The Tomography of Nucleon:

Lattice QCD calculation of the unpolarized transverse-momentum-dependent parton distributions

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Based on PRD109, 114513 (2024)
OUTLINE

➢ Motivation

➢ Lattice QCD calculation of TMDPDFs
  • Extract TMDPDFs from LaMET
  • Quasi TMDPDF matrix elements and their renormalization
  • From Quasi TMDPDF to physical TMDPDF
  • Numerical results

➢ Summary and Outlook
TMDPDFs: 3D tomography of the nucleon

TMD processes:

- **Drell-Yan**
  - $Q, q_T$
  - $\mu^-, \mu^+$
  - $q_T \ll Q$
  - LHC, Fermilab, RHIC, ...

- **Semi-Inclusive DIS**
  - $e \rightarrow e$
  - $P_{a, b}$
  - Parton Distribution
  - Fragmentation
  - HERMES, COMPASS, JLab, EIC, ...

- **Dihadron in $e^+e^-$**
  - $e^+ \rightarrow H_1$
  - $H_2$
  - $H_2 \rightarrow e^-$
  - BESIII, Babar, Belle, ...

Mathematical Formulas:

- $\sigma \sim f_{q/P}(x, k_T) f_{q/P}(x, k_T)$
- $\sigma \sim f_{q/P}(x, k_T) D_{h/q}(x, k_T)$
- $\sigma \sim D_{h_1/q}(x, k_T) D_{h_2/q}(x, k_T)$
TMDPDFs: 3D tomography of the nucleon

• Low-$q_T$ region of Drell-Yan Process:

Revealing the confined motion of partons inside the nucleon
TMDPDFs: 3D tomography of the nucleon

Leading Quark TMDPDFs

<table>
<thead>
<tr>
<th>Nucleon Polarization</th>
<th>Quark Polarization</th>
<th>Un-Polarized (U)</th>
<th>Longitudinally Polarized (L)</th>
<th>Transversely Polarized (T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>$f_1 = \bigcirc$</td>
<td></td>
<td>$g_{1L} = \bigcirc - \bigcirc$</td>
<td>$h_{1L}^{\perp} = \bigcirc - \bigcirc$</td>
</tr>
<tr>
<td></td>
<td>Unpolarized</td>
<td></td>
<td>Helicity</td>
<td>Boer-Mulders</td>
</tr>
<tr>
<td>L</td>
<td>$g_{1L} = \bigcirc - \bigcirc$</td>
<td>$h_{1L}^{\perp} = \bigcirc - \bigcirc$</td>
<td>Worm-gear</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>$f_{1T} = \bigcirc - \bigcirc$</td>
<td>$g_{1T} = \bigcirc - \bigcirc$</td>
<td>$h_{1T} = \bigcirc - \bigcirc$</td>
<td>Pretzelosity</td>
</tr>
</tbody>
</table>

TMD Handbook, TMD Collaboration, 2304.03302
Progress in the study of TMDPDFs

➢ Theoretical analysis

  • TMD factorization, evolution and resummation:
    Boussarie et al., TMD handbook, 2304.03302;
    Collins, Foundations of perturbative QCD; ……

➢ Phenomenological parametrizations and extractions

  • Unpolarized:
    Moos, JHEP05 (2024); Bacchetta, JHEP10 (2022); Bury, JHEP10 (2022);
    Scimem, JHEP06 (2020); Bacchetta, JHEP06 (2017); ……

  • Sivers, Boer-Mulders:
    Bury, PRL126 (2021), JHEP05 (2021); Cammarota, PRD102(2020);
    Zhang, PRD77 (2008), Lu, PRD81 (2010); ……

  • Others: worm-gear, gluon TMDs, ……
Lattice calculations

- **Lorentz-invariant approach**: ratios of Mellin moments
  
  Hagler, EPL88(2009); Musch, PRD85(2012); Engelhardt, PRD93(2016); Yoon, 1601.05717, PRD96(2017); ……

