}} = \{310, 220\}\text{MeV}$ ⇒ extrapolate to physical mass
- Large momentum: $P^z = \{1.72,\ 2.15,\ 2.58\}\text{GeV}$ ⇒ extrapolate to infinity
- Saturated length of Wilson link $L = 0.72\text{fm}$;
- $z_{\text{max}} = 1.44\text{fm}$, $b_{\perp\text{max}} = 0.6\text{fm}$.

Quasi TMDPDF matrix element

Bare quasi TMDPDF matrix element

$$\tilde{h}^0_\Gamma\left(z, b_\perp, P^z\right) = \lim_{L\to\infty}\left\langle P^z \middle| \bar{\psi}(b_\perp \hat{n}_\perp)\Gamma \; U_\sqsubset\left(b_\perp\hat{n}_\perp \leftarrow b_\perp\hat{n}_\perp + L\hat{n}_z; b_\perp\hat{n}_\perp + L\hat{n}_z \leftarrow L\hat{n}_z; L\hat{n}_z \leftarrow z\hat{n}_z\right) \; \psi(z\hat{n}_z) \middle| P^z \right\rangle$$

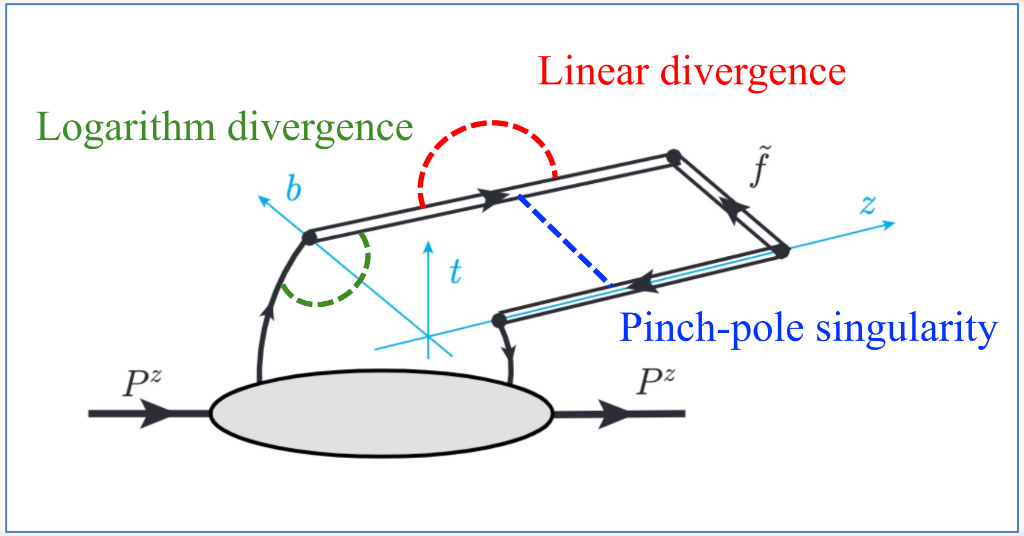




- **Extracted from 3- and 2-point functions**

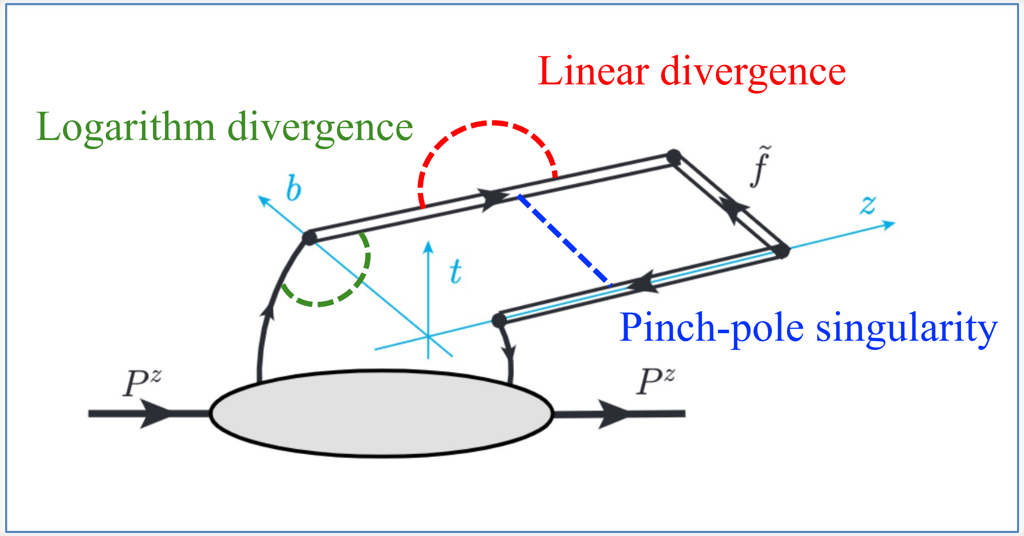
Quasi TMDPDF matrix element and renormalization

1. Divergences in bare quasi TMDPDF

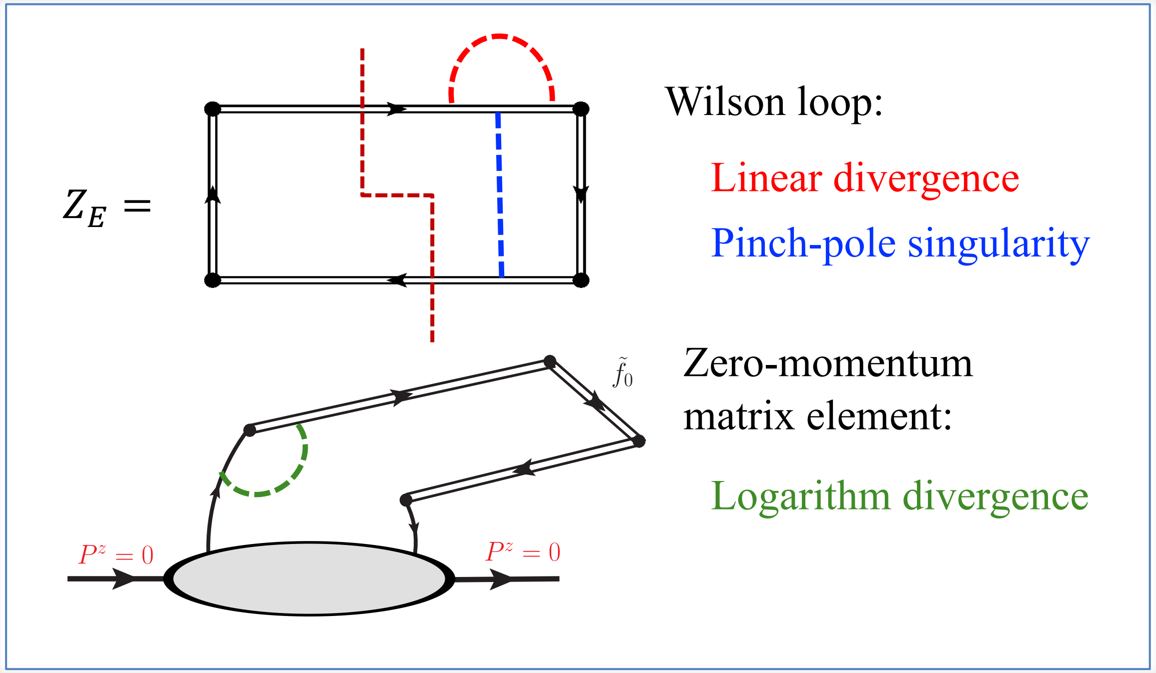


Quasi TMDPDF matrix element and renormalization

1. Divergences in bare quasi TMDPDF



2. Renormalization



$Z_E =$

Wilson loop:

Linear divergence

Pinch-pole singularity

Zero-momentum matrix element:

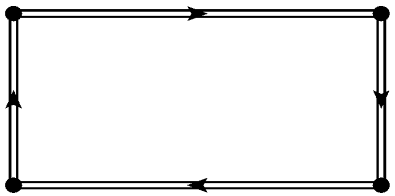
Logarithm divergence

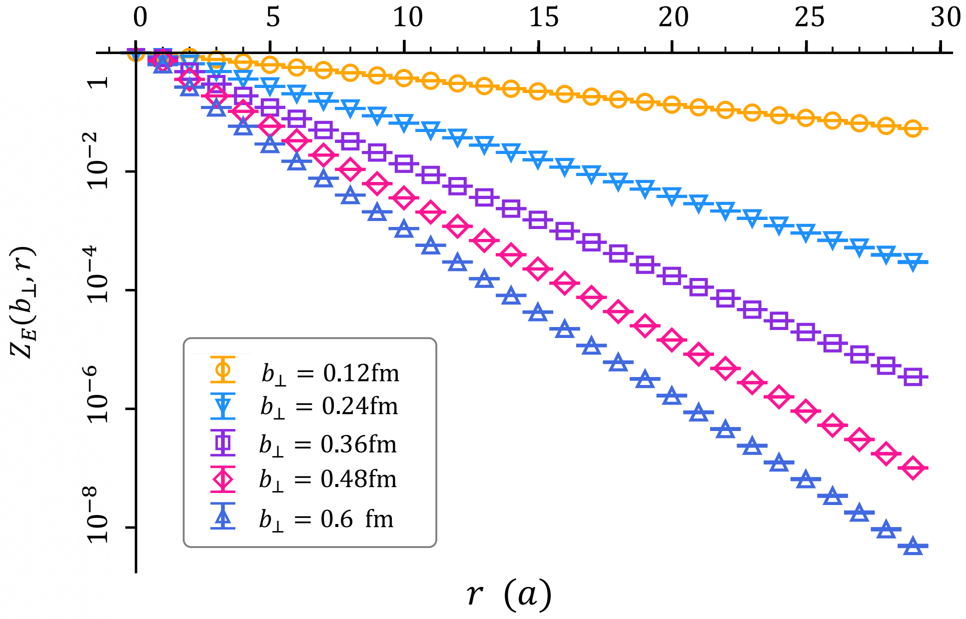
Ji, PRL120(2018), NPB964(2021), PLB257(1991); Zhang, PRD95(2017), NPB939(2019); Ishikawa, PRD96(2017); Green, PRL121(2018); Huo, NPB969(2021); Chen, NPB915(2017); Musch, PRD83(2011); ……

