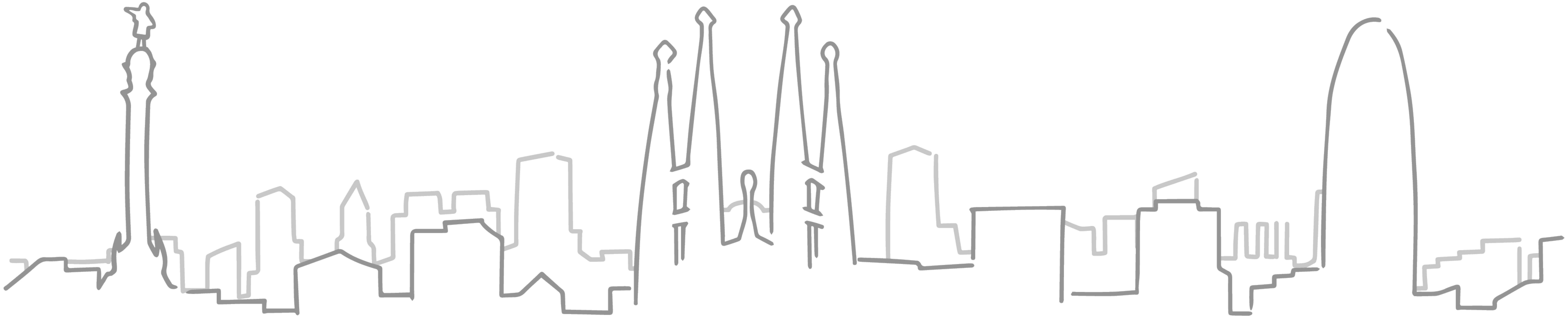
QNP 2024 BARCELONA

The role of convergence methods as fitting functions in the context of the MUonE experiment

Based on arXiv:2405.13638

Camilo Rojas

Work in collaboration with: Cristiane Yumi London, Diogo Boito and Pere Masjuan



July 9, 2024

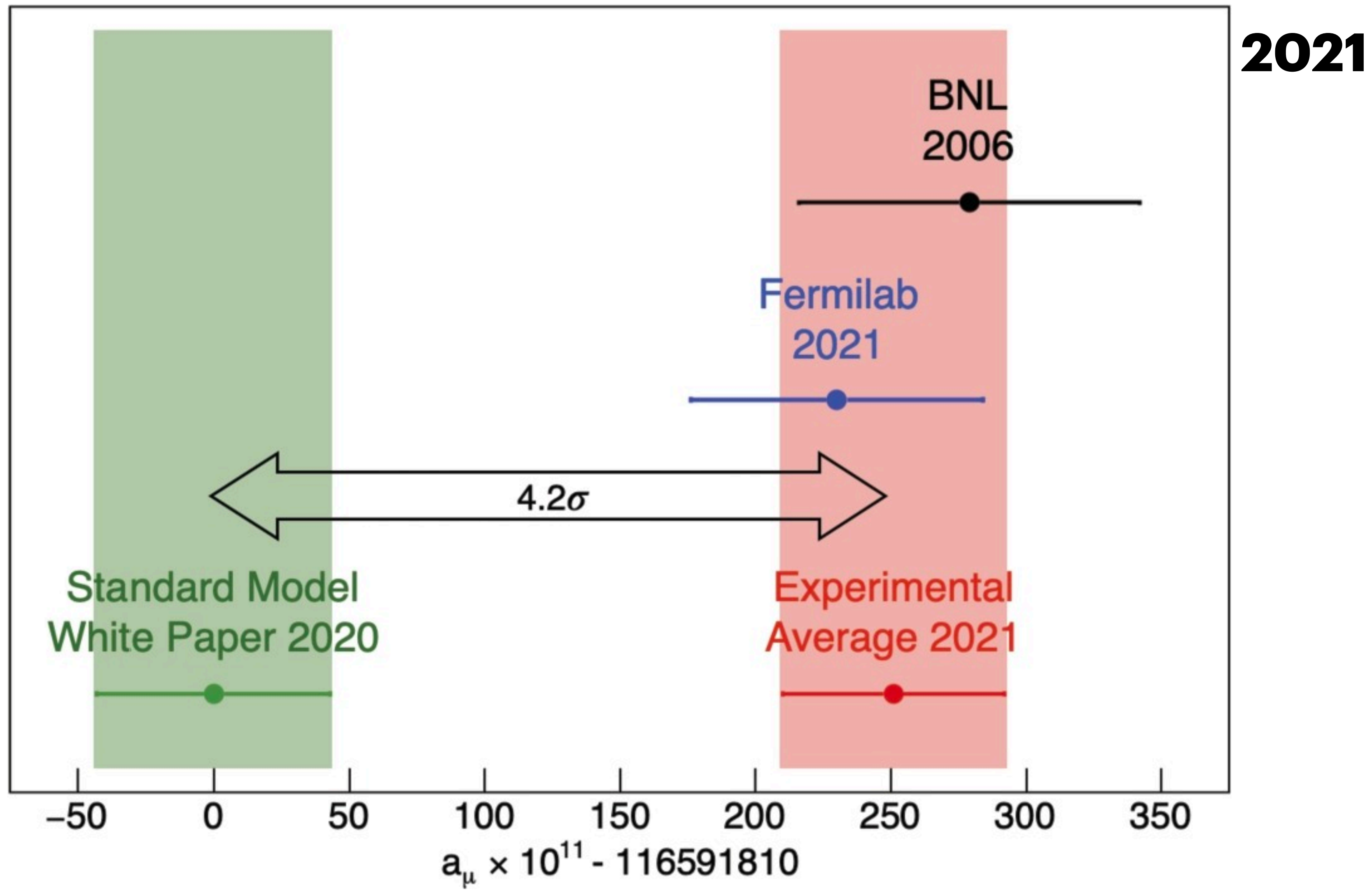
IFAE Institut de Física d'Altes Energies

UAB Universitat Autònoma de Barcelona

IFSC Universidade de São Paulo Instituto de Física de São Carlos

Motivation

- Huge discrepancy in the anomalous magnetic moment of the muon $a_\mu = (g_\mu - 2)/2$



$$\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{th}} = 251(59) \times 10^{-11}$$

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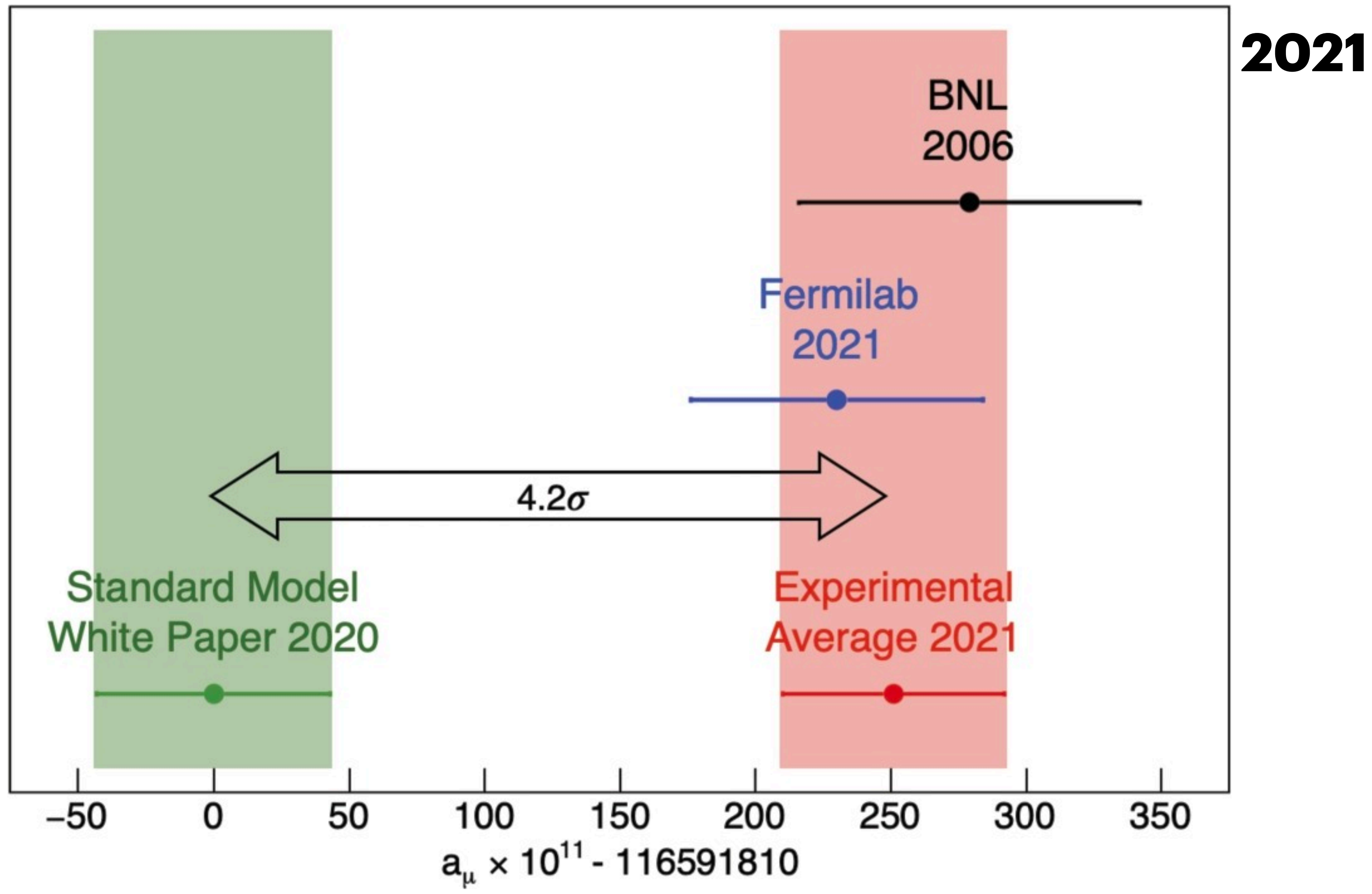

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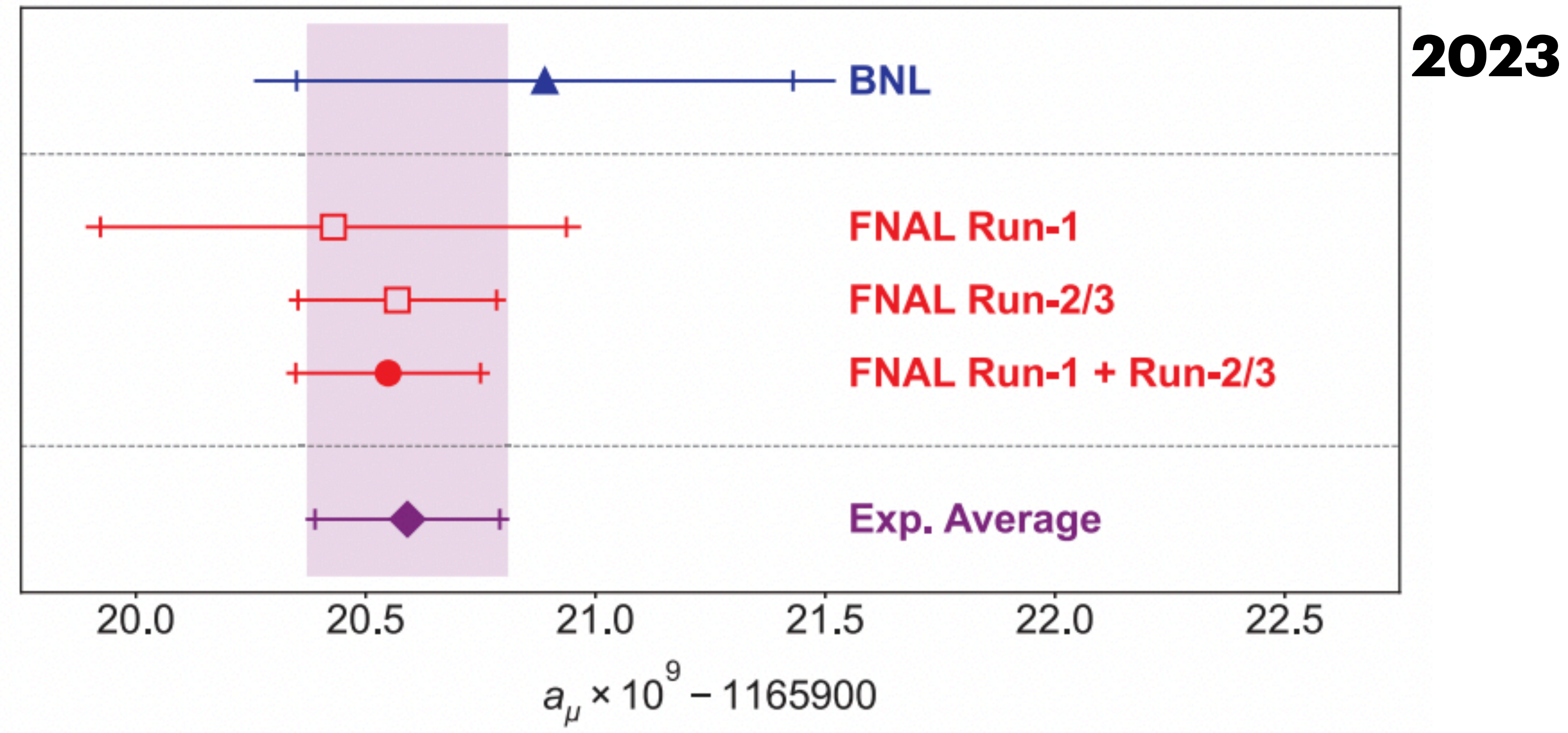

$$\Delta a_\mu = a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{th}} = 249(50) \times 10^{-11} \;\; ; \;\; [5.0\sigma]$$