- **LaMET formalism**:
  
  ✓ I: theoretical analysis of matching kernel, soft function, Collins-Soper kernel, ……
    
    Rio, PRD108(2023); Ji, JHEP08(2023), RMP93(2021), NPB955(2020), PLB811(2020);
    
    Ebert, JHEP04(2022); Deng, JHEP09(2022)……

  ✓ II: lattice calculation of intrinsic soft function, Collins-Soper kernel, beam function, ……
    
    LPC, JHEP08(2023), PRL125(2020); Li, PRL128(2022); LPC, PRD106(2022);
    
    Shanahan, PRD104(2021); Schlemmer, JHEP08(2021); ……

  ✓ III: Nonperturbative renormalization, resummation, ……
    
    Zhang, PLB884(2023); Ji, JHEP08(2023); Su, NPB991(2023); LPC, PRL129(2022); NPB991(2023)……

- IV: A real lattice calculation of TMD observable?
Extracting TMDs in LaMET formalism

- **Large-momentum effective theory:** connecting Euclidean lattice and physical observables

  - Lorentz boost
  - EFT

  Ji, PRL110(2013), RMP93(2021), ...

- Achieved great success in the studies of PDF:
  - Pion valance PDF, PRD106(2022)
  - Proton unpolarized PDF, in preparation
  - Proton transversity PDF, PRL131(2023)
  - Proton unpolarized PDF, in preparation

Preliminary
• Matching from quasi TMDs to TMDs

- Hadronic matrix element reduced from equal-time correlators:

\[ \tilde{h}_T^0 (z, b_\perp, P^z) = \lim_{L \to \infty} \left\langle P^z \right| \bar{\Psi}(b_\perp \hat{n}_\perp) \Gamma \times U_{\Gamma} (b_\perp \hat{n}_\perp \leftarrow b_\perp \hat{n}_\perp + L \hat{n}_z; b_\perp \hat{n}_\perp + L \hat{n}_z \leftarrow L \hat{n}_z; L \hat{n}_z \leftarrow z \hat{n}_z) \times \psi(z \hat{n}_z) \right| P^z \rangle \]

- Subtracted quasi TMDPDFs:

\[ \tilde{f}_T (x, b_\perp, P^z, \mu) \equiv \lim_{a \to 0} \int \frac{dz}{2\pi} e^{-iz(xP^z)} \tilde{h}_T^0 (z, b_\perp, P^z, a, L) \sqrt{Z_E (2L + z, b_\perp, a)Z_O (1/a, \mu, \Gamma)} \]
Lorentz boost
$L \to \infty$

Equal-time correlators, directly calculable on lattice

Space-like correlators, NO effective method for directly calculation

Ji, PLB811(2020); Ebert, JHEP04(2022)

Connected at large-momentum limit

$\tilde{f}_\Gamma (x, b_\perp, \zeta_z, \mu) \sqrt{S_I (b_\perp, \mu)} = H_\Gamma \left( \frac{\zeta_z}{\mu^2} \right) e^{\frac{1}{2} \ln \left( \frac{\zeta_z^2}{\zeta_f} \right)} K(b_\perp, \mu) f (x, b_\perp, \mu, \zeta) + O\left( \frac{\Lambda_{QCD}^2}{\zeta_z}, \frac{M^2}{(P^2)^2}, \frac{1}{b_\perp^2 \zeta_z} \right)$

Quasi TMDPDF  Intrinsic soft function  Collins-Soper kernel  Light-cone TMDPDF
Collins–Soper kernel and intrinsic soft function

• Collins-Soper kernel

**From quasi beam function:**

- Shanahan, *PRD104*(2021), *PRD102*(2020);
- Schlemmer, *JHEP08*(2021); ……

**From quasi TMDWF:**

- Chu, *JHEP08*(2023), *PRD106*(2022);
- Zhang, *PRL125*(2020); Li, *PRL128*(2022); ……
Collins–Soper kernel and intrinsic soft function