Quasi TMDPDF matrix element and renormalization

- **Wilson loop**

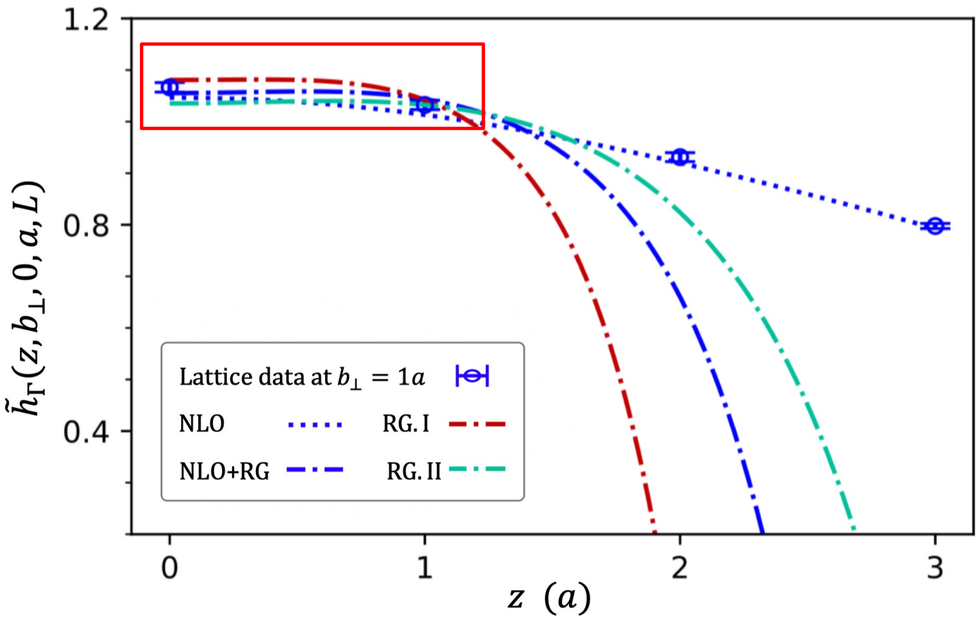
$Z_E =$



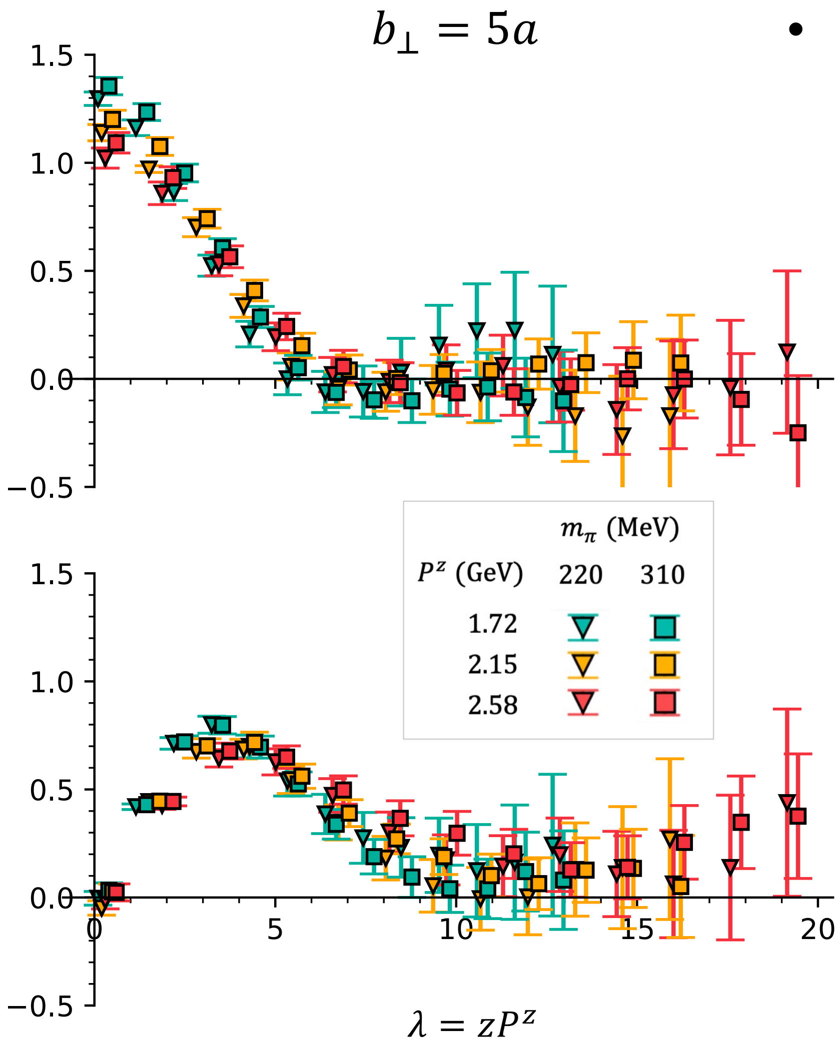


- **Logarithmic divergences factor**

$$Z_O(1/a,\mu,\Gamma)=\lim_{L\to\infty}\frac{\tilde{h}^{0}_{\Gamma}\left(z,b_\perp,0,a,L\right)}{\sqrt{Z_E\left(2L+z,b_\perp,a\right)}\tilde{h}^{\overline{\mathrm{MS}}}_{\Gamma}\left(z,b_\perp,\mu\right)}$$

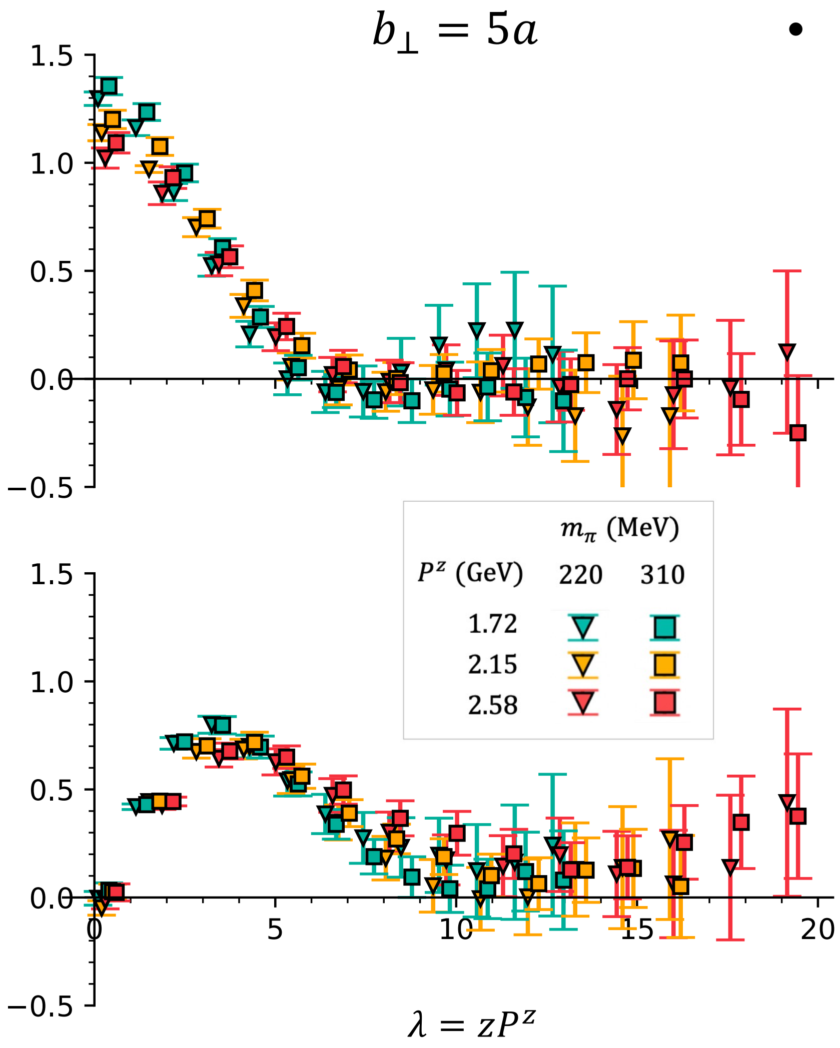


Quasi TMDPDF matrix element and λ extrapolation



- A brute-fore truncation at large λ will lead to **strong oscillation** after FT ⇒ need additional extrapolation….

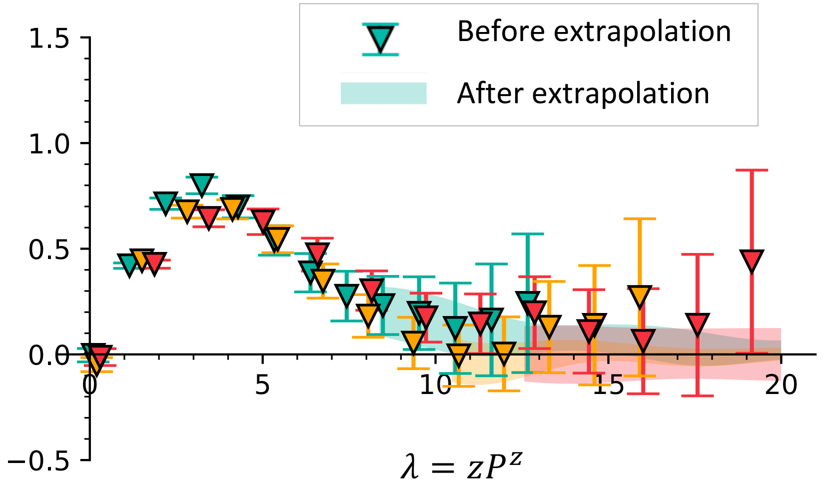
Quasi TMDPDF matrix element and λ extrapolation



- A brute-fore truncation at large λ will lead to **strong oscillation** after FT ⇒ need additional extrapolation….

$$\tilde{h}_{\text{extra}}(\lambda) = \left[\frac{c_1}{(-i\lambda)^{n_1}} + e^{i\lambda}\frac{c_2}{(i\lambda)^{n_2}}\right] e^{-\lambda/\lambda_0}$$

- end point power-law behavior $x^a(1-x)^b$;
- correlation function has a finite correlation length λ_0.



