The role of convergence methods as fitting functions in the context of the MUonE experiment

Camilo Rojas

Work in collaboration with: Cristiane Yumi London, Diogo Boito and Pere Masjuan

Based on arXiv:2405.13638
Motivation

- Huge discrepancy in the anomalous magnetic moment of the muon

$$a_\mu = \frac{(g_\mu - 2)}{2}$$

$$\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{th}} = 251(59) \times 10^{-11}$$
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\[ \Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{th}} = 249(50) \times 10^{-11} \; ; \; [5.0\sigma] \]

Muon g-2 Theory initiative | White Paper (arXiv:2006.04822v2)
The Muon g-2 coll. (10.1103/PhysRevLett.131.161802)

(Anna Driutti’s talk on Friday)
Motivation

• Huge discrepancy in the anomalous magnetic moment of the muon \( a_\mu = (g_\mu - 2)/2 \)

\[
\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{th}} = 251(59) \times 10^{-11}
\]

• MUonE experiment is a good candidate to clarify this discrepancy (space-like channel)

\[
\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{th}} = 249(50) \times 10^{-11} \quad ; \quad [5.0\sigma]
\]

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• The hadronic contribution dominates the current state of uncertainties.

\[
a_\mu^{\text{SM}} = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{HVP}} + a_\mu^{\text{HLbL}}
\]

\[a_\mu^{\text{HVP}} = 6845(40) \times 10^{-11}\]
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The Master Formula

\[ a_\mu^{\text{HVP,LO}} = \frac{\alpha^2}{\pi} \int_0^1 dx \ (1 - x) \Delta \alpha_{\text{had}}[t(x)] \]

- Requires a measurement of \( \Delta \alpha_{\text{had}} \) in the space-like region \( t = q^2 < 0 \)
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**Problem**

Finding a reliable method to fit the data + good extrapolation outside data region without using external information

---

We propose a technique based on Padé and D-Log Padé Approximants

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• Padé and DLog Approximants
• Fitting method & simulations
• Results with realistic errors
• Uncertainties Vs Extrapolation
• Outlook & Conclusions
Padé and D-Log Approximants
Padé Approximants

\[ P^N_M(z) = \frac{Q_N(z)}{R_M(z)} = \frac{q_0 + q_1 z + \cdots + q_N z^N}{1 + r_1 z + \cdots + r_M z^M} \]

**Advantages**

- Systematic and model-independent method
- Partial reconstruction of analytic (physical) properties
- Efficient approximation
- It is possible to provide a systematic error using the **convergence** properties

The standard method for constructing Padé Approximants (PAs)

\[ f(0) = P(0), \]
\[ f'(0) = P'(0), \]
\[ f''(0) = P''(0), \]
\[ \vdots \]
\[ f^{(m+n)}(0) = P^{(m+n)}(0). \]

- M+N+1 equations relate: PA coefficients \((q_i, r_i)\) with original function coefficients \((a_i)\).
- PAs return a better approximation for the M+N+1 term
Padé Approximants

Stieltjes function

$ f(z) = \int_0^\infty \frac{d\phi(u)}{1 + zu} \quad \phi(u) \text{ is a measure in } u \in [0, \infty) $

- $ \Delta\alpha_{\text{had}}(t) $ is a Stieltjes function!  
  [Masjuan, Peris ’09], [Aubin, Blum, Golterman, Peris ’12]

- There are convergence theorems for PAs to Stieltjes functions  
  [Baker ’96]

- Some convergence properties (thanks to analyticity and unitarity):
  * poles of $ P_N^{N+k}, \ k \geq -1, $ are located in the positive real axis;
  * PA sequences uniformly converge to the original function;
  * PA sequences act as bounds to the function

We can use the convergence theorem to bound our function

$ P_1^1(t) \leq P_2^2(t) \leq \cdots \leq \Delta\alpha_{\text{had}} \leq \cdots \leq P_1^2(t) \leq P_0^1(t) $
D-Log Padé Approximants