- Intrinsic/reduced soft function

From quasi TMDWF + 4-quark matrix element:

Chu, *PRD*109(2024); Ji, *NPB*955(2020);
Zhang, *PRL*125(2020); Li, *PRL*128(2022);
……
Matching kernel and RG resummation

\[ \tilde{f}_\Gamma (x, b_\perp, \zeta_z, \mu) \sqrt{S_I (b_\perp, \mu)} = H_\Gamma \left( \frac{\zeta_z}{\mu^2} \right) e^{\frac{1}{2} \ln \left( \frac{\zeta_z}{\zeta} \right)} \mathcal{K} (b_\perp, \mu) f (x, b_\perp, \mu, \zeta) + \mathcal{O} \left( \frac{\Lambda_{QCD}^2}{\zeta_z}, \frac{M^2}{(P^z)^2}, \frac{1}{b_\perp^2 \zeta_z} \right) \]

Matching kernel

- **NLO:** Ji, PLB811(2020); RMP93(2021)
- **NNLO:** Río, PRD108(2023); Ji, JHEP08(2023)

- Fixed order: $\mu = 2$GeV;
- RGR: RG evolution from lattice scale
  \[ \zeta_z = 2xP^z \] to $\overline{\text{MS}}$ scale $\mu = 2$GeV.
Lattice calculation of physical TMDPDF?

\[
\tilde{f}_\Gamma(x, b_\perp, \zeta, \mu) \sqrt{S_I(b_\perp, \mu)} = H_\Gamma \left( \frac{\zeta}{\mu^2} \right) e^{\frac{1}{2} \ln \left( \frac{\zeta_0}{\zeta} \right)} K(b_\perp, \mu) f(x, b_\perp, \mu, \zeta) + \mathcal{O} \left( \frac{\Lambda^2_{QCD}}{\zeta}, \frac{M^2}{(P^z)^2}, \frac{1}{b^2_\perp \zeta} \right)
\]

**Matching kernel**

**Quasi TMDPDF**  
**Intrinsic soft function**  
**Collins-Soper kernel**

Simulating quasi TMDPDF on a Euclidean lattice:

- MILC configuration: \(48^3 \times 64, a = 0.12\text{fm}\);
- Pion mass: \(m_\pi^{\text{sea}} = 130\text{MeV}, m_\pi^{\text{val}} = \{310, 220\}\text{MeV} \Rightarrow \) extrapolate to physical mass
- Large momentum: \(P^z = \{1.72, 2.15, 2.58\}\text{GeV} \Rightarrow \) extrapolate to infinity
- Saturated length of Wilson link \(L = 0.72\text{fm}\);
- \(z_{\text{max}} = 1.44\text{fm}, b_{\perp \text{max}} = 0.6\text{fm}\).
Quasi TMDPDF matrix element

Bare quasi TMDPDF matrix element

\[ \tilde{h}_{\Gamma}^0 (z, b_\perp, P^z) = \lim_{L \to \infty} \left< P^z \left| \bar{\psi}(b_\perp \hat{n}_\perp) \Gamma U_{\Gamma} (b_\perp \hat{n}_\perp \leftrightarrow b_\perp \hat{n}_\perp + L \hat{n}_x; b_\perp \hat{n}_\perp \leftrightarrow L \hat{n}_z; L \hat{n}_z \leftrightarrow z \hat{n}_x) \right| \right> \psi(z \hat{n}_x) \right| P^z \right> \]

- Extracted from 3- and 2-point functions
1. Divergences in bare quasi TMDPDF
Quasi TMDPDF matrix element and renormalization

1. Divergences in bare quasi TMDPDF

- Linear divergence
- Pinch-pole singularity
- Logarithm divergence

2. Renormalization

\[ Z_E = \]