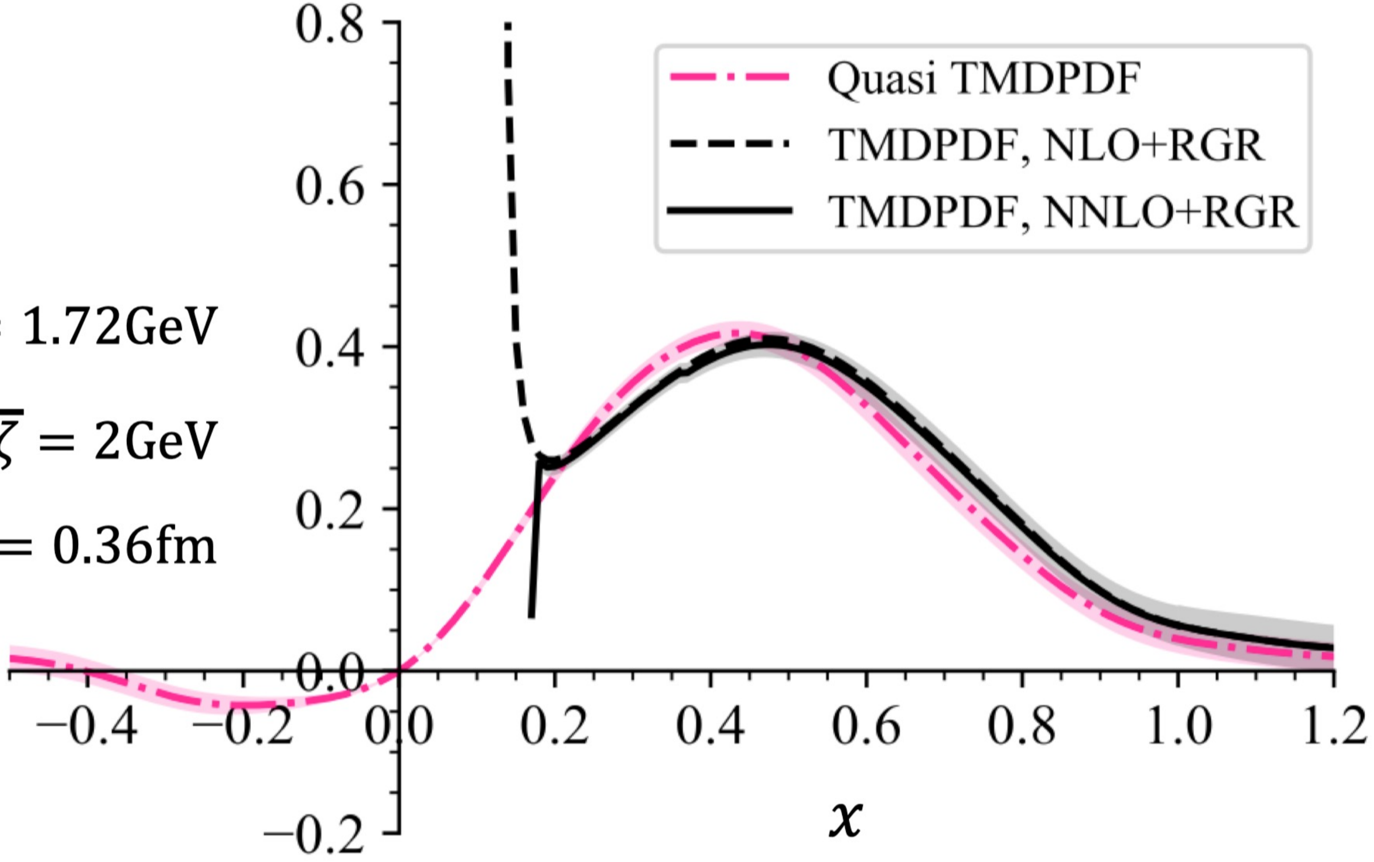
From Quasi TMDPDF to TMDPDF

$x\tilde{f}_{\gamma^t}$ v.s. xf_{γ^t}

$P^z = 1.72\text{GeV}$

$\mu = \sqrt{\zeta} = 2\text{GeV}$

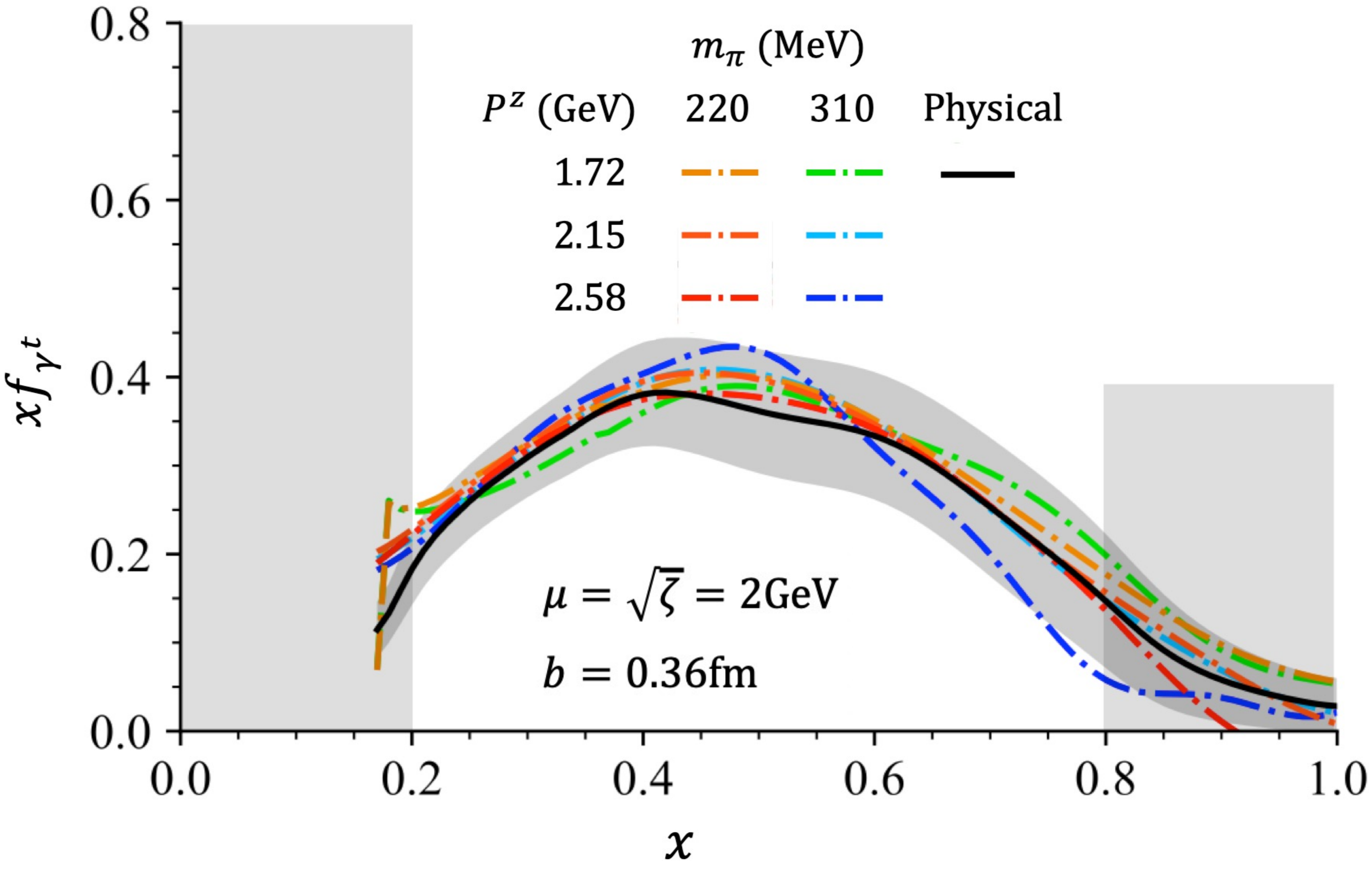
$b = 0.36\text{fm}$



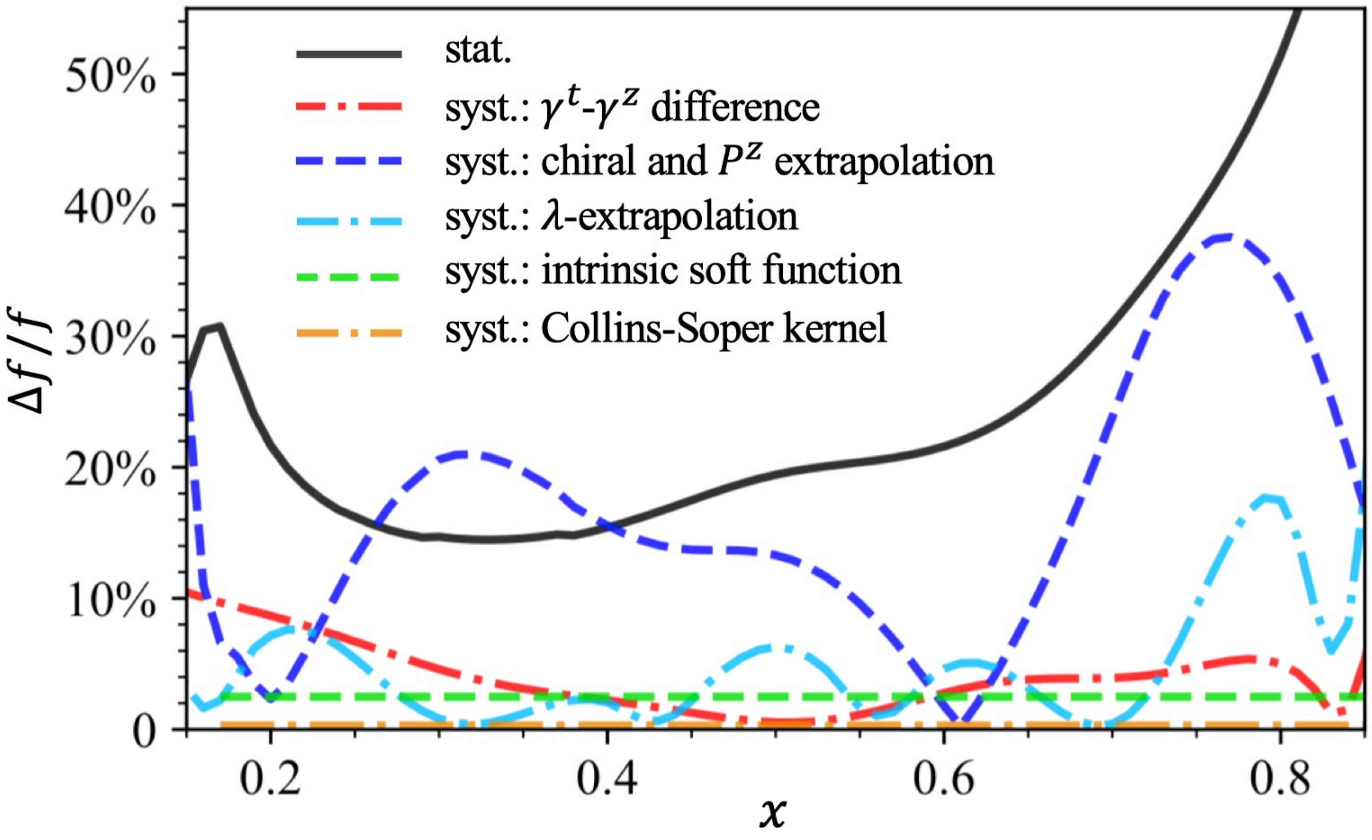
Physical TMDPDF

Chiral and large-P^z joint extrapolation:

$$d_0(m_\pi^2 - m_{\pi,\mathrm{phy}}^2) + \frac{d_1}{(P^z)^2}$$



Error estimation



All errors:

- **Statistical error;**

(1) From difference of γ^t and γ^z

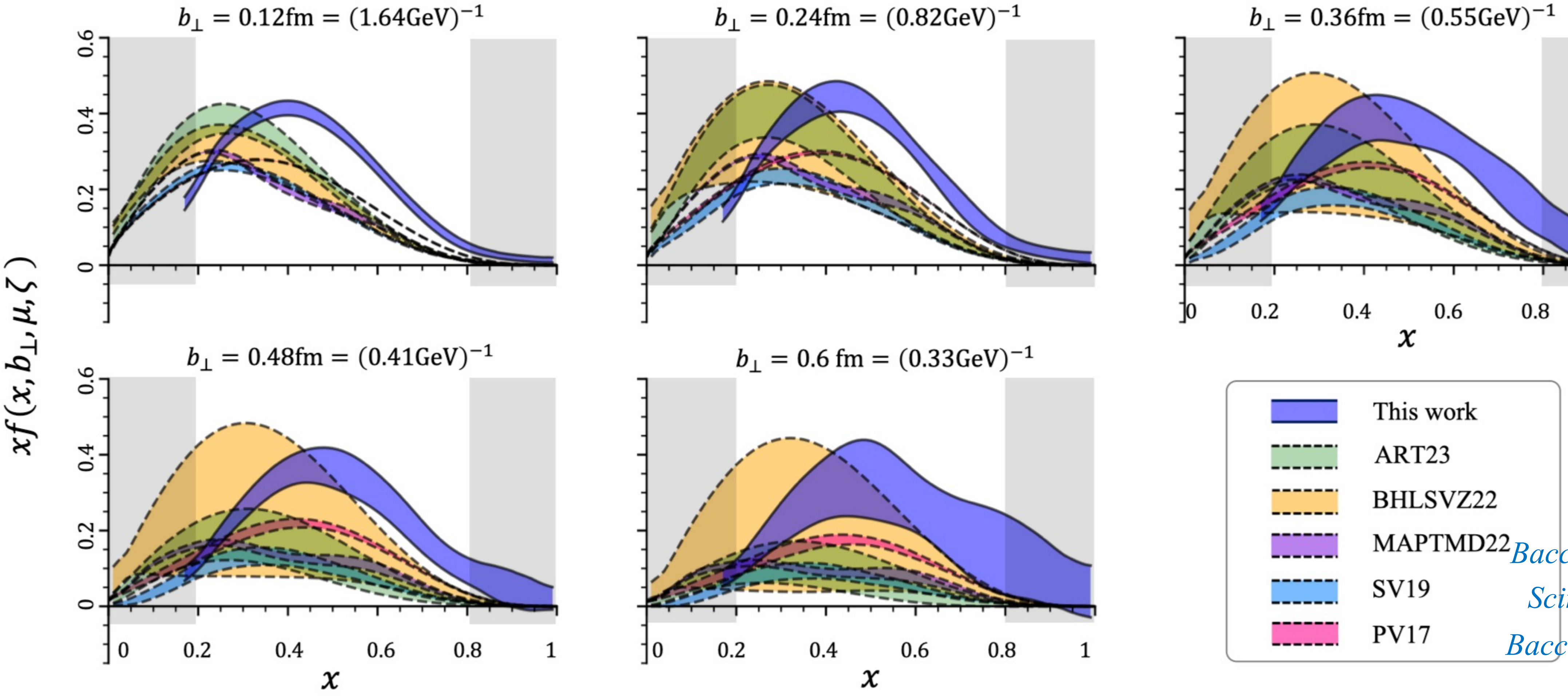
(2) From physical extrapolation

(3) From λ-extrapolation

(4) From soft function

(5) From Collins-Soper kernel

Final results and discussion



Moos, JHEP2024

Bury, JHEP2022

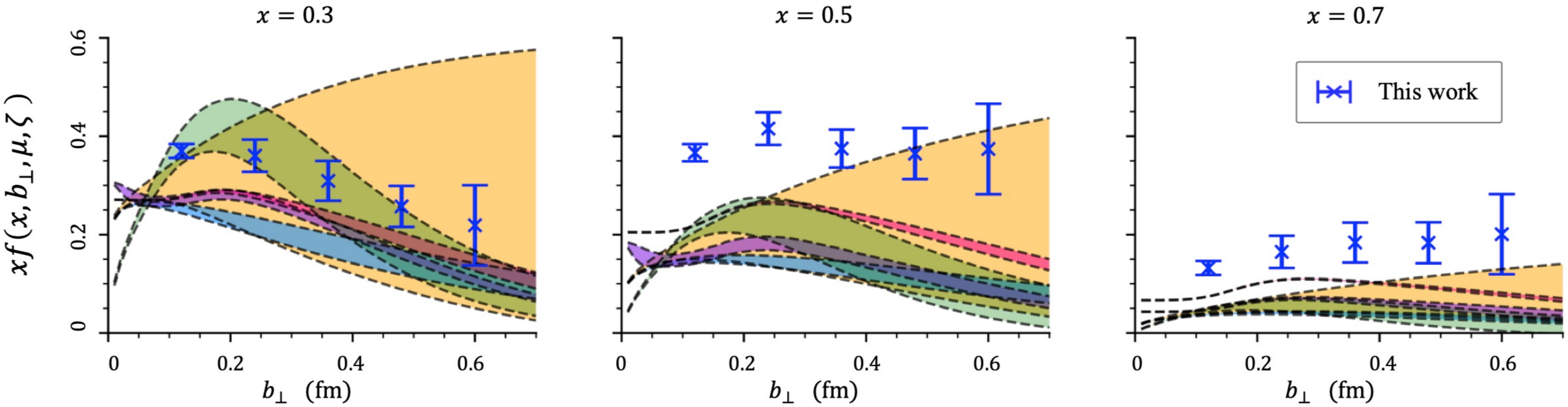
Bacchetta, JHEP2022

Scimemi, JHEP2020

Bacchetta, JHEP2019

Final results and discussion

Compare the $b_\perp$-dependence of lattice and phenomenological results:



Summary and Outlook

We present the lattice QCD calculation of TMDPDF at first attempt:

- ✓ The state-of-the-art techniques in renormalization and extrapolation on the lattice;
- ✓ The latest perturbative kernel up to 2-loop with RG evolution;
- ✓ Physical extrapolation include chiral-continuum and infinity momentum;
- ✓ Comparable results with phenomenological global fits.

Summary and Outlook

While there is still much room for further improvement:

- 🤔 Better control of uncertainties;
- 🤔 Continuum extrapolation: more lattice spacings;
- 🤔 Larger $b_\perp$ (up to nucleon radius?) to obtain a converge distribution in coordinate space;
- 🤔 Theoretical improvements:

  Power correction (small-x region), higher twist effects (operator mixing), ……

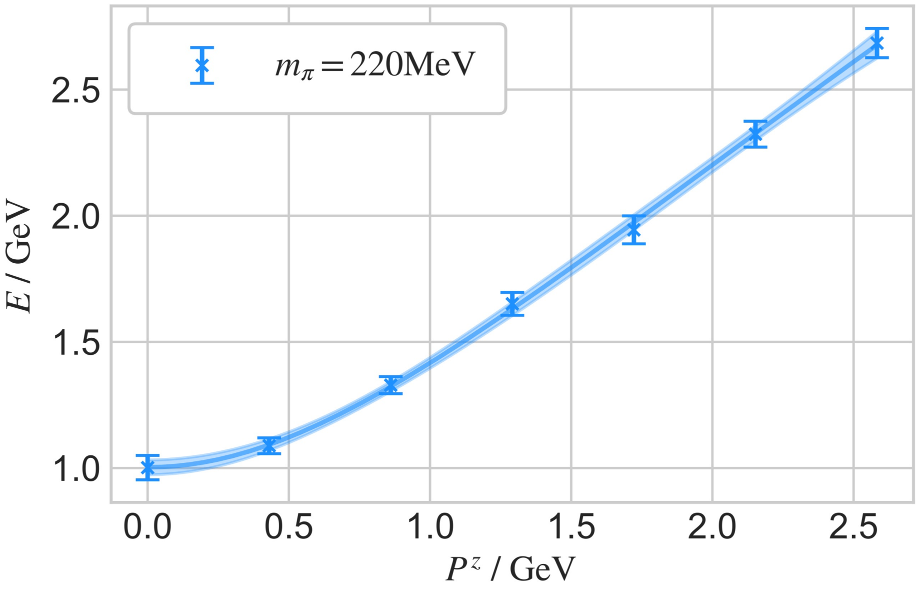
Thank you for your attention!