**Advantages respect to Padés**

- It is useful to reproduce not only poles but also cuts or branch points of the original function
- Faster convergence
- Also model independent to find the singularity position value and its multiplicity

Is not longer a rational approximant

Consider the following function:

\[ f(z) = A(z) \frac{1}{(\mu - z)^\gamma} + B(z) \quad \gamma \in \mathbb{R} \]

Set a new function applying the logarithm derivative

\[ F(z) = \frac{d}{dz} \ln f(z) \approx \frac{\gamma}{(\mu - z)} \]

- Approach \( F(z) \) with a Padé sequence: \( \tilde{P}_N^M [F(z)] \)
- Unfold the process: Integrate, exponentiate and normalize

\[ D_N^M(z) = f(0) \exp \left[ \int dz \tilde{P}_N^M(z) \right] \quad \text{d.o.f: M+N+2} \]

◆ Little is know about D-Logs, they work and we have a convergence conjecture. We find:

\[ D_1^1(t) \leq D_2^2(t) \leq \ldots \leq \Delta \alpha_{\text{had}} \leq \ldots \leq D_3^2(t) \leq D_2^1(t) \]
Fitting method & Simulations
From Taylor expansion

**Padé Approximants**

\[
\frac{\alpha (1 - z) \Delta\alpha_{\text{had}}(z)}{\pi} = P_N^1 + P_N^{N+1} + D_N^N
\]

Taylor expansion from the model @ Greynat and de Rafael '22
(arXiv:2202.10810 [hep-ph])

**D-Log Padé Approximants**
Fitting Method

- Data generation — simple model for $\Delta \alpha_{\text{had}}(t)$ as a Stieltjes function [Greynat, de Rafael (2022)]

\[
(x, \alpha \Delta \alpha_{\text{had}}(x) \times 10^5) \quad \text{with} \quad 0.2 \leq x \leq 0.93
\]

- Fitting parameters: starting with unknown Taylor series coefficients of $\Delta \alpha_{\text{had}}(t)$

\[
\Delta \alpha_{\text{had}}(t) = a_1 t + a_2 t^2 + a_3 t^3 + \cdots
\]

Example PA:

\[
P_1^1(t) = \frac{a_1 t}{1 - \frac{a_2}{a_1} t}
\]

Making change of variable $t \rightarrow x$

\[
t = -\frac{m_\mu^2 x^2}{1 - x}
\]

Then

\[
P_1^1(x) = -\frac{b_1 m_\mu^2 x^2}{1 - x + b_2 m_\mu^2 x^2}
\]

- PA parameters: $\chi^2$ minimization
- No cancelations between zeros and poles

30 equally spaced bins — data point are mid-value of each bin

- Model-independent constraints

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Fitting Method

- Data generation — simple model for $\Delta \alpha_{\text{had}}(t)$ as a Stieltjes function
  
  \[
  (x, \alpha \Delta \alpha_{\text{had}}(x) \times 10^5) \\
  0.2 \leq x \leq 0.93
  \]

- Fitting parameters: starting with unknown Taylor series coefficients of $\Delta \alpha_{\text{had}}(t)$

  \[
  \Delta \alpha_{\text{had}}(t) = a_1 t + a_2 t^2 + a_3 t^3 + \cdots
  \]

  Example DLog:

  \[
  D^1_2(t) = \frac{-f_0 t}{(r_1 - t)^{\gamma_1}} \rightarrow D^1_2(x) = \frac{f_0 m^2 \mu x^2 (1 - x)^{-1 + \gamma_1}}{(r_1 - r_1 x + m^2 \mu x^2)^{\gamma_1}}.
  \]

  \[
  D^2_2(t) = \frac{-f_0 t e^{\beta t}}{(r_1 - t)^{\gamma_1}} \rightarrow D^2_2(x) = \frac{f_0 m^2 \mu x^2 (1 - x)^{-1 + \gamma_1}}{(r_1 - r_1 x + m^2 \mu x^2)^{\gamma_1}} e^{\beta \frac{m^2 \mu x^2}{(x-1)}}.
  \]