- Wilson loop:
  - Linear divergence
  - Pinch-pole singularity
- Zero-momentum matrix element:
  - Logarithm divergence

Ji, PRL120(2018), NPB964(2021), PLB257(1991); Zhang, PRD95(2017), NPB939(2019); Ishikawa, PRD96(2017);
Green, PRL121(2018); Huo, NPB969(2021); Chen, NPB915(2017); Musch, PRD83(2011); ……
Quasi TMDPDF matrix element and renormalization

- Wilson loop

\[ Z_E = \]

- Logarithmic divergences factor

\[
Z_O(1/a, \mu, \Gamma) = \lim_{L \to \infty} \frac{\tilde{h}_\Gamma^0(z, b_\perp, 0, a, L)}{\sqrt{Z_E(2L + z, b_\perp, a)\tilde{h}_\Gamma^{\text{MS}}(z, b_\perp, \mu)}}
\]
Quasi TMDPDF matrix element and $\lambda$ extrapolation

$ b_\perp = 5a$

- A brute force truncation at large $\lambda$ will lead to **strong oscillation** after FT ⇒ need additional extrapolation....
Quasi TMDPDF matrix element and $\lambda$ extrapolation

$ b_\perp = 5a$

- A brute-force truncation at large $\lambda$ will lead to strong oscillation after FT $\Rightarrow$ need additional extrapolation,….

$$\tilde{h}_{\text{extra}}(\lambda) = \left[ \frac{c_1}{(-i\lambda)^{n_1}} + e^{i\lambda} \frac{c_2}{(i\lambda)^{n_2}} \right] e^{-\lambda/\lambda_0}$$

- end point power-law behavior $x^a(1-x)^b$;
- correlation function has a finite correlation length $\lambda_0$. 

Before extrapolation

After extrapolation
From Quasi TMDPDF to TMDPDF

$P^z = 1.72\text{GeV}$

$\mu = \sqrt{\bar{\tau}} = 2\text{GeV}$

$b = 0.36\text{fm}$
Chiral and large-$P^z$ joint extrapolation:

\[ d_0(m_{\pi}^2 - m_{\pi,\text{phy}}^2) + \frac{d_1}{(P^z)^2} \]
Error estimation

All errors:

- Statistical error;
- From difference of $\gamma^t$ and $\gamma^z$
- From physical extrapolation
- From $\lambda$-extrapolation
- From soft function
- From Collins-Soper kernel
Final results and discussion

\[ b_\perp = 0.12 \text{fm} = (1.64 \text{GeV})^{-1} \]

\[ b_\perp = 0.24 \text{fm} = (0.82 \text{GeV})^{-1} \]

\[ b_\perp = 0.36 \text{fm} = (0.55 \text{GeV})^{-1} \]

\[ b_\perp = 0.48 \text{fm} = (0.41 \text{GeV})^{-1} \]

\[ b_\perp = 0.6 \text{fm} = (0.33 \text{GeV})^{-1} \]
Final results and discussion

Compare the $b_\perp$-dependence of lattice and phenomenological results:
Summary and Outlook

We present the lattice QCD calculation of TMDPDF at first attempt:

- The state-of-the-art techniques in renormalization and extrapolation on the lattice;
- The latest perturbative kernel up to 2-loop with RG evolution;
- Physical extrapolation include chiral-continuum and infinity momentum;
- Comparable results with phenomenological global fits.
Summary and Outlook

While there is still much room for further improvement:

🤔 Better control of uncertainties;

🤔 Continuum extrapolation: more lattice spacings;

🤔 Larger $b_\perp$ (up to nucleon radius?) to obtain a converge distribution in coordinate space;

🤔 Theoretical improvements:

Power correction (small-$x$ region), higher twist effects (operator mixing), ……

Thank you for your attention!
Backup slides
Dispersion relation

\[ E = \sqrt{m^2 + c_1 (P^z)^2 + c_2 (P^z)^4 a^2} \]

\[ c_1 = 1.014(95), \quad c_2 = -0.014(17) \]

\[ c_1 = 1.066(80), \quad c_2 = -0.015(14) \]
Details of correlated joint fits

Fit quality:

• Utilizing bootstrap resampling to establish correlations among all datasets;
• Employing fully-correlated Bayesian constrained fits to extract ground-state matrix elements.
• Stability of the joint fits: $t_{\text{min}}$ dependence of the fit result, which fit range is $[t_{\text{min}}, t_{\text{max}}]$. 