Backup slides

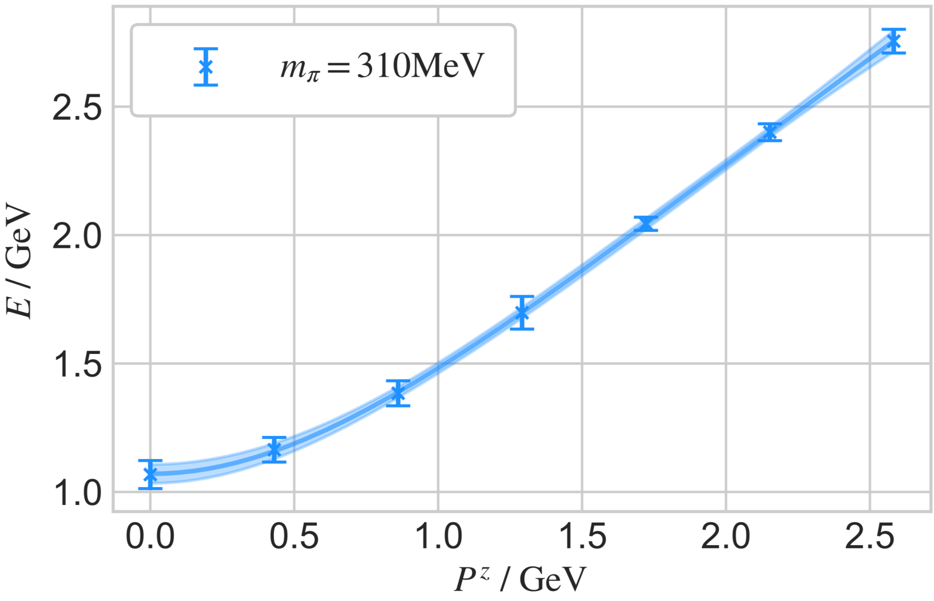
Dispersion relation

$$E = \sqrt{m^2 + c_1(P^z)^2 + c_2(P^z)^4 a^2}$$

$c_1 = 1.014(95),\ c_2 = -0.014(17)$



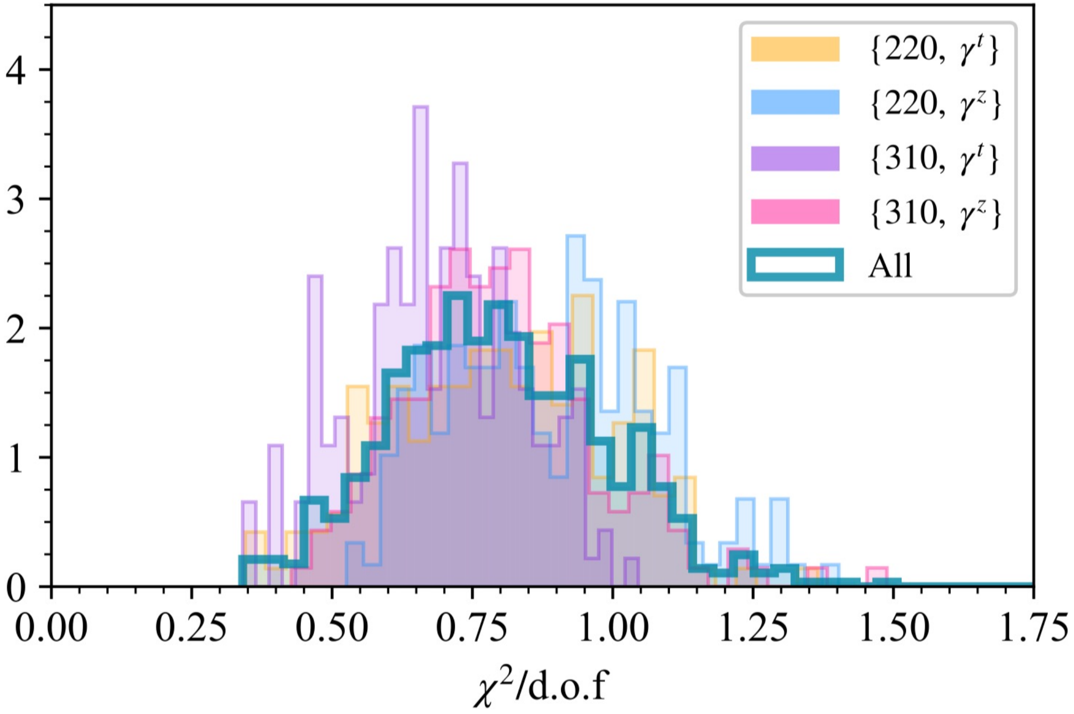
$c_1 = 1.066(80),\ c_2 = -0.015(14)$

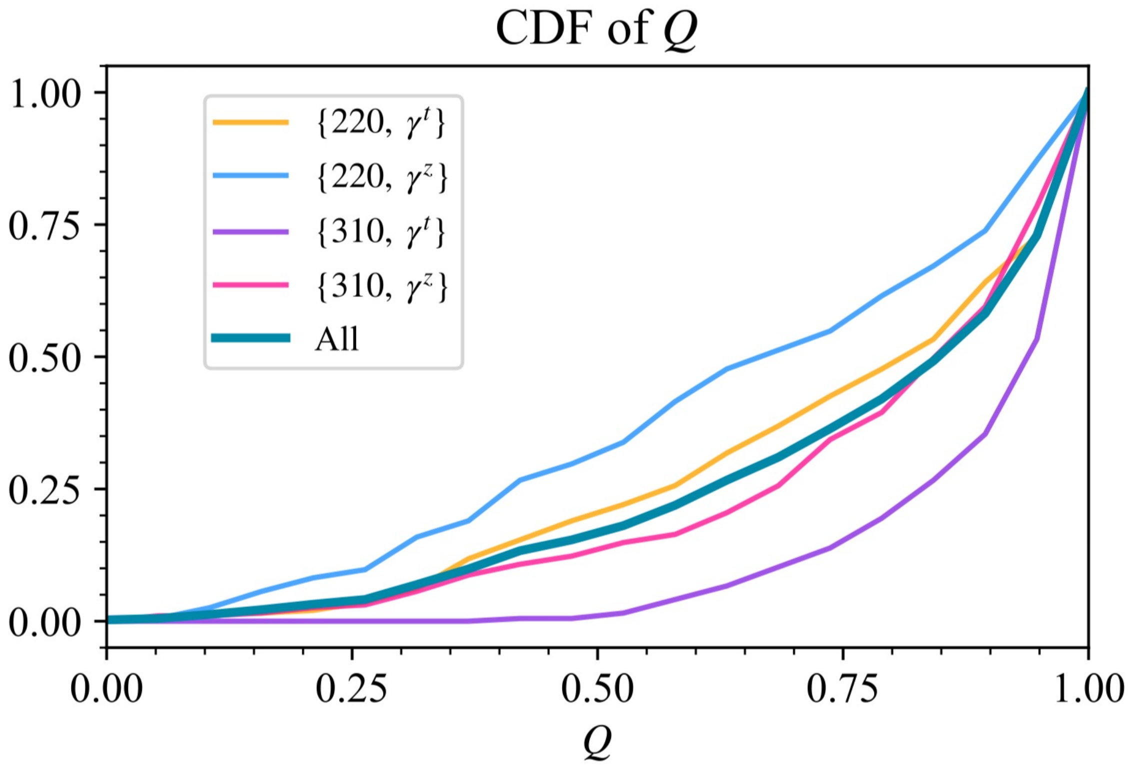


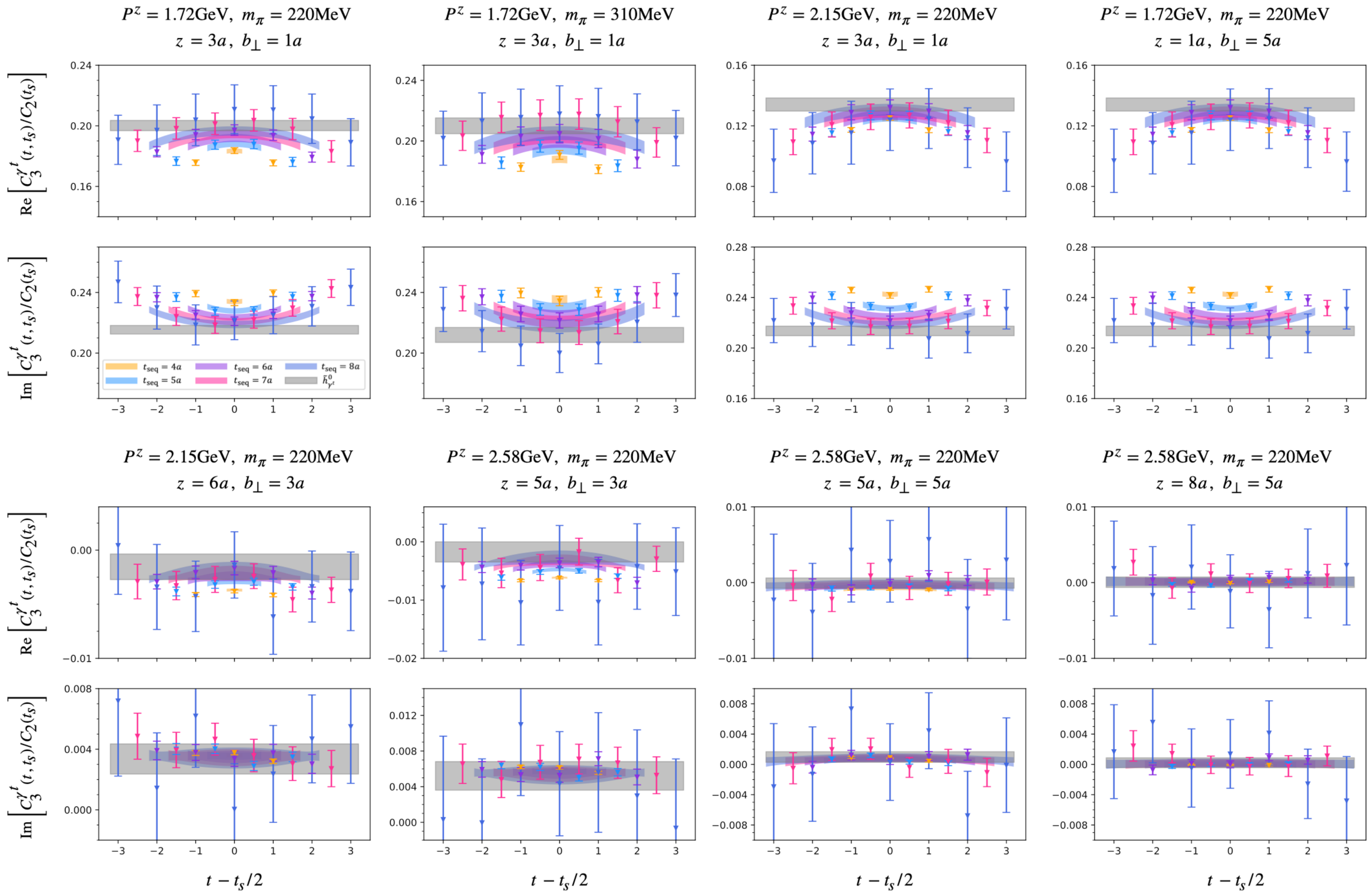
Details of correlated joint fits

Fit quality:

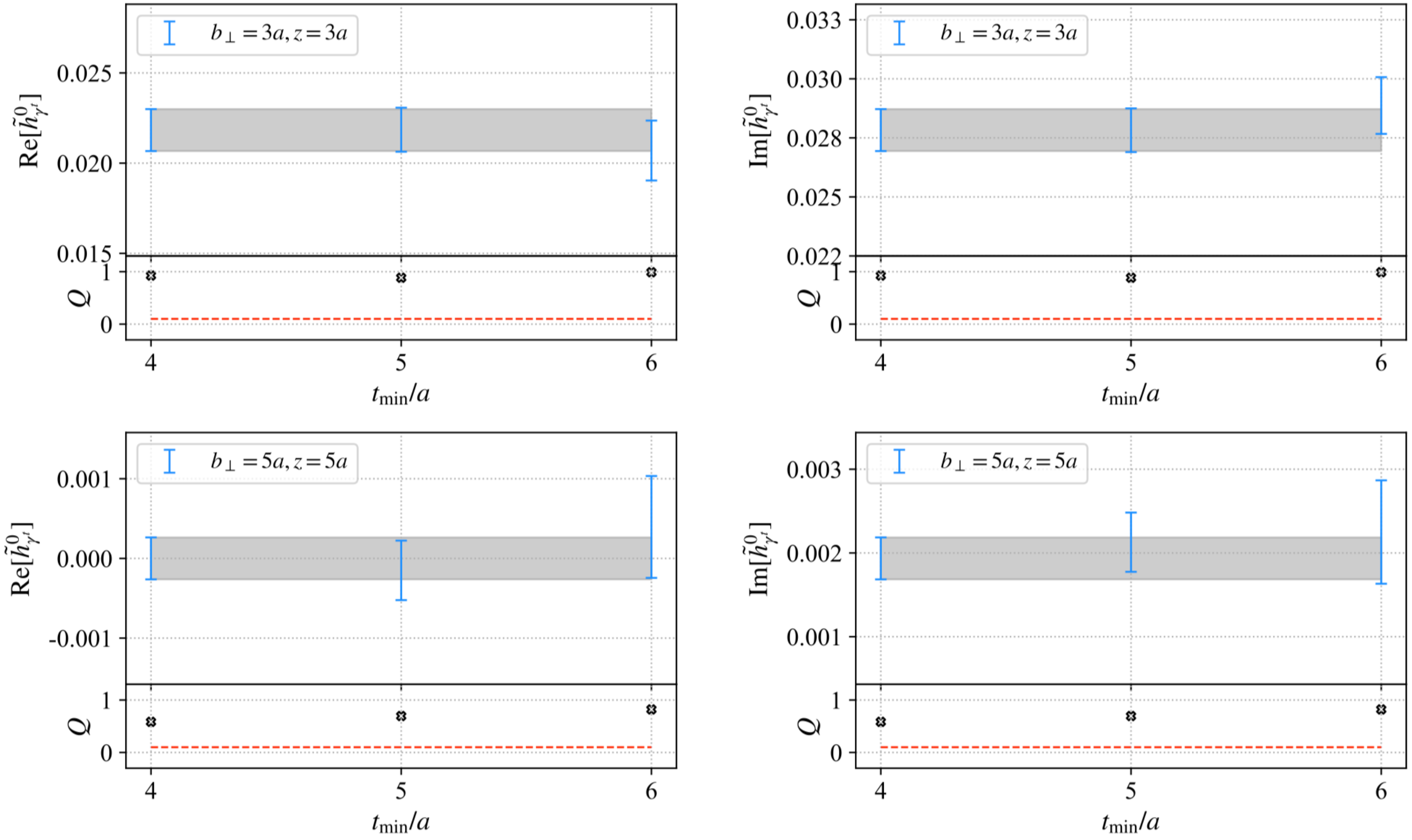
- Utilizing bootstrap resampling to establish correlations among all datasets;
- Employing fully-correlated Bayesian constrained fits to extract ground-state matrix elements.



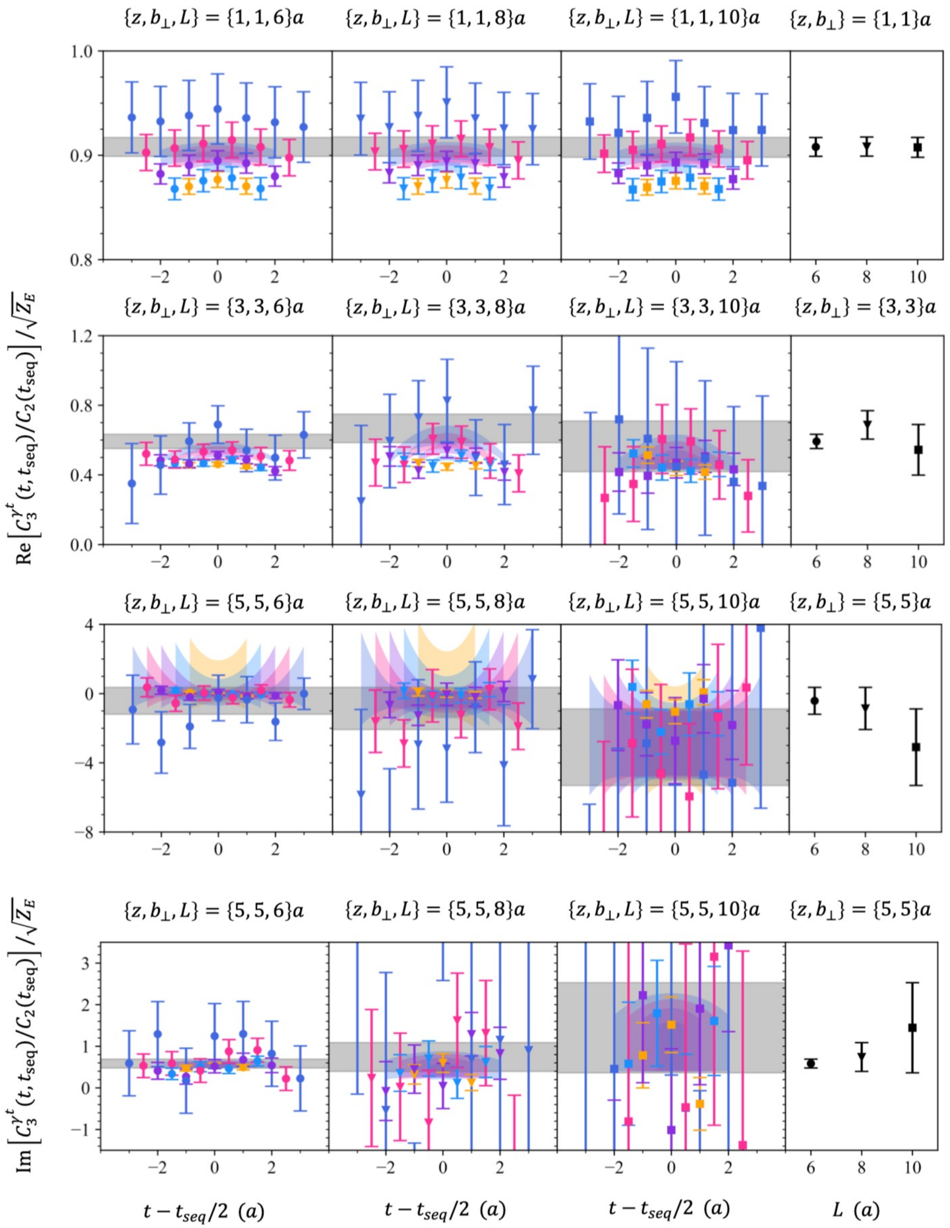

CDF of Q


$P^z = 1.72\text{GeV},\ m_\pi = 220\text{MeV}$ $z = 3a,\ b_\perp = 1a$
$P^z = 1.72\text{GeV},\ m_\pi = 310\text{MeV}$ $z = 3a,\ b_\perp = 1a$
$P^z = 2.15\text{GeV},\ m_\pi = 220\text{MeV}$ $z = 3a,\ b_\perp = 1a$
$P^z = 1.72\text{GeV},\ m_\pi = 220\text{MeV}$ $z = 1a,\ b_\perp = 5a$
$P^z = 2.15\text{GeV},\ m_\pi = 220\text{MeV}$ $z = 6a,\ b_\perp = 3a$
$P^z = 2.58\text{GeV},\ m_\pi = 220\text{MeV}$ $z = 5a,\ b_\perp = 3a$
$P^z = 2.58\text{GeV},\ m_\pi = 220\text{MeV}$ $z = 5a,\ b_\perp = 5a$
$P^z = 2.58\text{GeV},\ m_\pi = 220\text{MeV}$ $z = 8a,\ b_\perp = 5a$
$\text{Re}\left[C_3^{\gamma^t}(t,t_s)/C_2(t_s)\right]$
$\text{Im}\left[C_3^{\gamma^t}(t,t_s)/C_2(t_s)\right]$
$t_{\text{seq}} = 4a$
$t_{\text{seq}} = 5a$
$t_{\text{seq}} = 6a$
$t_{\text{seq}} = 7a$
$t_{\text{seq}} = 8a$
$\tilde{h}^0_{\gamma^t}$
$t - t_s/2$
−3 −2 −1 0 1 2 3
0.24 0.20 0.16
0.16 0.12 0.08
0.28 0.24 0.20 0.16
0.00 −0.01
0.00 −0.01 −0.02
0.01 0.00 −0.01
0.008 0.004 0.000
0.012 0.008 0.004 0.000
0.008 0.004 0.000 −0.004 −0.008