Muon g-2 Theory iniative / White Paper (arXiv:2006.04822v2)

The Muon g-2 coll. (10.1103/PhysRevLett.131.161802)

(Anna Driutti's talk on Friday)

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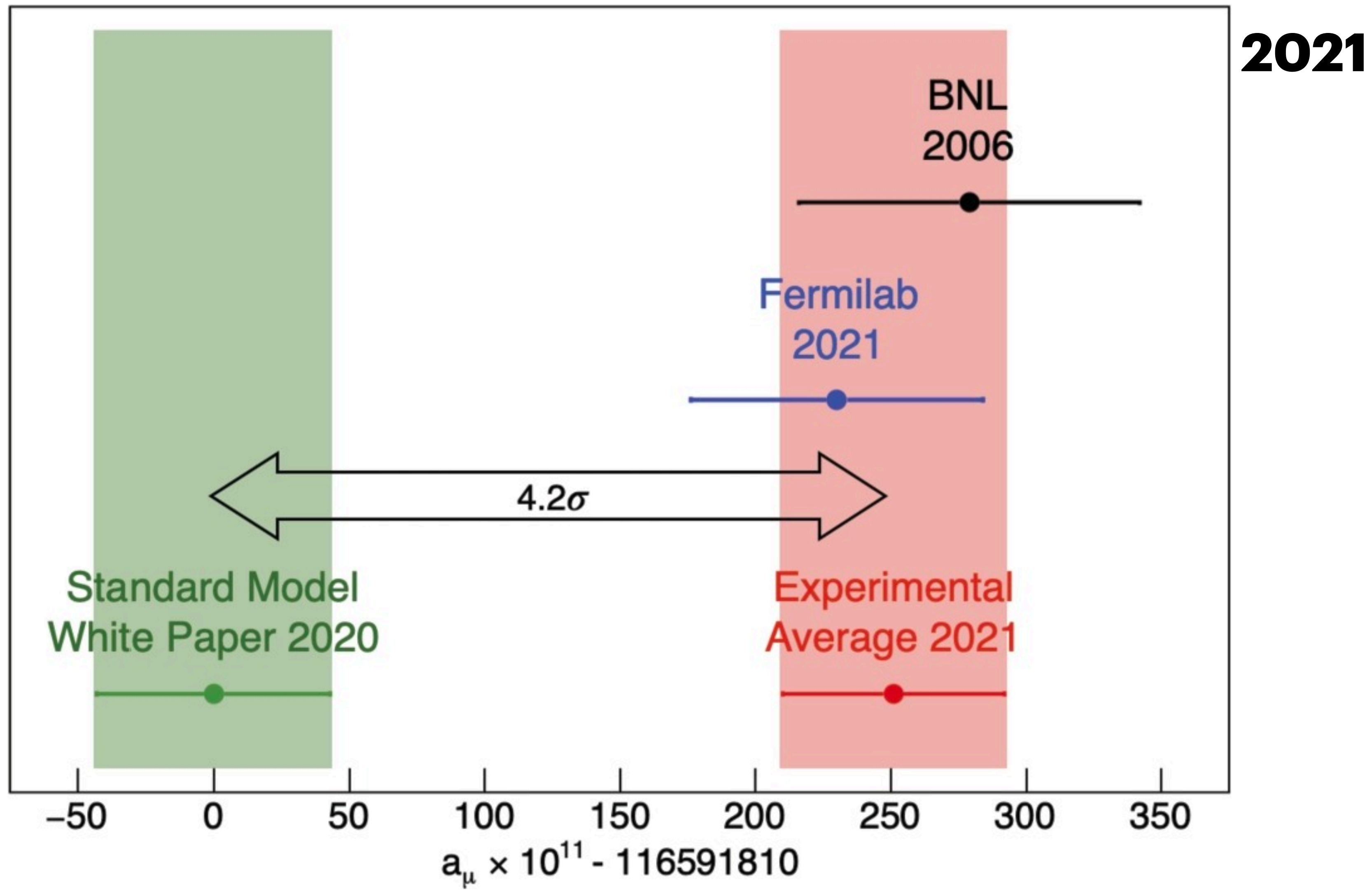

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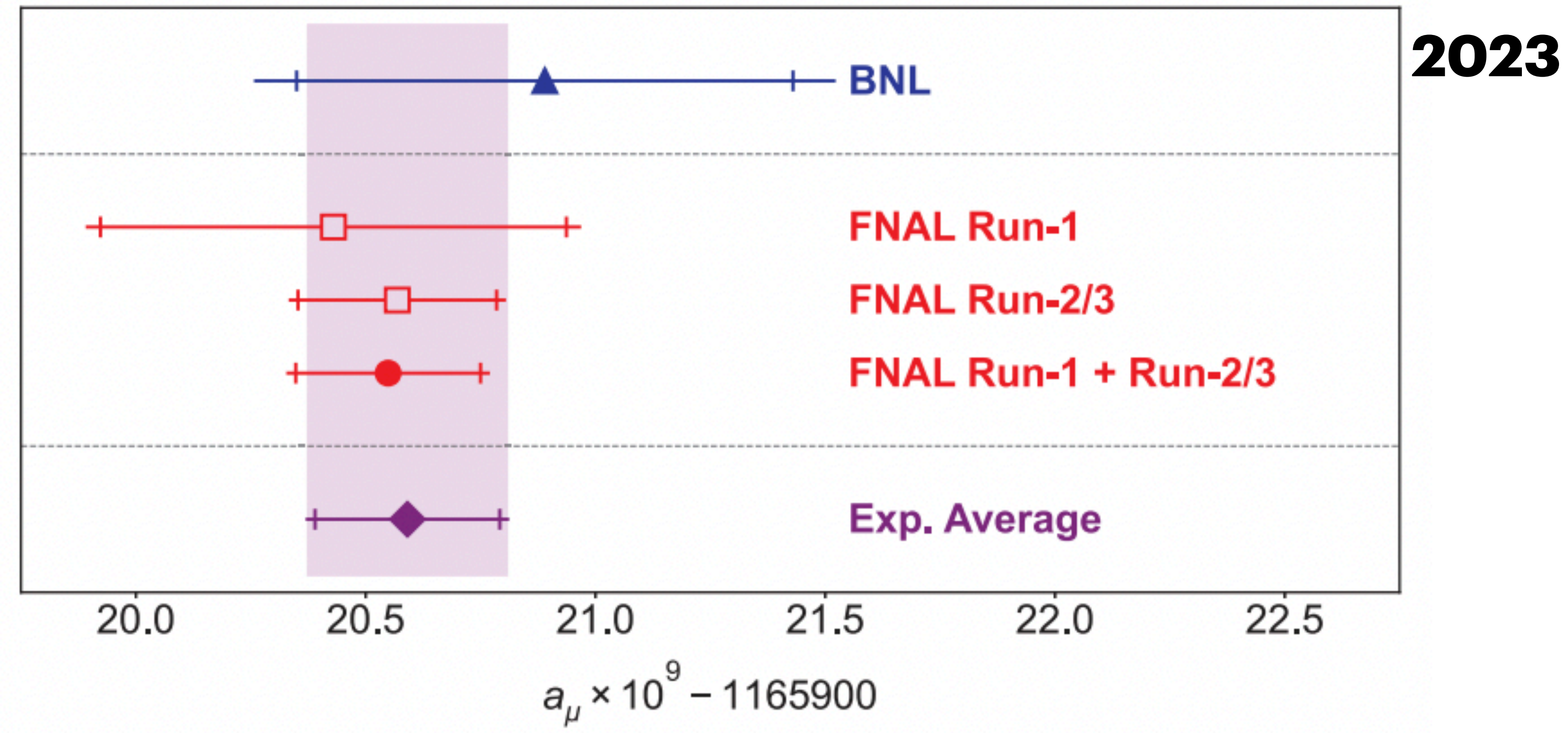

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- MUonE experiment is a good candidate to clarify this discrepancy (space-like channel)

(Anna Driutti's talk on Friday)

Motivation

- The hadronic contribution dominates the current state of uncertainties.

$$a_\mu^{\mathrm{SM}} = a_\mu^{\mathrm{QED}} + a_\mu^{\mathrm{EW}} + a_\mu^{\mathrm{HVP}} + a_\mu^{\mathrm{HLbL}}$$

$$a_\mu^{\mathrm{HVP}} = 6845(40) \times 10^{-11}$$

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The Master Formula

$$a_\mu^{\text{HVP, LO}} = \frac{\alpha^2}{\pi} \int_0^1 \mathrm{d}x \ (1-x)\, \Delta\alpha_{\text{had}}[t(x)]$$

- Requires a measurement of $\Delta\alpha_{\text{had}}$ in the space-like region ($t = q^2 < 0$)

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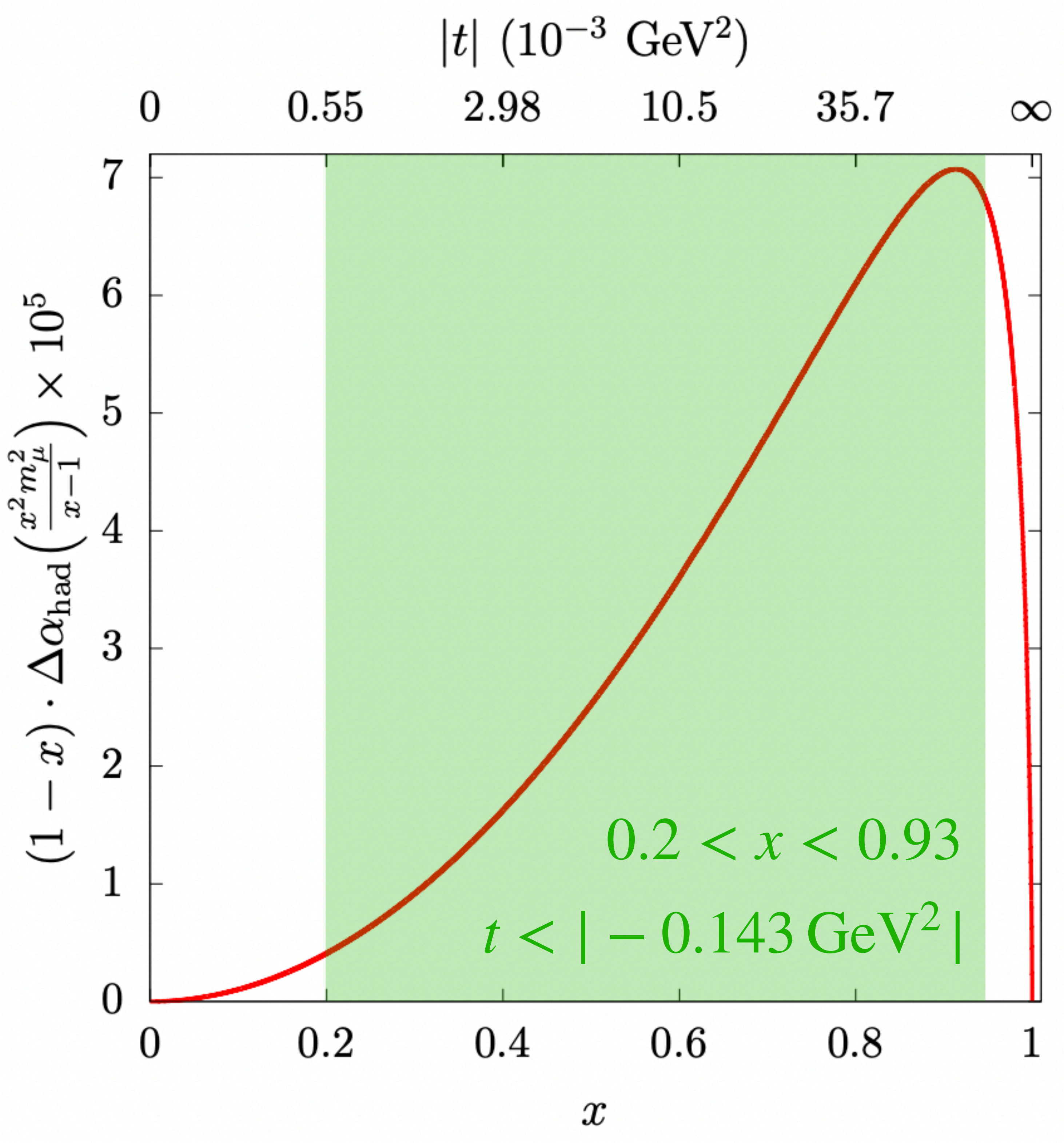