![Graphs showing stability of joint fits](image-url)
$L$-dependence

Saturation length of Wilson link:

$\tilde{K}_{\gamma}^{re} (z, b_1 = 1a)$

$\tilde{K}_{\gamma}^{re} (z, b_1 = 3a)$

$\tilde{K}_{\gamma}^{im} (z, b_1 = 5a)$
\( \lambda \)-extrapolation

Factorization of \( z \) and \( b_\perp \)?

\[
\tilde{h}_{\text{extra}}(\lambda) = \left[ \frac{c_1}{(-i\lambda)^{n_1}} + e^{i\lambda} \frac{c_2}{(i\lambda)^{n_2}} \right] e^{-\lambda/\lambda_0}
\]

(c) \( m_\pi = 220 \text{MeV}, P^z = 2.15 \text{GeV} \)

\( b_\perp = 3a \)

The power-law behavior and correlation length for each \( b_\perp \) should be similar,

but the joint fit will give a strict limit for large-\( b_\perp \) cases:

<table>
<thead>
<tr>
<th>( b_\perp (a) )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>Joint</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n_1 )</td>
<td>0.909(39)</td>
<td>0.943(61)</td>
<td>0.89(10)</td>
<td>0.801(78)</td>
<td>0.84(16)</td>
<td>0.887(28)</td>
</tr>
<tr>
<td>( n_2 )</td>
<td>1.31(34)</td>
<td>2.37(68)</td>
<td>1.71(31)</td>
<td>1.55(38)</td>
<td>1.22(44)</td>
<td>1.65(12)</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>2.63(38)</td>
<td>3.20(80)</td>
<td>2.42(85)</td>
<td>4.3(1.6)</td>
<td>4.4(2.8)</td>
<td>2.53(28)</td>
</tr>
<tr>
<td>( \chi^2/\text{d.o.f.} )</td>
<td>1.0</td>
<td>1.1</td>
<td>1.3</td>
<td>0.75</td>
<td>0.57</td>
<td>1.2</td>
</tr>
</tbody>
</table>
$\lambda$-extrapolation

Systematic uncertainty from fit region [$\lambda_L: \lambda_{\text{max}}$]

$$\tilde{h}_{\text{extra}}(\lambda) = \left[ \frac{c_1}{(-i\lambda)^{n_1}} + e^{i\lambda} \frac{c_2}{(i\lambda)^{n_2}} \right] e^{-\lambda/\lambda_0}$$

(a) $m_\pi = 220\text{MeV}, P^z = 2.15\text{GeV}$  
   $b_L = 5a$

(b) $m_\pi = 310\text{MeV}, P^z = 1.72\text{GeV}$  
   $b_L = 5a$

(c) $m_\pi = 220\text{MeV}, P^z = 2.15\text{GeV}$  
   $b_L = 3a$

$\lambda_L = 8a$

$\lambda_L = 6a$
Perturbative matching kernel and RG resummation

• Fixed-order perturbative results up to the 2-loop level:
  
  \[ h^{(1)} \left( \frac{\zeta_z}{\mu^2} \right) = \frac{\alpha_s C_F}{2\pi} \left( -2 + \frac{\pi^2}{12} + \ln \frac{\zeta_z}{\mu^2} - \frac{1}{2} \ln^2 \frac{\zeta_z}{\mu^2} \right), \]
  