- DLog parameters: $\chi^2$ minimization

  Model-independent constraints for $\beta, r_1, \gamma_1, \ldots$
Results with realistic errors
Realistic Data

- 1000 toy data sets
- \((x, \alpha \Delta \alpha_{\text{had}}(x) \times 10^5)\) — 30 data points equally spaced in \(0.2 \leq x \leq 0.93\)
- Central value randomly chosen from a gaussian distribution with expected error of
  MUnE experiment

  private communication with Abbiendi, Carloni Calame, Venanzoni

- Analysis of the fits for each Padé and DLog
- \(\chi^2\) penalties (\(\theta\) functions) if coefficients do not follow the expected hierarchy

- Value of \(a_{\mu}^{\text{HVP,LO}}\) calculated for each data set (extrapolation was done in the whole region, even in the data region)

Toy model result: 

\[
a_{\mu}^{\text{HVP,LO}} = \left(6991^{+22}_{-20}\right) \times 10^{-11}
\]
Realistic Data

- Good fit qualities

- Statistical and theoretical error of the same order but statistical is higher

\[
\Delta \alpha_{\text{QED-model}}(t) = KM \left[ -\frac{5}{9} - \frac{4M}{3t} + \frac{2}{9} \left( \frac{4M^2}{3t^2} + \frac{M}{3t} - \frac{1}{6} \right) \log \left| \frac{1 - \sqrt{1 - \frac{4M}{t}}}{1 + \sqrt{1 - \frac{4M}{t}}} \right| \right]
\]

Inner error bar — statistical error

Exterior error bar — statistical and systematic errors added in quadrature
Realistic Data

\[ D_1^1(t) \leq D_2^2(t) \leq \ldots \leq \Delta\alpha_{\text{had}} \leq \ldots \leq D_3^3(t) \leq D_2^1(t) \]

- Good fit qualities
- Statistical and theoretical error of the same order but statistical is higher

\[
\Delta\alpha_{\text{QED-model}}(t) = KM \left[ -\frac{5}{9} - \frac{4M}{3t} + \frac{2(4M^2 + \frac{M}{3t} - \frac{1}{6})}{\sqrt{1 - \frac{4M}{t}}} \log \left| \frac{1 - \sqrt{1 - \frac{4M}{t}}}{1 + \sqrt{1 - \frac{4M}{t}}} \right| \right]
\]

Inner error bar — statistical error
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## Realistic Data

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<th>( a_\mu^{\text{HVP,LO}} = \left( 6991^{+22}_{-20} \right) \times 10^{-11} )</th>
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| \( P_1^1 \)   | \( 6938 \pm 21 \) | \( 1.01^{+0.27}_{-0.25} \) |
| \( P_1^2 \)   | \( 7042^{+114}_{-104} \) | \( 1.01^{+0.28}_{-0.26} \) |
| \( P_2^2 \)   | \( 6980^{+46}_{-34} \) | \( 1.05^{+0.29}_{-0.27} \) |
| \( P_3^3 \)   | \( 6994^{+85}_{-49} \) | \( 1.11^{+0.29}_{-0.31} \) |
| **Final result** | \( 6987^{+46}_{-34} \) | — |

| \( D_2^1 \)   | \( 7052^{+66}_{-71} \) | \( 1.01^{+0.26}_{-0.26} \) |
| \( D_2^2 \)   | \( 6956^{+96}_{-65} \) | \( 1.05^{+0.28}_{-0.27} \) |
| \( D_3^2 \)   | \( 6999^{+48}_{-39} \) | \( 1.10^{+0.29}_{-0.28} \) |
| \( D_3^3 \)   | \( 6977^{+72}_{-53} \) | \( 1.14^{+0.30}_{-0.29} \) |
| **Final result** | \( 6988^{+48}_{-39} \) | — |