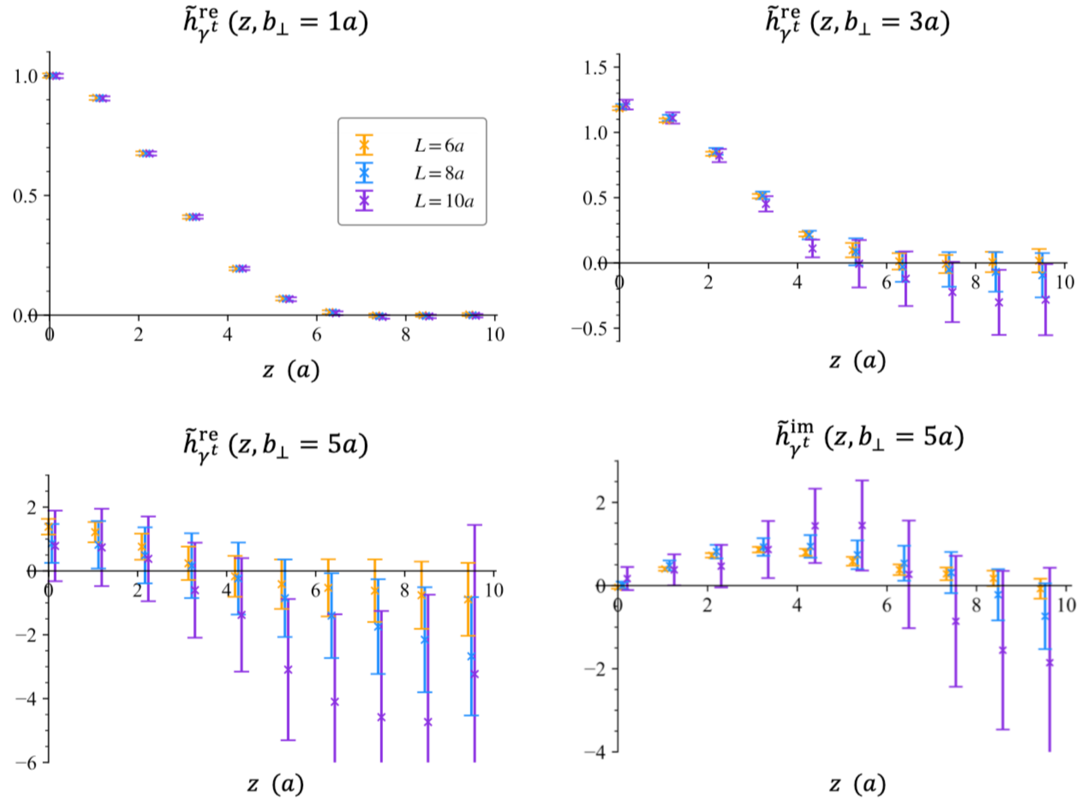
- Stability of the joint fits: t_{min} dependence of the fit result, which fit range is $[t_{\text{min}}, t_{\text{max}}]$.



L-dependence



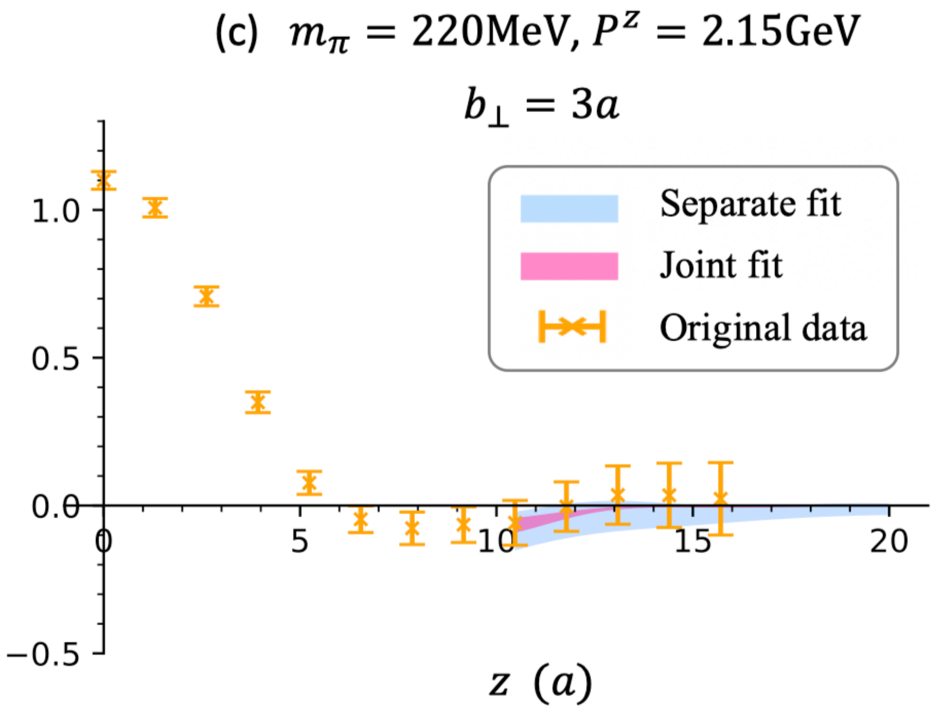
Saturation length of Wilson link:



λ-extrapolation

Factorization of z and $b_\perp$?

$$\tilde{h}_{\text{extra}}(\lambda) = \left[\frac{c_1}{(-i\lambda)^{n_1}} + e^{i\lambda}\frac{c_2}{(i\lambda)^{n_2}}\right] e^{-\lambda/\lambda_0}$$



The power-law behavior and correlation length for each $b_\perp$ should be similar,

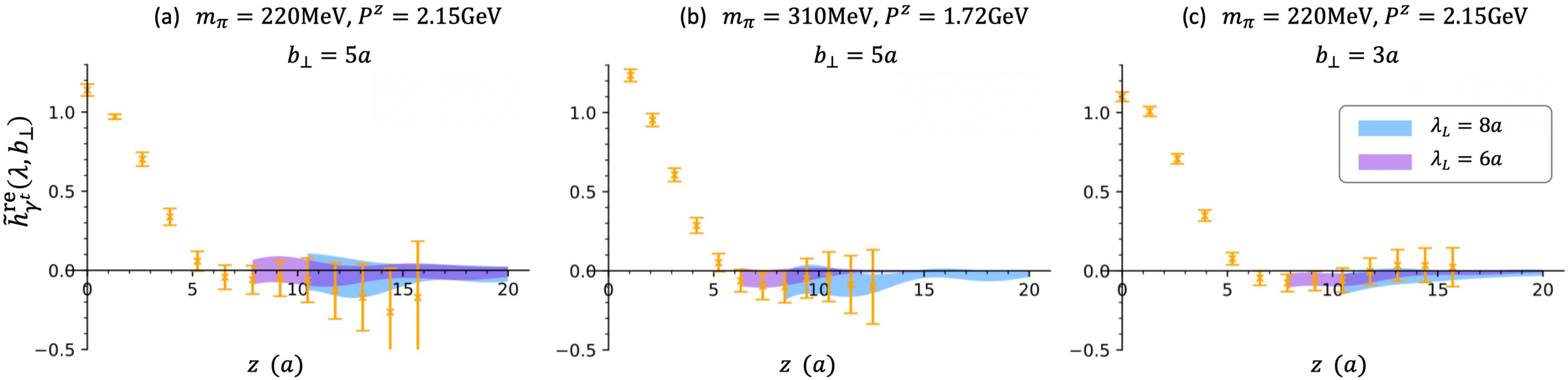
but the joint fit will give a strict limit for large-$b_\perp$ cases:

$b_\perp\ (a)$	1	2	3	4	5	Joint
n_1	0.909(39)	0.943(61)	0.89(10)	0.801(78)	0.84(16)	0.887(28)
n_2	1.31(34)	2.37(68)	1.71(31)	1.55(38)	1.22(44)	1.65(12)
λ	2.63(38)	3.20(80)	2.42(85)	4.3(1.6)	4.4(2.8)	2.53(28)
χ^2/d.o.f.	1.0	1.1	1.3	0.75	0.57	1.2

λ-extrapolation

Systematic uncertainty from fit region $[\lambda_L : \lambda_{\max}]$

$$\tilde{h}_{\text{extra}}(\lambda) = \left[\frac{c_1}{(-i\lambda)^{n_1}} + e^{i\lambda}\frac{c_2}{(i\lambda)^{n_2}}\right]e^{-\lambda/\lambda_0}$$

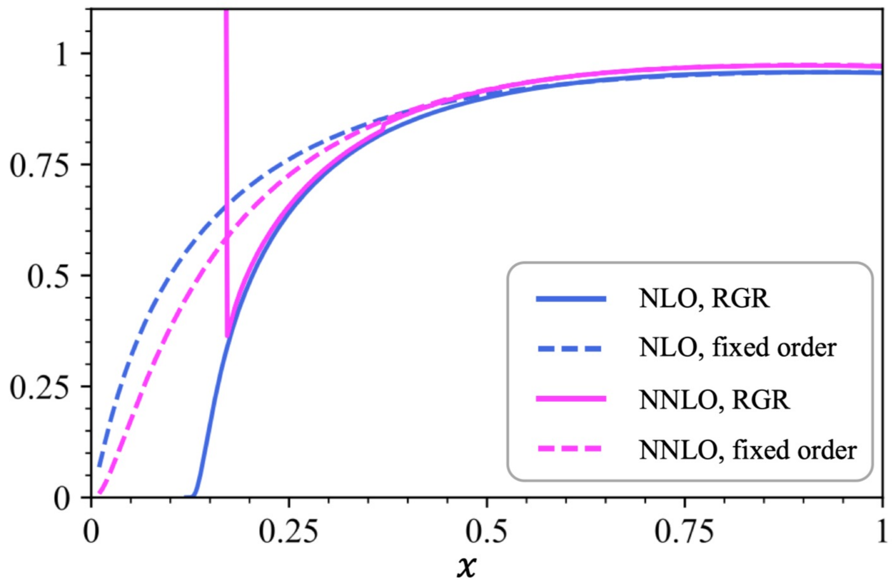


Perturbative matching kernel and RG resummation

- **Fixed-order perturbative results up to the 2-loop level:**

$$h^{(1)}\left(\frac{\zeta_z}{\mu^2}\right) = \frac{\alpha_s C_F}{2\pi}\left(-2+\frac{\pi^2}{12}+\ln\frac{\zeta_z}{\mu^2}-\frac{1}{2}\ln^2\frac{\zeta_z}{\mu^2}\right),$$

$$h^{(2)}\left(\frac{\zeta_z}{\mu^2}\right) = \alpha_s^2\left[c_2-\frac{1}{2}\left(\gamma_C^{(2)}-\beta_0 c_1\right)\ln\frac{\zeta_z}{\mu^2}-\frac{1}{4}\left(\Gamma_{\text{cusp}}^{(2)}-\frac{\beta_0 C_F}{2\pi}\right)\ln^2\frac{\zeta_z}{\mu^2}-\frac{\beta_0 C_F}{24\pi}\ln^3\frac{\zeta_z}{\mu^2}\right]$$