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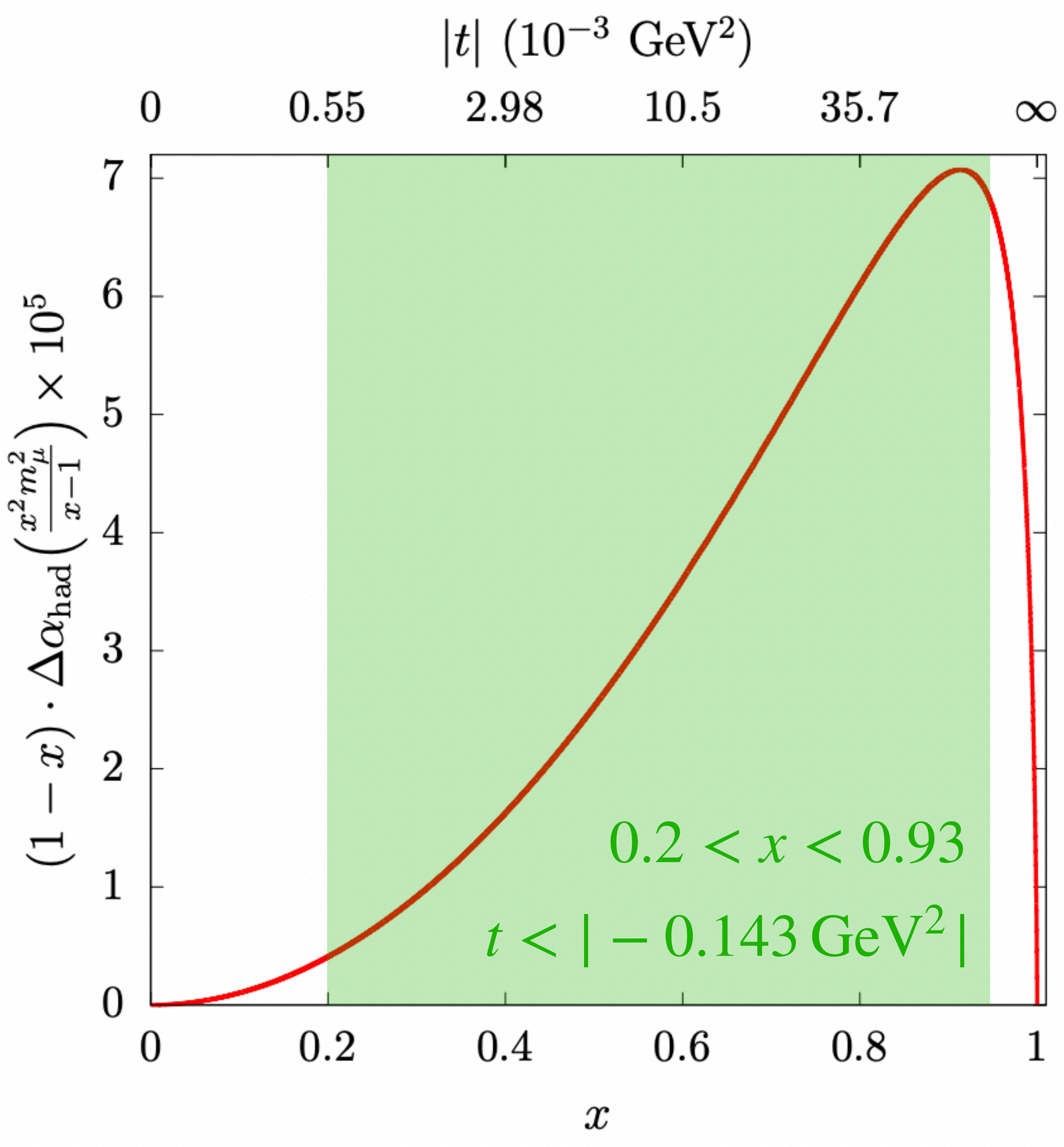


Problem

Finding a reliable method to fit the data + good extrapolation outside data region without using external information

We propose a technique based on Padé and D-Log Padé Approximants

Outline

- Padé and DLog Approximants
- Fitting method & simulations
- Results with realistic errors
- Uncertainties Vs Extrapolation
- Outlook & Conclusions

Padé and D-Log Approximants

Padé Approximants

$$P_M^N(z) = \frac{Q_N(z)}{R_M(z)} = \frac{q_0 + q_1\, z + \cdots + q_N\, z^N}{1 + r_1\, z + \cdots + r_M\, z^M}$$

Advantages

- Systematic and model-independent method
- Partial reconstruction of analytic (physical) properties
- Efficient approximation
- It is possible to provide a systematic error using the **convergence** properties

The standard method for constructing Padé Approximants (PAs)

$$\begin{aligned} f(0) &= P(0), \\ f'(0) &= P'(0), \\ f''(0) &= P''(0), \\ &\vdots \\ f^{(m+n)}(0) &= P^{(m+n)}(0)\,. \end{aligned}$$

- M+N+1 equations relate: PA coefficients (q_i , r_i) with original function coefficients (a_i).
- PAs return a better approximation for the M+N+1 term

Padé Approximants

Stieltjes function

What about Padé Convergence ??

$$f(z) = \int_0^\infty \frac{d\phi(u)}{1+zu} \qquad \phi(u) \text{ is a measure in } u \in [0, \infty)$$

Useful for us

✔ Stieltjes

$f(z) \equiv \Delta\alpha_{had}(t)$

$-\infty < t < 0$

To ensure that poles are real and t-positive axis

- $\Delta\alpha_{had}(t)$ is a Stieltjes function! [Masjuan, Peris '09], [Aubin, Blum, Golterman, Peris '12]
- There are convergence theorems for PAs to Stieltjes functions [Baker '96]
- Some convergence properties (thanks to analyticity and unitarity):
  - ✱ poles of P_N^{N+k}, $k \geq -1$, are located in the positive real axis;
  - ✱ PA sequences uniformly converge to the original function;
  - ✱ PA sequences act as bounds to the function

We can use the convergence theorem to bound our function

$$P_1^1(t) \leq P_2^2(t) \leq \ldots \leq \Delta\alpha_{had} \leq \ldots \leq P_1^2(t) \leq P_0^1(t)$$

D-Log Padé Approximants

Advantages respect to Padés

- It is useful to reproduces not only poles but also cuts or branch points of the original function
- Faster convergence
- Also model independent to find the singularity position value and its multiplicity

Is not longer a rational approximant

Consider the following function:

$$f(z) = A(z)\frac{1}{(\mu - z)^{\gamma}} + B(z) \quad \gamma \in \Re$$

Set a new function applying the logarithm derivative

$$F(z) = \frac{\mathrm{d}}{\mathrm{d}z}\ln f(z) \approx \frac{\gamma}{(\mu - z)}$$

- Approach $F(z)$ with a Padé sequence : $\tilde{P}^M_N\left[F(z)\right]$
- Unfold the process: Integrate, exponentiate and normalize

$$\mathrm{D}^M_N(z) = f(0)\exp\left[\int \mathrm{d}z \tilde{P}^M_N(z)\right]$$

d.o.f: M+N+2

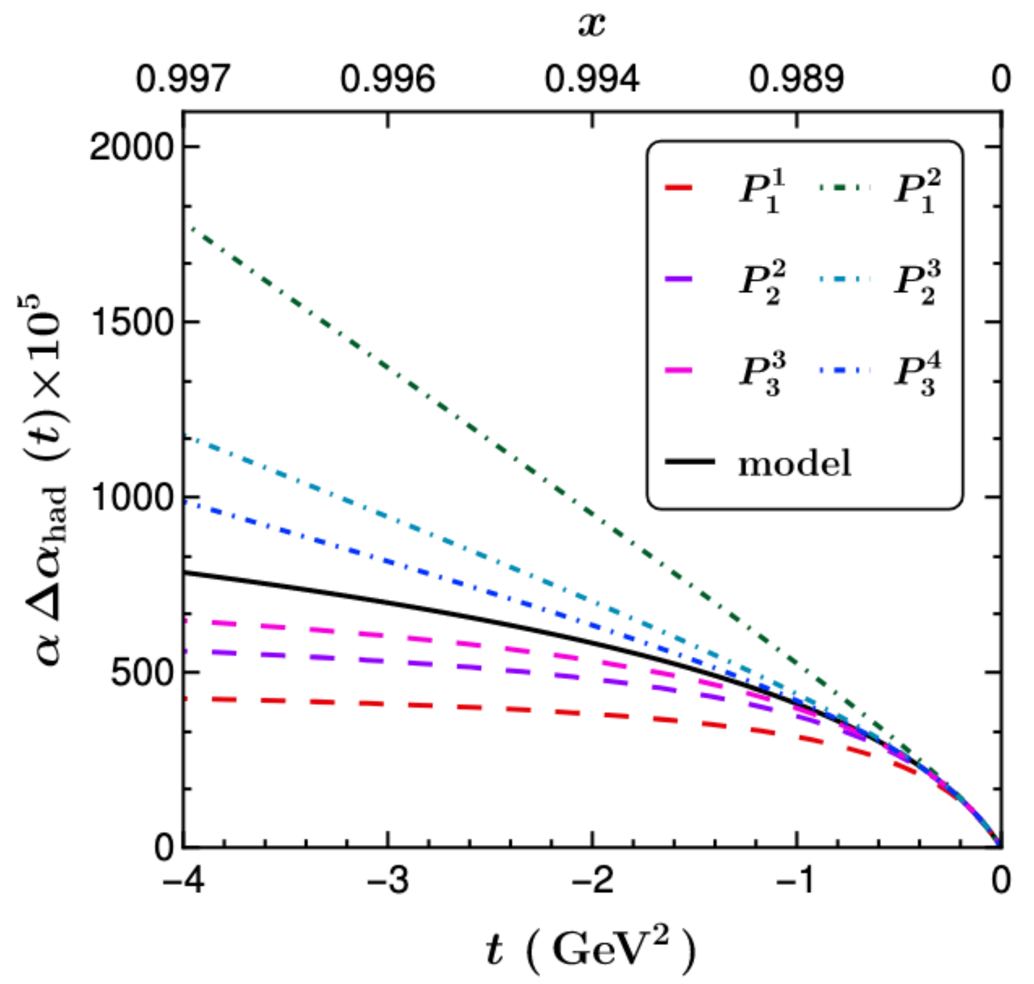
✦ Little is know about D-Logs, they work and we have a convergence conjecture. We find:

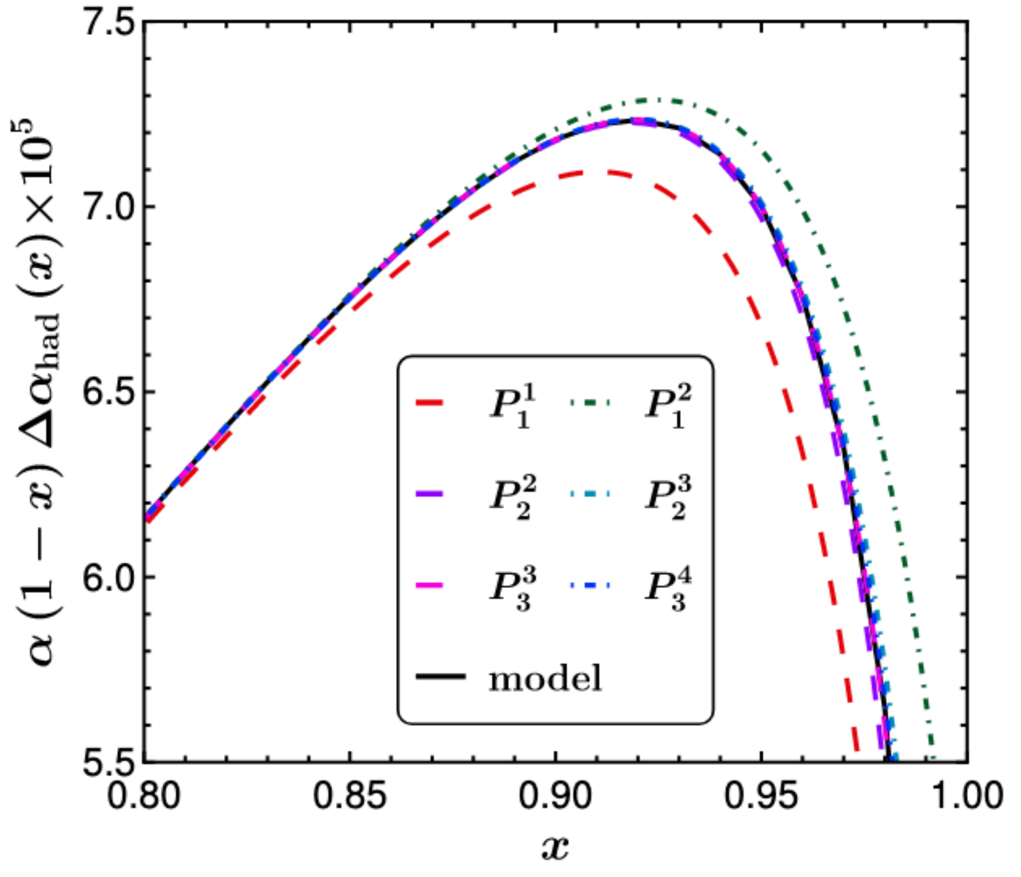
$$D^1_1(t) \leq D^2_2(t) \leq \ldots \leq \Delta\alpha_{\mathrm{had}} \leq \ldots \leq D^2_3(t) \leq D^1_2(t)$$

Fitting method & Simulations

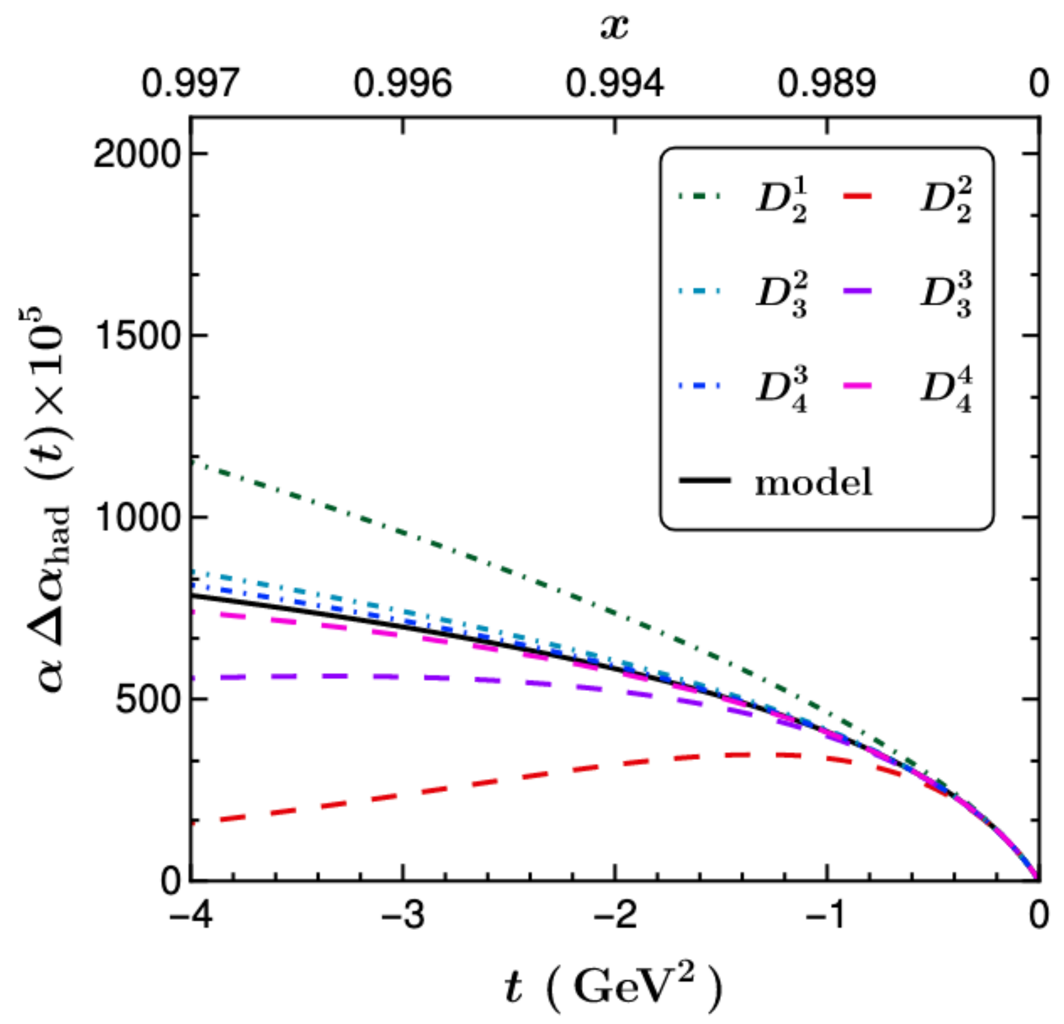
From Taylor expansion

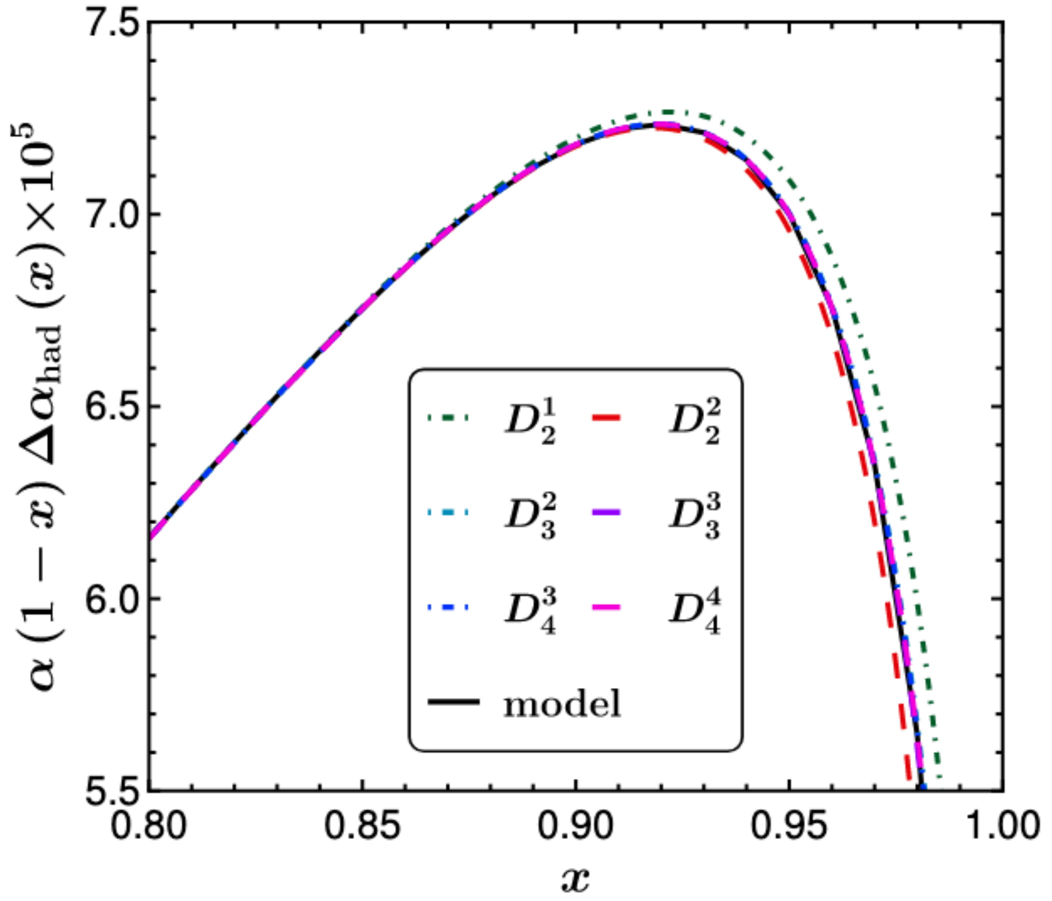
Padé Approximants

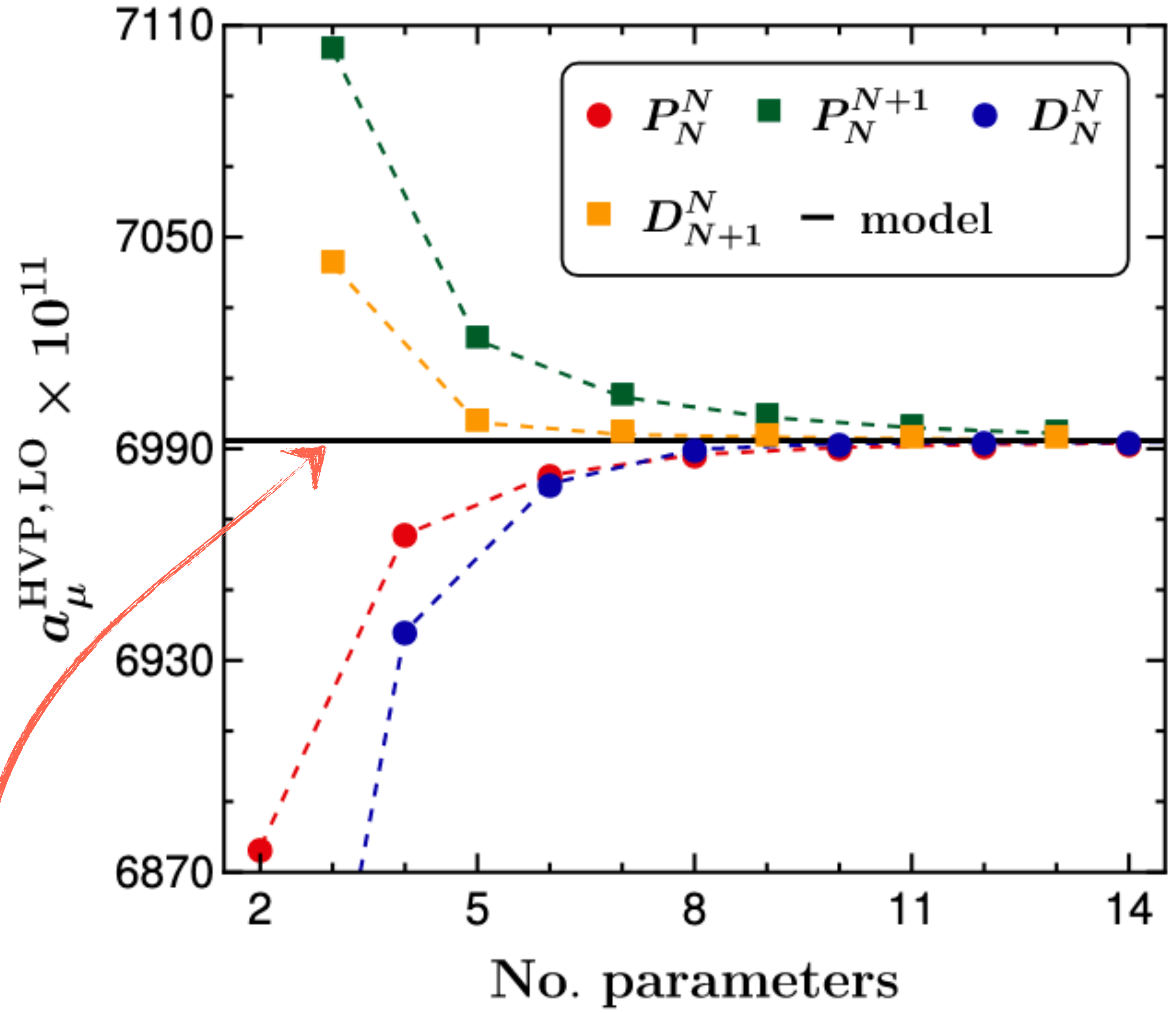




D-Log Padé Approximants







$$a_\mu^{\rm HVP,\,LO} = \frac{\alpha^2}{\pi}\int_0^1 {\rm d}x\;(1-x)\,\Delta\alpha_{\rm had}[t(x)]$$