*best result we can expect from PAs and DLogs predictions*
Uncertainties Vs Extrapolation
Realistic Data: extrapolation up to $x_{\text{max}}$

Fits up to $x \sim 0.93$ and extrapolated up to $x_{\text{max}}$: 

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First error - statistical error

Second error - systematic (extrapolation) error

$$\Delta \alpha_{\text{QED-model}}(t) = KM \left[ \frac{5}{9} - \frac{4M}{3t} + 2 \left( \frac{4M^2}{3t^2} + \frac{M}{3t} - \frac{1}{6} \right) \log \left| \frac{1 - \sqrt{1 - \frac{4M}{t}}}{1 + \sqrt{1 - \frac{4M}{t}}} \right| \right]$$
Fits up to $x \sim 0.93$ and extrapolated up to $x_{\text{max}}$:

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Conclusions
Conclusions

- Discrepancy for $a_\mu$ between experimental measurements and predictions reached the 5.0\(\sigma\) level

- D-Logs and Padé approximant sequences are a model-independent method to fit and extrapolate the data from the MUonE experiment. Approximants up to 6 coefficients seem ok!

- The method uses fundamental knowledge about the analytic structure of $\Delta\alpha_{\text{had}}(t)$. It is a Stieltjes function (we use analyticity and unitarity, yet inclusive) ⇒ beyond the unitary cut!

⚠️ PAs and DLogs can provide a lower and upper bound for the true value!

- Uncertainties may be reduced if:
  - Knowledge about the structure of $\Delta\alpha_{\text{had}}(t)$ is included (or either extracted from fit)
  - Extrapolate to certain $x_{\text{max}}$ (corresponding to “large enough energy”) and then match to pQCD or $e^+e^-$ (with DLogs we can access the time-like: $\pi^+\pi^-$ or $\pi^0\gamma$ production thresholds, $\rho$-meson!)
Thanks
Backup slides
Muon Anomalous magnetic moment
The anomalous magnetic moment of the muon

- The magnetic moment (for charged leptons)

\[ \mu_l = g_l \frac{e}{2m_l} \cdot \vec{S} \]

- The magnetic anomaly (deviation from Dirac value)

\[ a_l = (g_l - 2)/2 \]

This observable can be both precisely measured experimentally and predicted in the Standard Model, providing a stringent test of the SM.

- The first order correction (by J. Schwinger)

\[ a_{e^{QED,LO}} = \alpha/2\pi \approx 1.16 \times 10^{-3} \]

\( \alpha \) is the fine structure constant.
The anomalous magnetic moment of the muon

- E821 @BNL measurement with an error of 0.54 ppm
  
  \[ a_{\mu}^{\text{exp}} = 116592089(63) \times 10^{-11} \]

  Merge with FNAL measurement

  \[ a_{\mu}^{\text{exp}} = 116592089(63) \times 10^{-11} \] [0.35 ppm]

- The theoretical calculation with SM (approved consensus)
  
  \[ a_{\mu}^{\text{SM}} = 116591810(43) \times 10^{-11} \]

  By the Muon g − 2 Theory Initiative


Current discrepancy limited by:
- Experimental uncertainty: New run test at FNAL and J-PARC x4 accuracy
- In the theoretical frame: How to calibrate Hadronic Contributions uncertainties.

\[ \Delta a_{\mu} = a_{\mu}^{\text{exp}} - a_{\mu}^{\text{th}} = 251(59) \times 10^{-11} \]
The anomalous magnetic moment of the muon

\[ a_{\mu}^{\text{SM}} = a_{\mu}^{\text{QED}} + a_{\mu}^{\text{EW}} + a_{\mu}^{\text{HVP}} + a_{\mu}^{\text{HLbL}} \]