- **RG equation of the matching kernel:**

$$\mu^2\frac{d}{d\mu^2}\ln H\left(\frac{\zeta_z}{\mu^2}\right) = \frac{1}{2}\Gamma_{\text{cusp}}\left(\alpha_s\right)\ln\frac{\zeta_z}{\mu^2}+\frac{\gamma_C\left(\alpha_s\right)}{2},$$

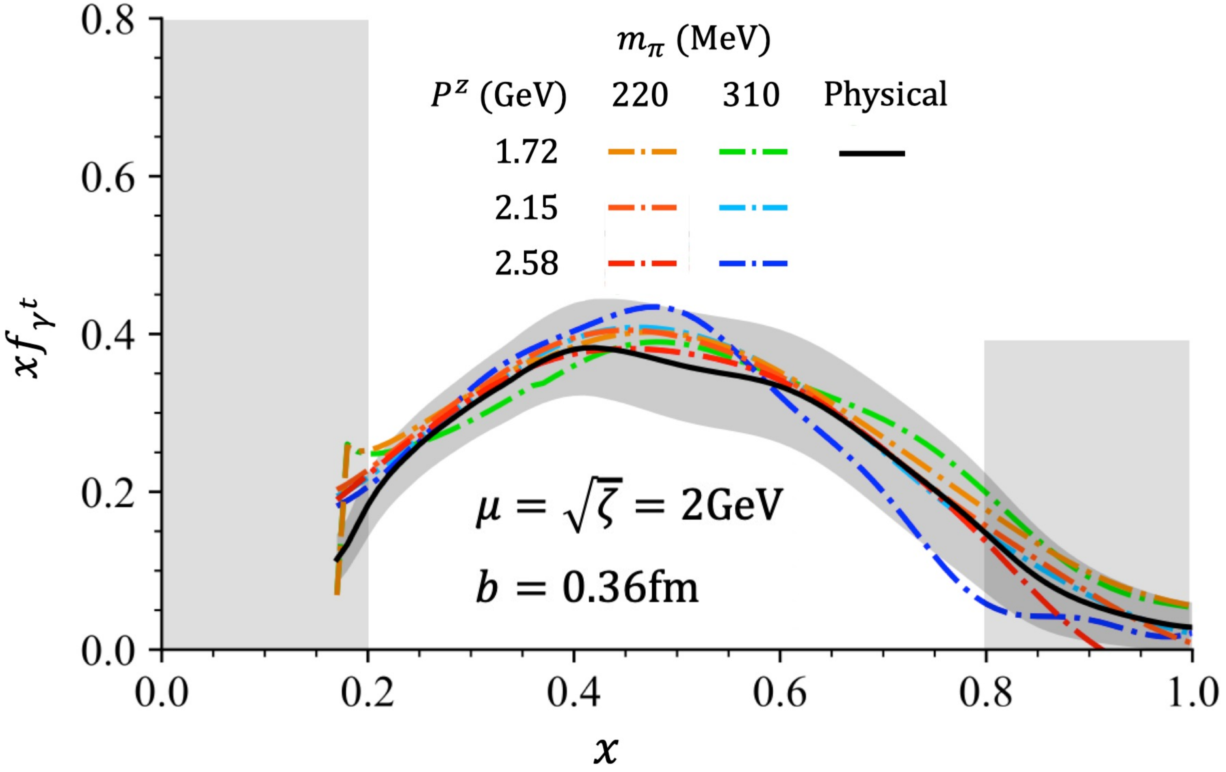
and its solution:

$$H\left(\zeta_z/\mu^2\right) = H\left(\zeta_z/\mu_0^2\right)\exp\left[\int_{\mu_0}^{\mu}\frac{d\mu}{\mu}\left(\Gamma_{\text{cusp}}^{(1)}\ln\frac{\zeta_z}{\mu^2}\alpha_s(\mu)+\gamma_C^{(1)}\alpha_s(\mu)+\Gamma_{\text{cusp}}^{(2)}\ln\frac{\zeta_z}{\mu^2}\alpha_s^2(\mu)\right.\right.$$
$$\left.\left.+\gamma_C^{(2)}\alpha_s^2(\mu)+\Gamma_{\text{cusp}}^{(3)}\ln\frac{\zeta_z}{\mu^2}\alpha_s^3(\mu)+\gamma_C^{(3)}\alpha_s^3(\mu)+\Gamma_{\text{cusp}}^{(4)}\ln\frac{\zeta_z}{\mu^2}\alpha_s^4(\mu)\right)\right].$$

Physical TMDPDF

Chiral and large-P^z joint extrapolation:

$$d_0(m_\pi^2 - m_{\pi,\text{phy}}^2) + \frac{d_1}{(P^z)^2}$$

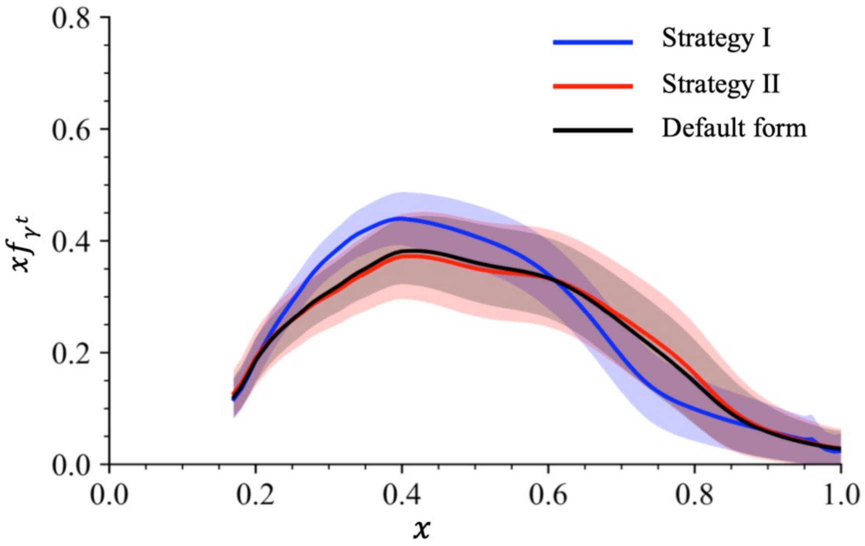


Systematic from chiral extrapolation (strategy I):

$$d_0(m_\pi^2 - m_{\pi,\text{phy}}^2)^2 + \frac{d_1}{(P^z)^2}$$

from large-P^z extrapolation (strategy II):

$$d_0(m_\pi^2 - m_{\pi,\text{phy}}^2) + \frac{d_1}{(P^z)^2} + \frac{d_2}{P^z}$$



Power correction

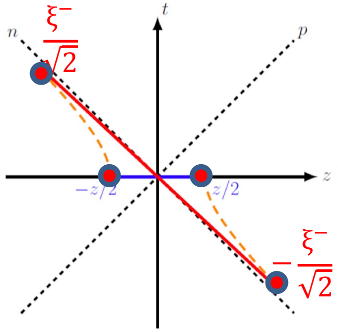
After Lorentz boost:

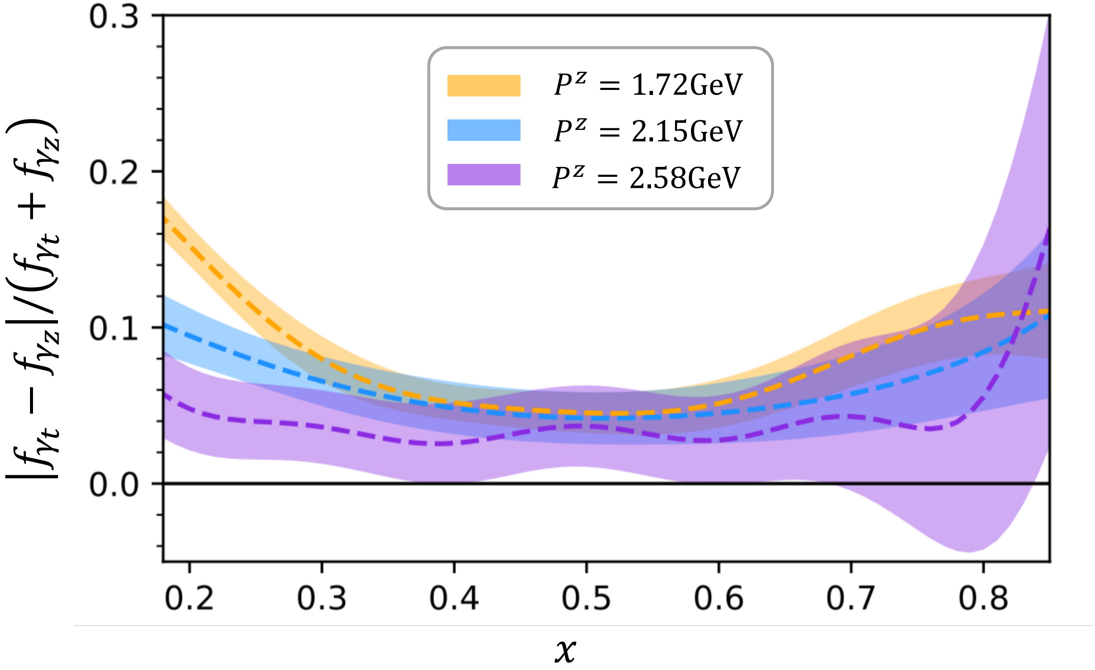
Leading power

Higher power

$$\bar\psi(z)\gamma^t\psi(0) = \boxed{\frac{1}{2}\bar\psi(z)\gamma^+\psi(0)} + \boxed{\frac{1}{2}\bar\psi(z)\gamma^-\psi(0)}$$

$$\bar\psi(z)\gamma^z\psi(0) = \boxed{\frac{1}{2}\bar\psi(z)\gamma^+\psi(0)} - \boxed{\frac{1}{2}\bar\psi(z)\gamma^-\psi(0)}$$





- ➢ **Ratios denote the deviations from light-like correlator with specific P^z;**
- ➢ **Ratio becomes smaller with P^z increasing.**

Final results and discussion

🤔 **The unpolarized TMDPDFs seem not converge in $b_\perp$-space?**

Of course not! Perhaps there will be abrupt change at the edge of nucleon

⇒ Need larger $b_\perp$ and more statistics!

🤔 **Lattice discretization and finite-volume systematics are still absent in this preliminary work…**

- **It is a challenging work for calculating the TMDPDF at small lattice spacing**
- **From the previous experience of PDF (*Lin, 2011.14971*), we can roughly estimate that:**

Finite-volume effect is less than 1%;

Discretization effects overall within 2 standard deviations.