Taylor expansion from the model @ Greynat and de Rafael '22

(arXiv:2202.10810 [hep-ph])

Camilo Rojas

Fitting Method

- Data generation — simple model for $\Delta\alpha_{\rm had}(t)$ as a Stieltjes function [Greynat, de Rafael (2022)]

$$(x, \alpha\, \Delta\alpha_{\rm had}(x) \times 10^5)$$
$$0.2 \leq x \leq 0.93$$

30 equally spaced bins — data point are mid-value of each bin

- Fitting parameters: starting with unknown Taylor series coefficients of $\Delta\alpha_{\rm had}(t)$

$$\Delta\alpha_{\rm had}(t) = a_1\, t + a_2\, t^2 + a_3\, t^3 + \cdots$$

Example PA:

$$P_1^1(t) = \frac{a_1\, t}{1 - \frac{a_2}{a_1} t}$$

$t \to x$

Making change of variable $t = -\dfrac{m_\mu^2\, x^2}{1-x}$

$$P_1^1(x) = -\frac{b_1\, m_\mu^2\, x^2}{1 - x + b_2\, m_\mu^2\, x^2}$$

$$b_1 = a_1 < 0$$
$$b_2 = \frac{a_2}{a_1} > 1$$

model-independent constraints

- PA parameters: χ^2 minimization
- No cancelations between zeros and poles

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Example DLog:

$$D_2^1(t) = \frac{-f_0\,t}{(r_1 - t)^{\gamma_1}} \to D_2^1(x) = \frac{f_0\,m_\mu^2\,x^2(1-x)^{-1+\gamma_1}}{(r_1 - r_1 x + m_\mu^2 x^2)^{\gamma_1}}$$

$t \to x$

$$D_2^2(t) = \frac{-f_0\,t\,e^{\beta t}}{(r_1 - t)^{\gamma_1}} \to D_2^2(x) = \frac{f_0\,m_\mu^2\,x^2(1-x)^{-1+\gamma_1}}{(r_1 - r_1 x + m_\mu^2 x^2)^{\gamma_1}} e^{\beta \frac{m_\mu^2 x^2}{(x-1)}}$$

$t \to x$

- DLog parameters: χ^2 minimization

Model-independent constraints for $\beta, r_1, \gamma_1, \ldots$

Results with realistic errors

Realistic Data

- 1000 toy data sets
- $(x, \alpha\,\Delta\alpha_{\rm had}(x)\times 10^5)$ — 30 data points equally spaced in $0.2 \leq x \leq 0.93$
- Central value randomly chosen from a gaussian distribution with expected error of MUonE experiment private communication with Abbiendi, Carloni Calame, Venanzoni
- Analysis of the fits for each Padé and DLog
- χ^2 penalties (θ functions) if coefficients do not follow the expected hierarchy

$$\Delta\alpha_{\rm had}(t) = -\sum_{i=1}^{\infty} a_i\, t^i$$

$$0 < a_i < a_{i+1}$$

- Value of $a_\mu^{\rm HVP,LO}$ calculated for each data set (extrapolation was done in the whole region, even in the data region)

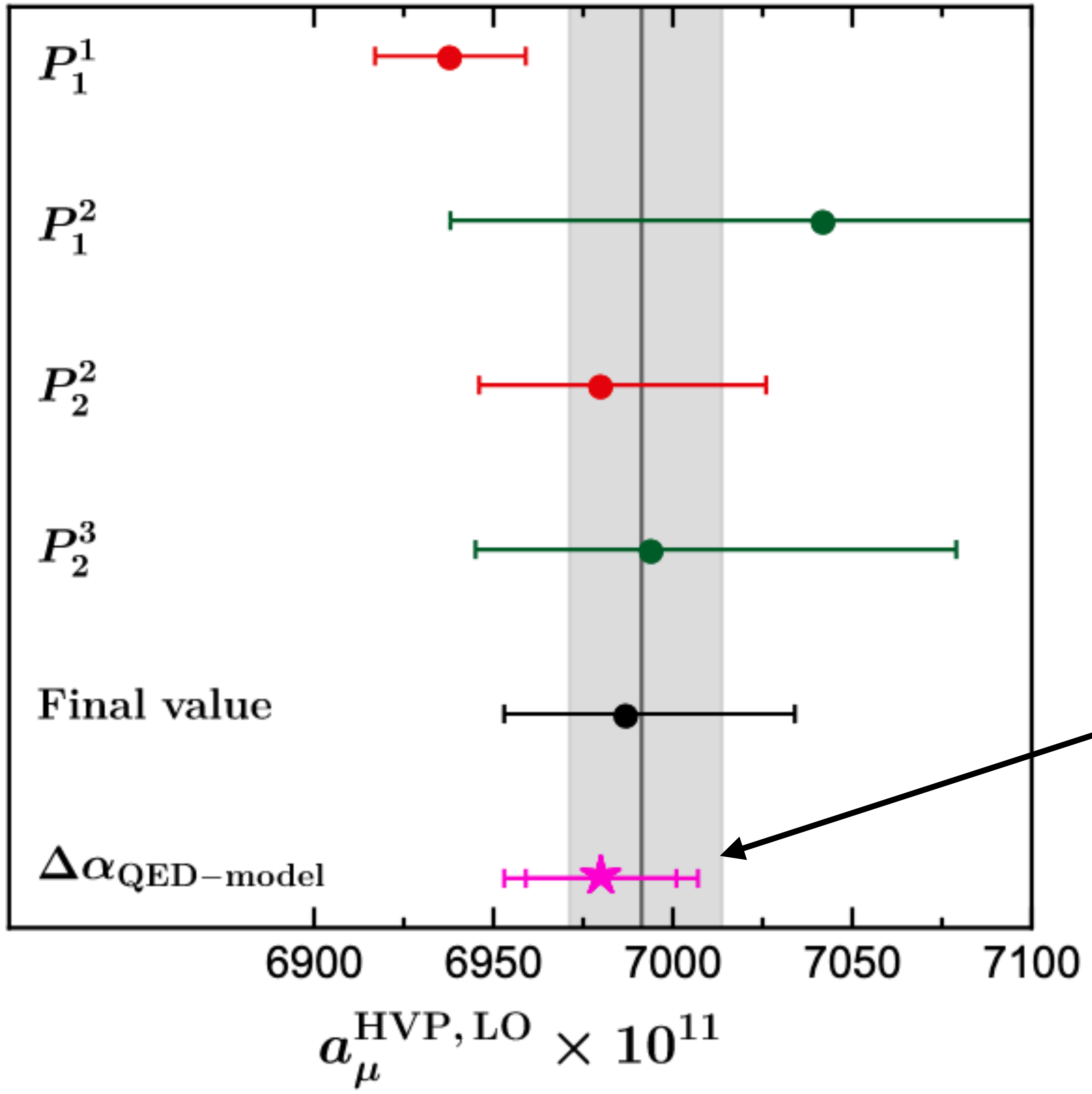
Toy model result:

$$a_\mu^{\rm HVP,LO} = \left(6991^{+22}_{-20}\right)\times 10^{-11}$$

best result we can expect from PAs and DLogs predictions

Realistic Data

$$P_1^1(t) \leq P_2^2(t) \leq \ldots \leq \Delta\alpha_{\text{had}} \leq \ldots \leq P_2^3(t) \leq P_1^2(t)$$



Convergence pattern preserved for the central values

- Good fit qualities
- Statistical and theoretical error of the same order but statistical is higher

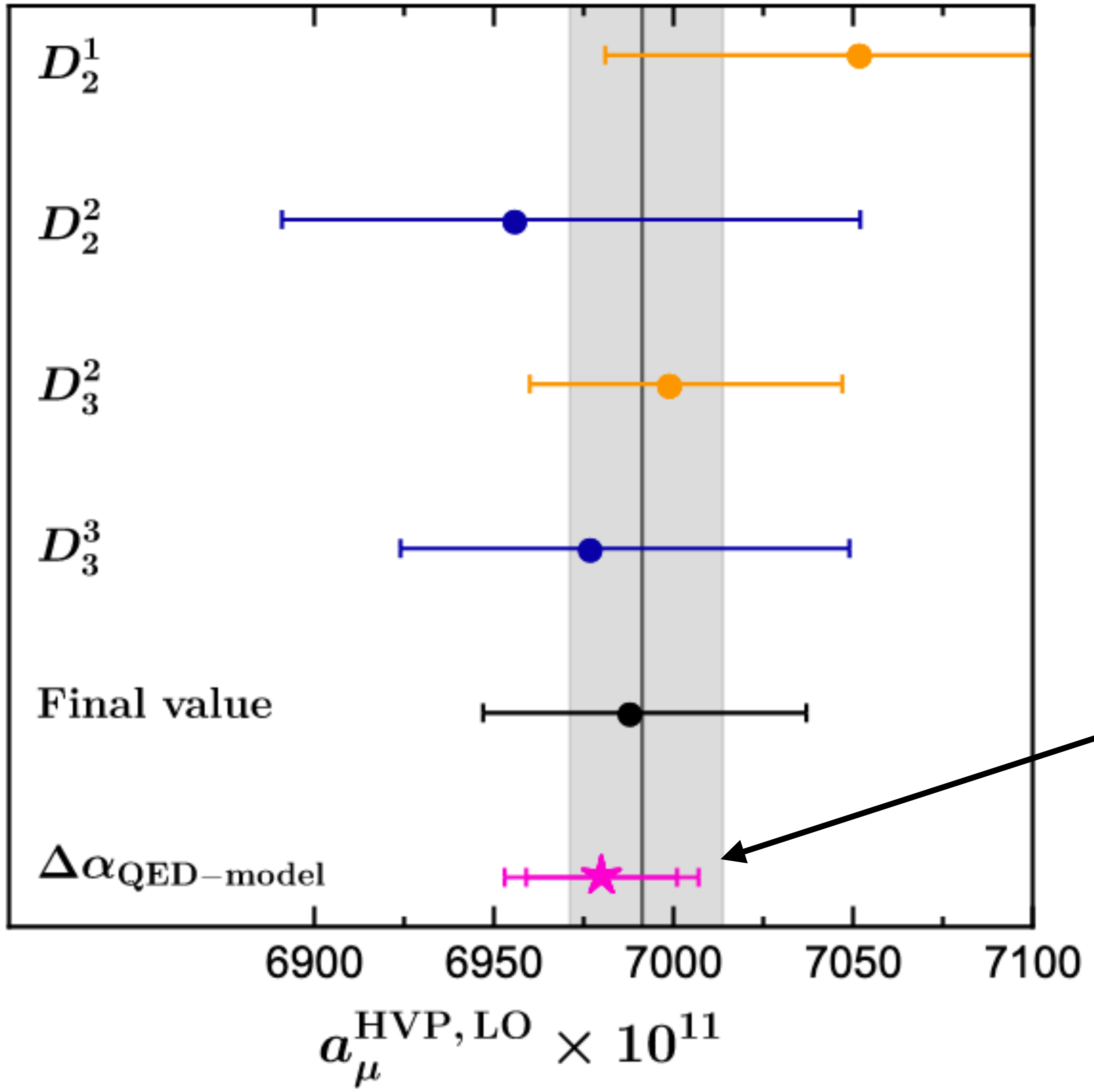
$$\Delta\alpha_{\text{QED-model}}(t) = KM\left[-\frac{5}{9} - \frac{4M}{3t} + \frac{2\left(\frac{4M^2}{3t^2} + \frac{M}{3t} - \frac{1}{6}\right)}{\sqrt{1-\frac{4M}{t}}}\log\left|\frac{1-\sqrt{1-\frac{4M}{t}}}{1+\sqrt{1-\frac{4M}{t}}}\right|\right]$$

Inner error bar — statistical error

Exterior error bar — statistical and systematic errors added in quadrature

Realistic Data

$$D_1^1(t) \leq D_2^2(t) \leq \ldots \leq \Delta\alpha_{\text{had}} \leq \ldots \leq D_3^2(t) \leq D_2^1(t)$$