- QED corrections known up to 5 loops with related precision \( \sim 7 \times 10^{-10} \)
  \[ a_{\mu}^{\text{QED}} = 116584718.931(104) \times 10^{-11} \]
- EW corrections up to 2 loops with precision \( \sim 10^{-9} \) rel. uncertainty < 1%
  \[ a_{\mu}^{\text{EW}} = 153.6(1.0) \times 10^{-11} \]
- Hadronic contribution \( \sim 7 \times 10^{-8} \); the dominant theoretical uncertainty
  \[ a_{\mu}^{\text{HVP}} = 6845(40) \times 10^{-11} \]

\&gt; not calculable by pQCD \&lt;\n
Data-Driven approach
(Timelike domain)

Lattice QCD

Experiment MUonE
(Spacelike domain)

Main contribution: LO Vacuum Polarization estimated
rel. uncertainty 0.35% - 0.6%
\[ a_{\mu}^{\text{HVP}} = a_{\mu}^{\text{LO}, \text{HVP}} + a_{\mu}^{\text{NLO}, \text{HVP}} + a_{\mu}^{\text{NNLO}, \text{HVP}} \]
How to calculate the HVP contribution?

Using the dispersive relation integral:

\[ a_{\mu}^{\text{HVP, LO}} = \alpha \frac{2}{\pi^2} \int_0^\infty \frac{ds}{s} K(s) \frac{\sigma_{e^+e^- \to \text{hadrons}}(s) \text{Im} \Pi_{\text{had}}(s + i\epsilon)}{s^2} \]

Kernel has the lepton information!

- Using the optical theorem
- Involving the total hadronic cross section measured experimentally at e^+e^- machines

Data-Driven

Alternative representation usually use in:

In terms of the Euclidean space

\[ \tilde{\Pi}(Q^2) \quad Q^2 = -q^2 < 0 \]

This representation would be useful for MUonE experiment since it is in the space-like domain

Tension between **Data-driven** Vs **Lattice QCD**
MUonE Experiment
MUonE Experiment ($\mu e \rightarrow \mu e$) @CERN

It is a new experimental proposal @ CERN:

- Scattering $\mu$'s on e's in a low Z target looks like an ideal process (fixed target experiment)
- It is a pure t-channel process at tree level
- The M2 muon beamline ($E_\mu \simeq 150$ GeV) is available at CERN
- Useful C.M. energy to test dominant region of $a_{\mu}^{LO,HVP} \sqrt{s} \simeq 0.4$ GeV $\rightarrow q^2 < 0.11$ GeV$^2$
- With $\sim 3$ years of data taking, a statistical accuracy of 0.35% on $a_{\mu}^{LO,HVP}$ can be achieved
- Easy selection based on the correlation of the electron and muon scattering angles.

\[ \frac{d\sigma}{dt} \simeq \frac{d\sigma_0}{dt} \left| \frac{a(t)}{a_0} \right|^2 \]

\[ \Delta \alpha(t) = \Delta \alpha_{lep}(t) + \Delta \alpha_{had}(t) \]

- $\theta_e < 32$ mrad ($E_e > 1$ GeV)
- $\theta_\mu < 5$ mrad
- Known from QED
- To be measured

Since it is related with HVP contribution of $a_\mu$
Scattering $\mu$'s on e's in a low Z target looks like an ideal process (fixed target experiment)

- It is a pure t-channel process at tree level
- The M2 muon beam is available at CERN
- Useful cms energy to test dominant region of
- Easy selection based on the correlation of the electron and muon scattering angles.