  \[ h^{(2)} \left( \frac{\zeta_z}{\mu^2} \right) = \alpha_s^2 \left[ c_2 - \frac{1}{2} \left( \gamma_C^{(2)} - \beta_0 c_1 \right) \ln \frac{\zeta_z}{\mu^2} - \frac{1}{4} \left( \Gamma_{\text{cusp}}^{(2)} - \frac{\beta_0 C_F}{2\pi} \right) \ln^2 \frac{\zeta_z}{\mu^2} - \frac{\beta_0 C_F}{24\pi} \ln^3 \frac{\zeta_z}{\mu^2} \right], \]

• RG equation of the matching kernel:
  
  \[ \mu^2 \frac{d}{d\mu^2} \ln H \left( \frac{\zeta_z}{\mu^2} \right) = \frac{1}{2} \Gamma_{\text{cusp}} \ln \frac{\zeta_z}{\mu^2} + \frac{\gamma_C (\alpha_s)}{2}, \]

and its solution:

\[ H \left( \frac{\zeta_z}{\mu^2} \right) = H \left( \frac{\zeta_z}{\mu_0^2} \right) \exp \left[ \int_{\mu_0}^{\mu} \frac{d\mu}{\mu} \left( \Gamma_{\text{cusp}}^{(1)} \ln \frac{\zeta_z}{\mu^2} \alpha_s(\mu) + \gamma_C^{(1)} \alpha_s(\mu) + \Gamma_{\text{cusp}}^{(2)} \ln \frac{\zeta_z}{\mu^2} \alpha_s^2(\mu) + \gamma_C^{(2)} \alpha_s^2(\mu) + \Gamma_{\text{cusp}}^{(3)} \ln \frac{\zeta_z}{\mu^2} \alpha_s^3(\mu) + \gamma_C^{(3)} \alpha_s^3(\mu) + \Gamma_{\text{cusp}}^{(4)} \ln \frac{\zeta_z}{\mu^2} \alpha_s^4(\mu) \right) \right]. \]
Chiral and large-$P^z$ joint extrapolation:

\[ d_0 \left( m_\pi^2 - m_{\pi,\text{phy}}^2 \right) + \frac{d_1}{(P^z)^2} \]

Systematic from chiral extrapolation (strategy I):

\[ d_0 \left( m_\pi^2 - m_{\pi,\text{phy}}^2 \right)^2 + \frac{d_1}{(P^z)^2} \]

from large-$P^z$ extrapolation (strategy II):

\[ d_0 \left( m_\pi^2 - m_{\pi,\text{phy}}^2 \right) + \frac{d_1}{(P^z)^2} + \frac{d_2}{P^z} \]
Power correction

After Lorentz boost:

\[
\begin{align*}
\bar{\psi}(z) \gamma^t \psi(0) &= \frac{1}{2} \bar{\psi}(z) \gamma^+ \psi(0) + \frac{1}{2} \bar{\psi}(z) \gamma^- \psi(0) \\
\bar{\psi}(z) \gamma^z \psi(0) &= \frac{1}{2} \bar{\psi}(z) \gamma^+ \psi(0) - \frac{1}{2} \bar{\psi}(z) \gamma^- \psi(0)
\end{align*}
\]

- Ratios denote the deviations from light-like correlator with specific \( P^z \);
- Ratio becomes smaller with \( P^z \) increasing.
Final results and discussion

🤔 The unpolarized TMDPDFs seem not converge in $b_\perp$-space?

Of course not! Perhaps there will be abrupt change at the edge of nucleon

⇒ Need larger $b_\perp$ and more statistics!

🤔 Lattice discretization and finite-volume systematics are still absent in this preliminary work…

• It is a challenging work for calculating the TMDPDF at small lattice spacing

• From the previous experience of PDF (Lin, 2011.14971), we can roughly estimate that:

  Finite-volume effect is less than 1%;
  Discretization effects overall within 2 standard deviations.