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Realistic Data

	$a_\mu^{\mathrm{HVP,LO}} \times 10^{11}$	χ^2/n_{dof}		$a_\mu^{\mathrm{HVP,LO}} \times 10^{11}$	χ^2/n_{dof}
P_1^1	6938 ± 21	$1.01^{+0.27}_{-0.25}$	D_2^1	7052^{+66}_{-71}	$1.01^{+0.26}_{-0.26}$
P_1^2	7042^{+114}_{-104}	$1.01^{+0.28}_{-0.26}$	D_2^2	6956^{+96}_{-65}	$1.05^{+0.28}_{-0.27}$
P_2^2	6980^{+46}_{-34}	$1.05^{+0.29}_{-0.27}$	D_3^2	6999^{+48}_{-39}	$1.10^{+0.29}_{-0.28}$
P_2^3	6994^{+85}_{-49}	$1.11^{+0.29}_{-0.31}$	D_3^3	6977^{+72}_{-53}	$1.14^{+0.30}_{-0.29}$
Final result	6987^{+46}_{-34}	—	Final result	6988^{+48}_{-39}	—

Toy model result:

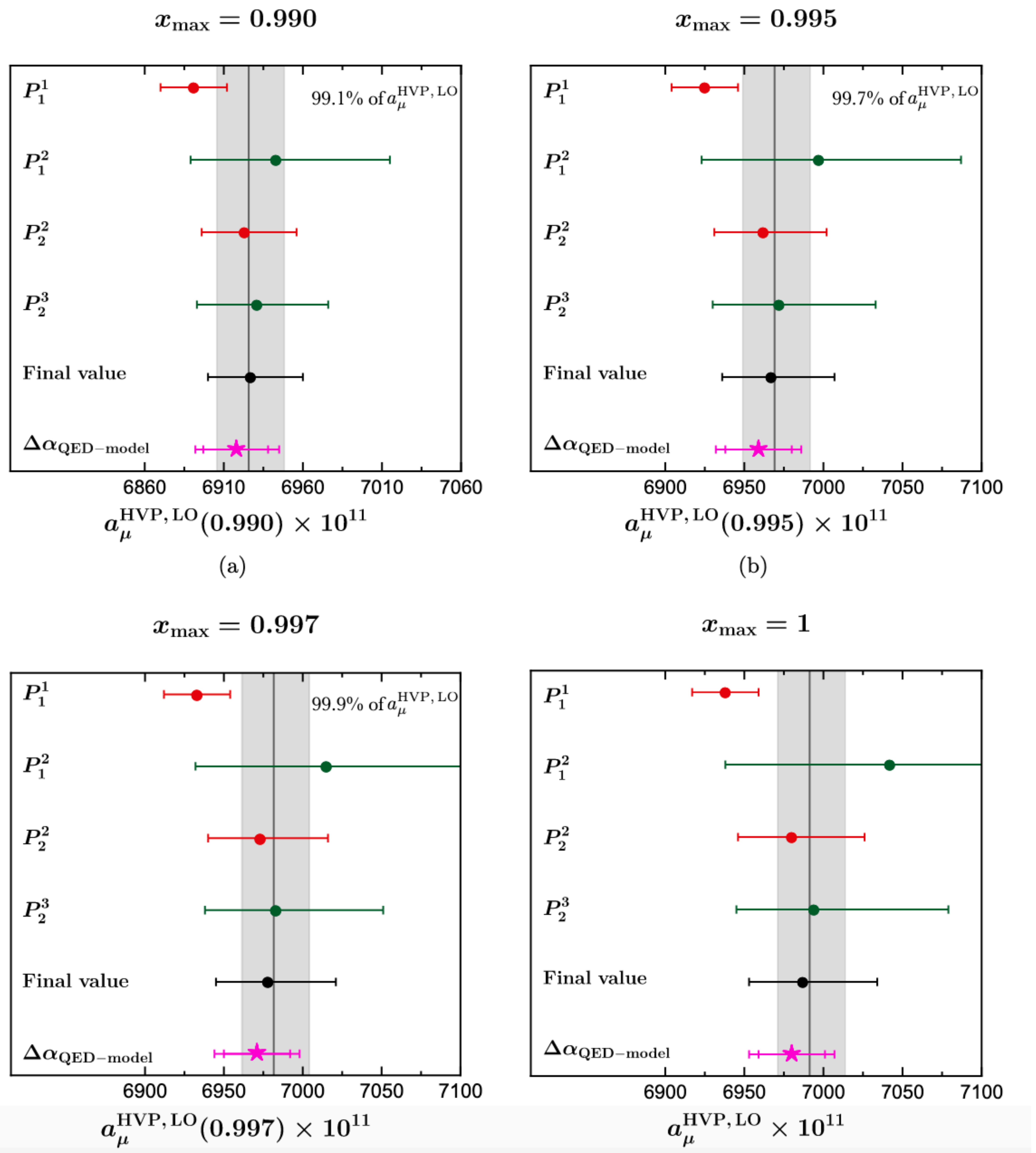
$$a_\mu^{\mathrm{HVP,LO}} = \left(6991^{+22}_{-20}\right) \times 10^{-11}$$

best result we can expect from PAs and DLogs predictions

Uncertainties Vs Extrapolation

Realistic Data: extrapolation up to x_{max}

Fits up to $x \sim 0.93$ and extrapolated up to x_{max}:



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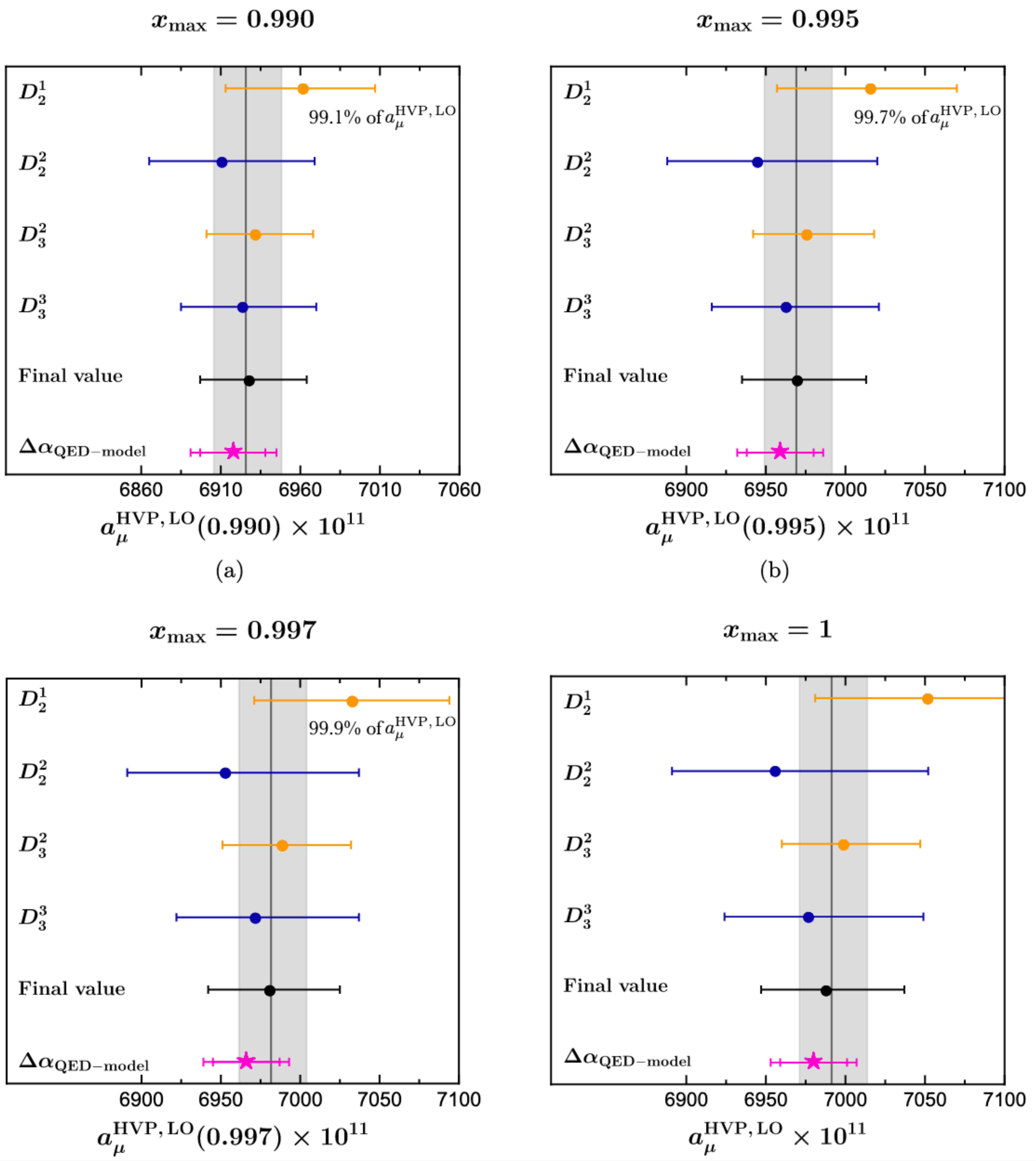

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Fits up to $x \sim 0.93$ and extrapolated up to $x_{\max}$:

$x_{\max}$	$a_{\mu,\,\mathrm{PAs}}^{\mathrm{HVP,LO}}$	$a_{\mu,\,\mathrm{Dlogs}}^{\mathrm{HVP,LO}}$	$a_{\mu,\,\mathrm{QED-model}}^{\mathrm{HVP,LO}}$	$a_{\mu,\,\mathrm{data-sets}}^{\mathrm{HVP,LO}}$
0.990	$6927\,\left(^{+33}_{-27}\right)\,(\pm 4)$	$6928\,\left(^{+36}_{-31}\right)\,(\pm 4)$	$6918\,\left(^{+21}_{-20}\right)\,(\pm 4)$	$6926\,\left(^{+22}_{-20}\right)$
0.995	$6967\,\left(^{+40}_{-31}\right)\,(\pm 5)$	$6970\,\left(^{+42}_{-34}\right)\,(\pm 7)$	$6959(\pm 21)(\pm 17)$	$6969\,\left(^{+22}_{-20}\right)$
0.997	$6978\,\left(^{+43}_{-33}\right)\,(\pm 5)$	$6981\,\left(^{+43}_{-38}\right)\,(\pm 9)$	$6971(\pm 21)(\pm 17)$	$6982\,\left(^{+22}_{-20}\right)$
1.000	$6987\,\left(^{+46}_{-34}\right)\,(\pm 7)$	$6988\,\left(^{+48}_{-39}\right)\,(\pm 11)$	$6980(\pm 21)(\pm 17)$	$6991\,\left(^{+22}_{-20}\right)$

First error - statistical error

Second error - systematic (extrapolation) error

$$\Delta\alpha_{\mathrm{QED-model}}(t) = KM\left[-\frac{5}{9} - \frac{4M}{3t} + \frac{2\left(\frac{4M^2}{3t^2} + \frac{M}{3t} - \frac{1}{6}\right)}{\sqrt{1-\frac{4M}{t}}}\log\left|\frac{1-\sqrt{1-\frac{4M}{t}}}{1+\sqrt{1-\frac{4M}{t}}}\right|\right]$$

Realistic Data: extrapolation up to x_{max}

Fits up to $x \sim 0.93$ and extrapolated up to $x_{\max}$:

$x_{\max}$	$a_{\mu,\,\mathrm{PAs}}^{\mathrm{HVP,LO}}$	$a_{\mu,\,\mathrm{Dlogs}}^{\mathrm{HVP,LO}}$	$a_{\mu,\,\mathrm{QED-model}}^{\mathrm{HVP,LO}}$	$a_{\mu,\,\mathrm{data-sets}}^{\mathrm{HVP,LO}}$
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First error - statistical error

Second error - systematic (extrapolation) error

Extrapolation seems to be under control!

$$\Delta\alpha_{\mathrm{QED-model}}(t) = KM\left[-\frac{5}{9} - \frac{4M}{3t} + \frac{2\left(\frac{4M^2}{3t^2} + \frac{M}{3t} - \frac{1}{6}\right)}{\sqrt{1-\frac{4M}{t}}}\log\left|\frac{1-\sqrt{1-\frac{4M}{t}}}{1+\sqrt{1-\frac{4M}{t}}}\right|\right]$$

Conclusions

Conclusions

- Discrepancy for a_μ between experimental measurements and predictions reached the 5.0σ level
- D-Logs and Padé approximant sequences are a model-independent method to fit and extrapolate the data from the MUonE experiment. Approximants up to 6 coefficients seem ok!
- The method uses fundamental knowledge about the analytic structure of $\Delta\alpha_{\rm had}(t)$. It is a Stieltjes function (we use analyticity and unitarity, yet inclusive) ⇒ beyond the unitary cut!