With $\sim 3$ years of data taking, a statistical accuracy of 0.35% on $\Delta \alpha$ can be achieved

$$E_\mu \approx 150 \text{ GeV}$$

Running of $\alpha$

Known from QED

To be measured

$\theta_e < 32 \text{ mrad } (E_e > 1 \text{ GeV})$

$\theta_\mu < 5 \text{ mrad}$


**MUonE experiment - measurement in the space-like momentum**

\[ \Delta \alpha(q^2) = - \text{Re} \bar{\Pi}(q^2) \]
\[ \text{Im} \bar{\Pi}(q^2 < 0) = 0 \]

\[
d_{\mu}^{\text{HVP, LO}} = \frac{\alpha^2}{\pi} \int_0^1 dx \ (1 - x) \Delta \alpha_{\text{had}}[t(x)]
\]

\[
t(x) = - \frac{x^2 m_\mu^2}{1 - x} \quad 0 \leq -t < \infty
\]
\[	x(t) = \frac{t}{2 m_\mu^2} \left( 1 - \sqrt{1 - \frac{4 m_\mu^2}{t}} \right) \quad 0 \leq x < 1
\]

\[ t < -0.143 \text{ GeV}^2 \quad <. \quad 0.2 < x < 0.93 \]

- \( a^{\text{LO, HVP}}_{\mu} \) is given by the integral of the curve (smooth behaviour)
- Requires a measurement of \( \Delta \alpha_{\text{had}} \) in the space-like region \( t = q^2 < 0 \)
- It enhances the contribution from low \( q^2 \) region (below 0.11 GeV^2)
- They expect to cover 87% of the \( a^{\text{LO, HVP}}_{\mu} \) with the space-like integral
  - But they need to extrapolate: \( x \rightarrow 1 \), (13% missing)
- Its precision is determined by the uncertainty on \( \Delta \alpha_{\text{had}} \) in this region

\[ x_{\text{peak}} \approx 0.914 \]
\[ t_{\text{peak}} \approx -0.108 \text{ GeV}^2 \]

---

Recovering the Master Formula

using the output of the routine hadr5n12 (which uses time-like hadron production data and perturbative QCD)

Here is our motivation!!

Finding a reliable method to fit the data + good **extrapolation** outside data region

**This fitting method must be:**

- A very precise fitting
- A fast convergence method to the original function
- It must have the same analytical properties as the original function
Stieltjes function

\[ f(z) = \int_0^\infty \frac{d\phi(u)}{1 + zu} \quad \phi(u) \text{ is a measure in } u \in [0, \infty) \]

\[ f(z) = \sum_{i=0}^{\infty} f_i (-z)^i, \quad f_i = \int_0^\infty u^i \, d\phi(u) \]

\[
\begin{vmatrix}
 f_m & f_{m+1} & \cdots & f_{m+n} \\
 f_{m+1} & f_{m+2} & \cdots & f_{m+n+1} \\
 \vdots & \vdots & \ddots & \vdots \\
 f_{m+n} & f_{m+n+1} & \cdots & f_{m+2n}
\end{vmatrix} > 0 \quad m \geq 0, \quad n \geq 0
\]

Aubin, Blum, Golterman, Peris (2012)
Masjuan, Peris (2009)

- \( \Delta \alpha_{\text{had}}(t) \) is a Stieltjes function in \( t \in (-\infty, 0] \) since HVP correlator is a Stieltjes function

\[ \Delta \alpha_{\text{had}}(t) = \sum_{i=1}^{\infty} a_i t^i \quad 0 < a_i < a_{i+1}, \quad i \in \mathbb{N} \]
Model of Greynat and de Rafael

$$\Delta\alpha_{\text{had}}[q^2] = \hat{\Pi}(q^2) = q^2 \int_{4m^2_{\pi}}^{\infty} ds \frac{\text{Im } \Pi(s)}{s(s - q^2 + i\epsilon)}$$

$$\text{Im } \Pi_{\text{had}}(s) = \frac{1}{4\pi} \left( 1 - \frac{4m^2_{\pi}}{s} \right)^{3/2} \left( \frac{|F(s)|^2}{12} + \sum_{i=u,d,...} Q_i^2 \Theta(s, s_c, \Delta) \right) \theta(s - 4m^2_{\pi})$$