⚠ PAs and DLogs can provide a lower and upper **bound** for the true value!

- Uncertainties may be reduced if:
  - Knowledge about the structure of $\Delta\alpha_{\rm had}(t)$ is included (or either extracted from fit)
  - Extrapolate to certain x_{max} (corresponding to "large enough energy") and then match to pQCD or e^+e^- (with DLogs we can access the time-like: $\pi^+\pi^-$ or $\pi^0\gamma$ production thresholds, ρ-meson!)

Thanks

Backup slides

Muon Anomalous magnetic moment

The anomalous magnetic moment of the muon

- The magnetic moment (for charged leptons)

$$\vec{\mu_l} = g_l \frac{e}{2m_l} \cdot \vec{S}$$

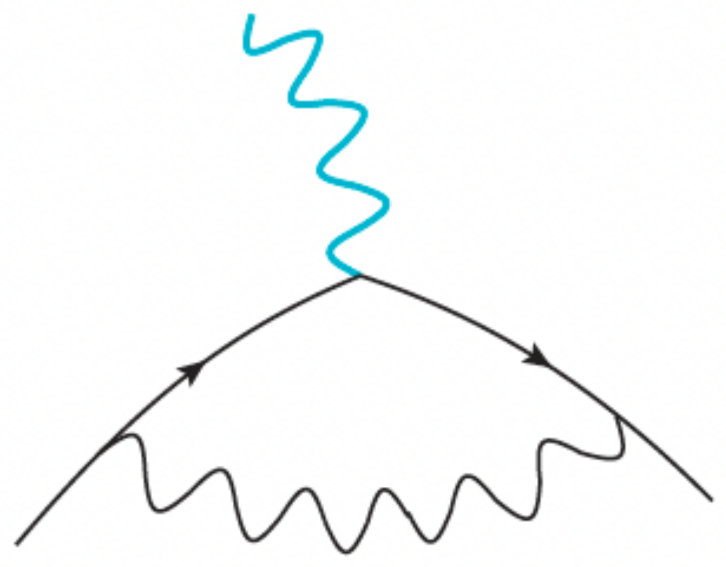
g_l is gyromagnetic ratio

Dirac Theory → Spin = 1/2 → $g_l = 2$ → Radiative correction of QFT → $g_l \neq 2$

- The magnetic anomaly (deviation from Dirac value)

$$a_l = (g_l - 2)/2$$

This observable can be both precisely measured experimentally and predicted in the Standard Model, providing a stringent test of the SM.



"These quantum fluctuations modify g"

- The first order correction (by J. Schwinger)

$$a_e^{\text{QED,LO}} = \alpha/2\pi \approx 1.16 \times 10^{-3}$$

α is fine structure constant

The anomalous magnetic moment of the muon

- E821 @BNL measurement with an error of 0.54 ppm

$$a_\mu^{\text{exp}} = 116592089(63) \times 10^{-11}$$

Bennet et al, PRD73,072003 (2006)

Merge with FNAL measurement

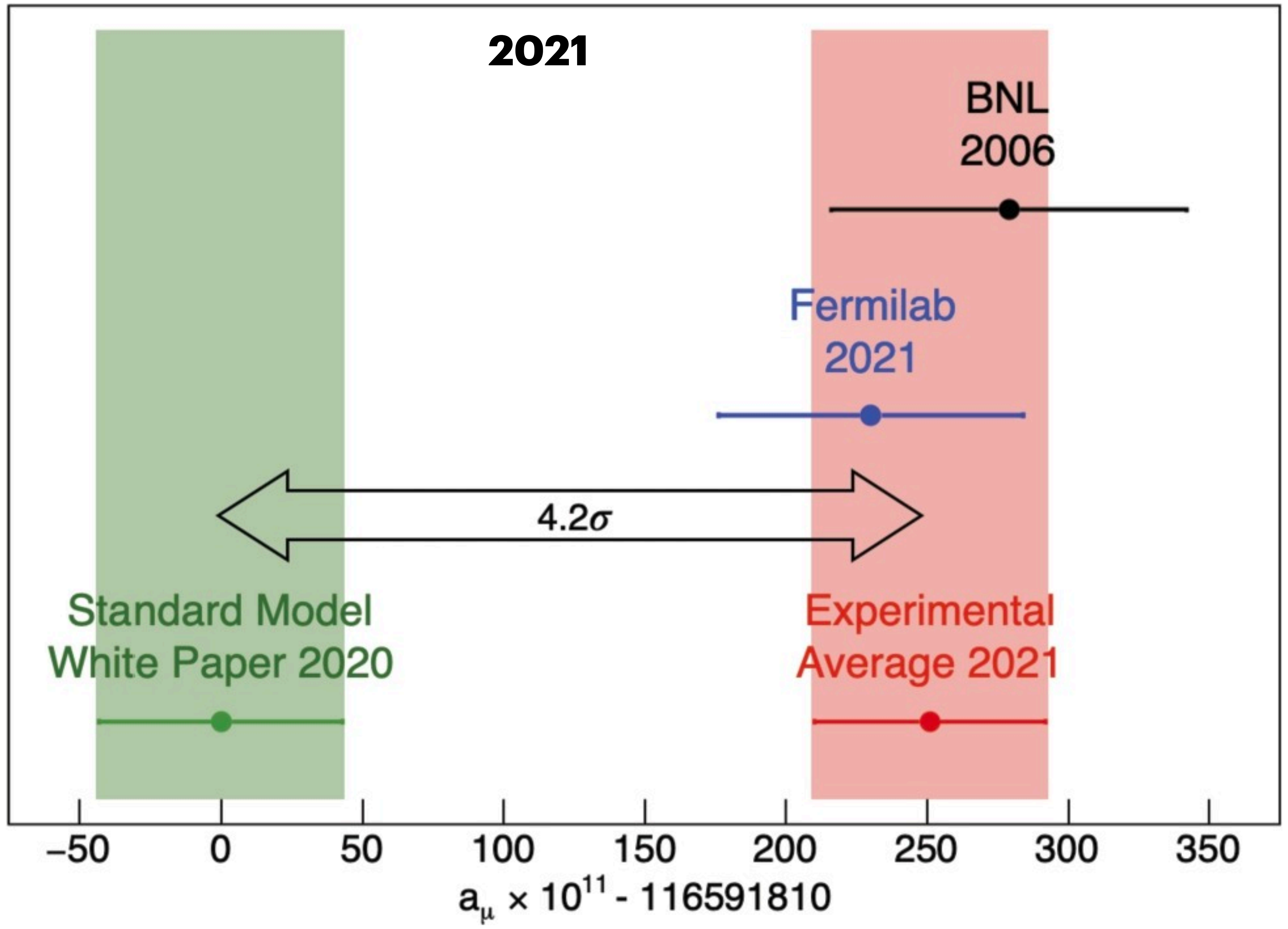
$$a_\mu^{\text{exp}} = 116592089(63) \times 10^{-11}$$ [0.35 ppm]

B. Abi et al, Phys. Rev. Lett.126,141801(2021)

- The theoretical calculation with SM (approved consensus)

$$a_\mu^{\text{SM}} = 116591810(43) \times 10^{-11}$$

By the Muon $g-2$ Theory Initiative
T Aoyama et al, Phys. Rep., 887 (2020)



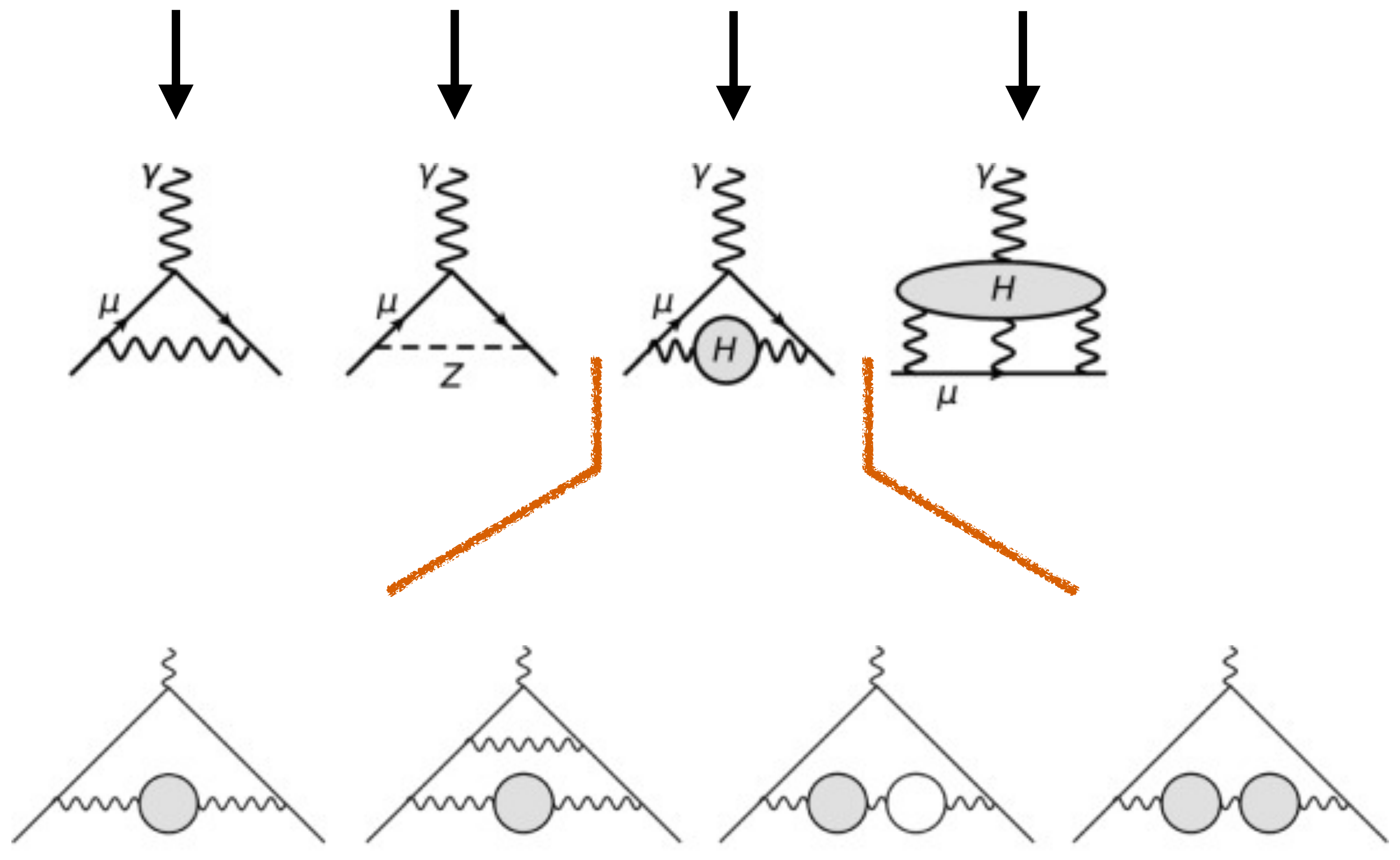
Current discrepancy limited by:

- Experimental uncertainty: New run test at FNAL and J-PARC x4 accuracy
- In the theoretical frame: How to calibrate Hadronic Contributions uncertainties.

$$\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{th}} = 251(59) \times 10^{-11}$$

The anomalous magnetic moment of the muon

$$a_\mu^{SM} = a_\mu^{QED} + a_\mu^{EW} + a_\mu^{HVP} + a_\mu^{HLbL}$$



(a) LO. (b) NLO.