model used to generate toy data

$$|F(s)|^2 = \frac{m^4_{\rho}}{(m^2_{\rho} - s)^2 + m^2_{\rho} \Gamma(s)^2}$$

$$\Gamma(s) = \frac{m_{\rho} s}{96\pi f^2_{\pi}} \left[ \left( 1 - \frac{4m^2_{\pi}}{s} \right)^{3/2} \theta(s - 4m^2_{\pi}) + \frac{1}{2} \left( 1 - \frac{4m^2_{k}}{s} \right)^{3/2} \theta(s - 4m^2_{k}) \right]$$

$$\Theta(s) = \frac{2}{\pi} \left[ \frac{\arctan \left( \frac{s - s_c}{\Delta} \right) - \arctan \left( \frac{4m^2_{\pi} - s_c}{\Delta} \right)}{\frac{s}{2} - \arctan \left( \frac{4m^2_{\pi} - s_c}{\Delta} \right)} \right]$$

Greynat, de Rafael (2022)
Fitting Method

- Fitting function from PAs — example
  \[
  \Delta \alpha_{\text{had}}(t) = a_1 t + a_2 t^2 + \ldots
  \]
  unknown Taylor series coefficients

  \[
  P_1(t) = \frac{q_0 + q_1 t}{1 + r_1 t} \approx q_0 + (q_1 - q_0 r_1)t + (q_0 r_1^2 - q_1 r_1)t^2 + \ldots
  \]

  \[
  P_1(x) = -\frac{a_2^2 m^2_{\mu} x^2}{a_1 - a_1 x + a_2 m^2_{\mu} x^2} = -\frac{b_1 m^2_{\mu} x^2}{1 - x + b_2 m^2_{\mu} x^2}
  \]

- Modified \( \chi^2 \) function with penalties
  \[
  \chi^2 = \sum_{i,j=1}^{30} \left[ \alpha \Delta \alpha_{\text{had}}(x_i) \times 10^5 - P^M_N(x_i) \right] \left[ C^{-1} \right]_{ij} \left[ \alpha \Delta \alpha_{\text{had}}(x_j) \times 10^5 - P^M_N(x_j) \right] + n_{\text{dof}} \sum_{i=2}^{N+M} \theta (a_i - a_{i-1})
  \]

- Matching coefficients
  \[
  P_1(t) = \frac{a_1^2 t}{1 - a_2 t}
  \]

- Model-independent constraints
  \[
  b_1 = a_1 < 0 \quad b_2 = \frac{a_2}{a_1} > 1
  \]
Time-like $\rightarrow$ Space-like

Graph on left: $R_{had}$ vs. $E$ (GeV)
Graph on right: $\Delta Q_{had}(E)$ vs. $z$
Model parametrisation used by MUonE team

Physics-inspired from the calculable contribution of lepton-pairs and top quarks at $t<0$

$$q^2 = t < 0 \quad \Delta \alpha_{\text{had}}(t) = k \left\{ \frac{5}{9} - \frac{4M}{3t} + \left( \frac{4M^2}{3t^2} + \frac{M}{3t} - \frac{1}{6} \right) \frac{2}{\sqrt{1 - \frac{4M}{t}}} \log \left| \frac{1 - \sqrt{1 - \frac{4M}{t}}}{1 + \sqrt{1 - \frac{4M}{t}}} \right| \right\}$$

$M$ with dimension of mass squared, related to the mass of the fermion in the vacuum polarization loop
$k$ depending on the coupling $\alpha(0)$, the electric charge and the colour charge of the fermion

Ref: Giovanni Abbiendi  arXiv:2201.13177v1 [physics.ins-det]
D-Log Padé Approximants

How is our D-Log type function?

Make a Taylor series from a model in t-variable

\[ f(t) \]

Construct a PA sequence in the D-log space

\[ \tilde{P}_N^M[F(t)] \]

Get the D-log Padé approximants to the original function in t

\[ D_N^M(t) \]

Analysing the structure of approximants and propose a general function to fit

\[ f(t) \cong D_N^M(t) = f(0) \frac{t e^{\sum_{i=1}^{M-N+1} a_i t^i}}{(\mu_1 - t)^{\gamma_1} \cdots (\mu_{N-1} - t)^{\gamma_{N-1}}} \]

Making change of variable

\[ t \to x \]

Fitting function

\[
\begin{align*}
D_M^1(t) & = \frac{-f_0 t}{(r_1-t)^{\gamma_1}} \\
D_M^2(t) & = \frac{-f_0 t e^{\beta t}}{(r_1-t)^{\gamma_1}} \\
D_M^3(t) & = \frac{-f_0 t e^{\beta t}}{(r_1-t)^{\gamma_1}(r_2-t)^{\gamma_2}}
\end{align*}
\]

\[
\begin{align*}
D_M^1(x) & = \frac{f_0 m_\mu^2 x^2 (1-x)^{-1+\gamma_1}}{(r_1-r_1 x + m_\mu^2 x^2)^{\gamma_1}} \\
D_M^2(x) & = \frac{f_0 m_\mu^2 x^2 (1-x)^{-1+\gamma_1} e^{\beta m_\mu^2 x^2}}{(r_1-r_1 x + m_\mu^2 x^2)^{\gamma_1}} \\
D_M^3(x) & = \frac{f_0 m_\mu^2 x^2 (1-x)^{-1+\gamma_1} e^{\beta m_\mu^2 x^2}}{(r_1-r_1 x + m_\mu^2 x^2)^{\gamma_1}(r_2-r_2 x + m_\mu^2 x^2)^{\gamma_2}}
\end{align*}
\]
An example of Padé's approximant (dataset without errors)

First we can consider a data point set without uncertainties just to see the behaviour of the Padé approximant

Example: Padé Approximant (2 dofs)

\[ \tilde{a}_1^{PA} = 6933 \times 10^{-11} \quad (0.9\%) \]

\[ \tilde{a}_2 = -1489 \text{ GeV}^{-4} \quad (15\%) \]

\[ \chi^2_{\text{min}}/\text{dof} = 1.18 \times 10^{-3} \]

\[ p\text{-value} = 1 \]

\( t \rightarrow x \)

\[ P_1(t) = \frac{a_1 t}{1 - a_2 t} \]

\[ \tilde{a}_1^{PA} = 6933 \times 10^{-11} \quad (0.9\%) \]

\[ \tilde{a}_2 = -1489 \text{ GeV}^{-4} \quad (15\%) \]

\[ \chi^2_{\text{min}}/\text{dof} = 1.18 \times 10^{-3} \]

\[ p\text{-value} = 1 \]

\( t \rightarrow x \)

\[ P_1(t) = \frac{a_1 t}{1 - a_2 t} \]

The fitting functions for the two sequences proposed \( P_N \) and \( P_N^{N+1} \) can be obtained in the same way (results in the right plot)

We compute a weighted average for the final value of \( a_\mu^{\text{HLO}} \) (weights = N)

\[ \tilde{a}_\mu^{\text{HLO}} = 6991 \times 10^{-11} \quad (0.02\%) \]

\[ a_\mu^{\text{LO-HVP}} \] computed using the parametrisation used by the MUonE team
Fitting to data with no errors

- We confirmed the pattern of convergence
- Some defect effects at order 6 and 7 in PAs
- The systematic uncertainty that we can achieve at each order

\[ \Delta \alpha_{\text{QED-model}}(t) = KM \left[ -\frac{5}{9} \frac{4M}{3t} + \frac{2}{\sqrt{1 - \frac{4M}{t}}} \left( \frac{4M^2}{3t^2} + \frac{M}{3t} - \frac{1}{6} \right) \log \left( 1 - \frac{\sqrt{1 - \frac{4M}{t}}}{1 + \sqrt{1 - \frac{4M}{t}}} \right) \right] \]

[Camilo Rojas] [arXiv:2201.13177v1]