Main contribution: LO Vacuum Polarization estimated
rel. uncertainty 0.35% - 0.6%

$$a_\mu^{HVP} = a_\mu^{LO,HVP} + a_\mu^{NLO,HVP} + a_\mu^{NNLO,HVP}$$

- QED corrections known up to 5 loops with related precision $\sim 7 \times 10^{-10}$

$$a_\mu^{QED} = 116584718.931(104) \times 10^{-11}$$ *Aoyama, Hayakawa, Kinoshita,Nio (2012)*

- EW corrections up to 2 loops with precision $\sim 10^{-9}$ rel. uncertainty $< 1\,\%$

$$a_\mu^{EW} = 153.6(1.0) \times 10^{-11}$$ *Gnendiger, D. Stöckinger, H. Stöckinger-Kim (2013)*

- Hadronic contribution $\sim 7 \times 10^{-8}$; the dominant theoretical uncertainty

$$a_\mu^{HVP} = 6845(40) \times 10^{-11}$$ *T Aoyama et al, Phys. Rep., 887 (2020)*

>> not calculable by pQCD <<

Data-Driven approach
(Timelike domain)

Experiment MUonE
(Spacelike domain)

Lattice QCD

How to calculate the HVP contribution ?

Using the dispersive relation integral:

$$a_\mu^{\mathrm{HVP,\,LO}} = \frac{\alpha}{\pi^2}\int_0^\infty \frac{\mathrm{d}s}{s}\, K(s)\, \mathrm{Im}\Pi_{\mathrm{had}}(s+i\epsilon) \quad ; \quad K(s) = \int_0^1 \frac{x^2(1-x)}{x^2+(1-x)\frac{s}{m_\mu^2}}\,\mathrm{d}x \sim \frac{1}{s}$$

$\mathrm{Im}\Pi_{\mathrm{had}}(s)$ Is the hadronic contribution to the photon vacuum polarisation function

Kernel has the lepton information !

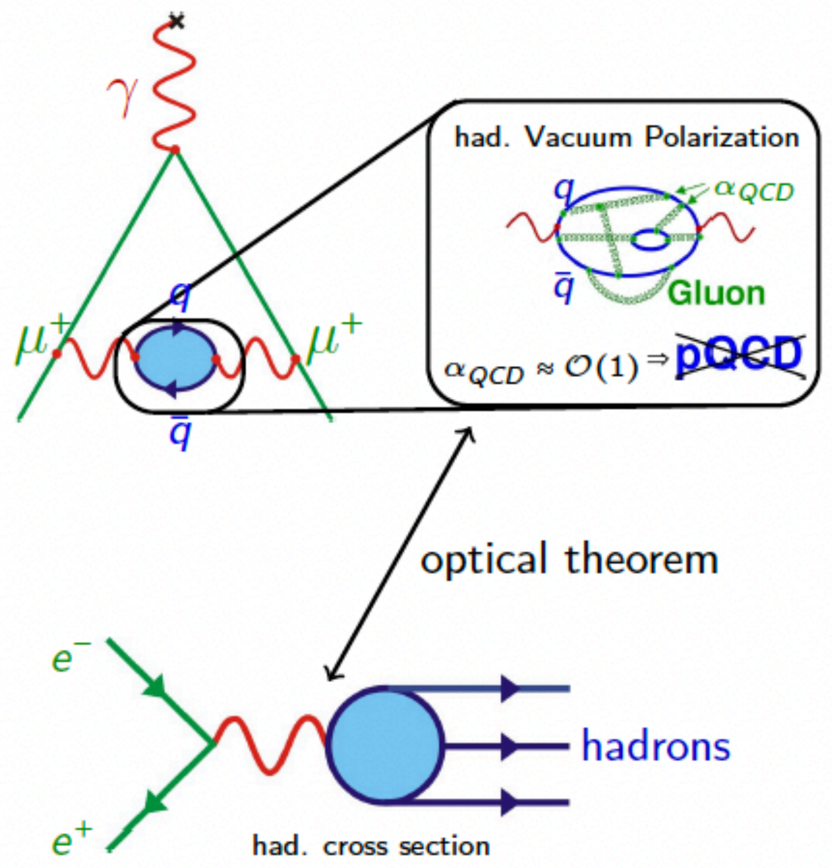
- Using the optical theorem
- Involving the total hadronic cross section measured experimentally at e +e − machines

Data-Driven

$$a_\mu^{\mathrm{HVP,\,LO}} = \left(\frac{\alpha m_\pi}{3\pi}\right)^2 \int_{4m_\pi^2}^\infty \mathrm{d}s\, \frac{K(s)R(s)}{s^2}$$

$$R(s) = \left(\frac{3s}{4\pi\alpha^2}\right) \sigma_{e^+e^-\to\mathrm{hadrons}}(s) = 12\pi \mathrm{Im}\Pi_{\mathrm{had}}$$

$$a_\mu^{\mathrm{HVP,\,LO}} = \left(\frac{\alpha m_\pi}{3\pi}\right)^2 \int_{4m_\pi^2}^\infty \mathrm{d}s\, K(s)\,\sigma_{e^+e^-\to\mathrm{hadrons}}(s)$$



Alternative representation usually use in:

Lattice

$$\int \mathrm{d}s \int \mathrm{d}x \rightarrow \int \mathrm{d}x \int \mathrm{d}s$$

In terms of the Euclidean space $\bar{\Pi}\left(Q^2\right)$; $Q^2 = -q^2 < 0$

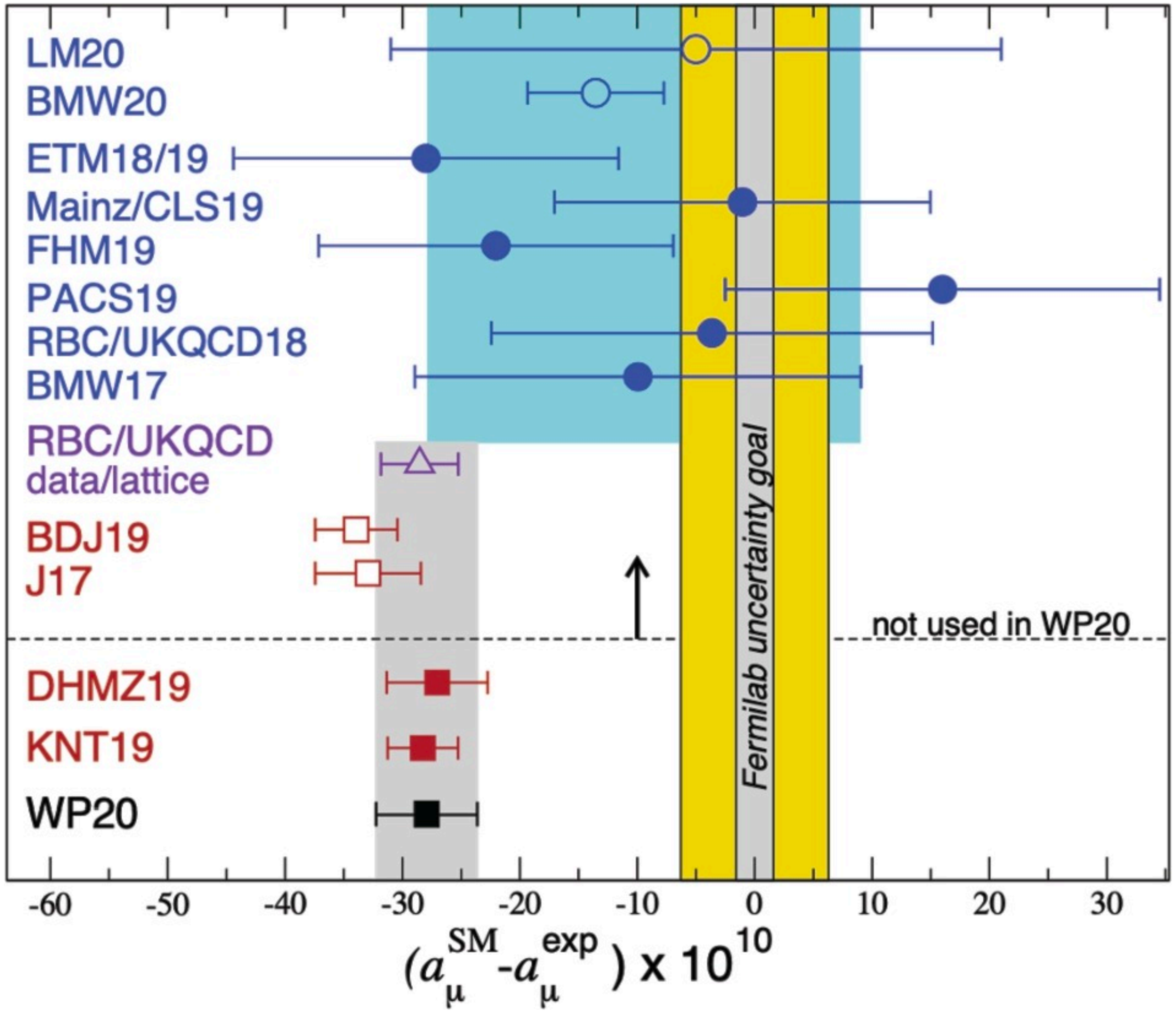
$$a_\mu^{\mathrm{HVP,\,LO}} = \frac{\alpha}{\pi}\int_0^1 \mathrm{d}x\ (x-1)\,\bar{\Pi}^{\mathrm{HVP}}\left(Q^2\right)$$

$$\bar{\Pi}^{\mathrm{HVP}}\left(Q^2\right) = \int_{s_0}^\infty \frac{\mathrm{d}s}{s}\,\frac{Q^2}{(s+Q^2)}\,\frac{1}{\pi}\mathrm{Im}\Pi_{\mathrm{had}}(s)$$

This representation would be useful for MUonE experiment since it is in the space-like domain

C. Aubin, T. Blum, Phys. Rev. D, 75 (2007)
P. Boyle, L. Del Debbio, E. Kerrane, J. Zanotti, Phys. Rev. D, 85 (2012)
X. Feng, K. Jansen, M. Petschlies, D.B. Renner, Phys. Rev. Lett., 107 (2011)

Tension between Data-driven Vs Lattice QCD

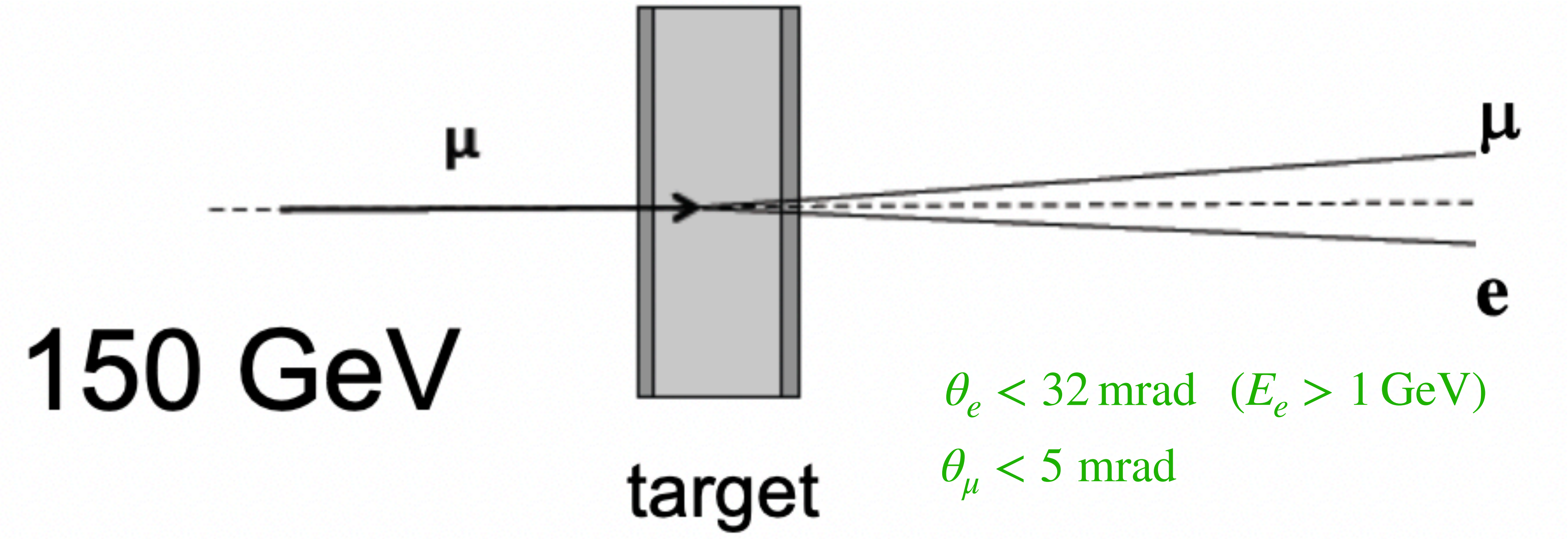


MUonE Experiment

MUonE Experiment ($\mu e \to \mu e$) @CERN

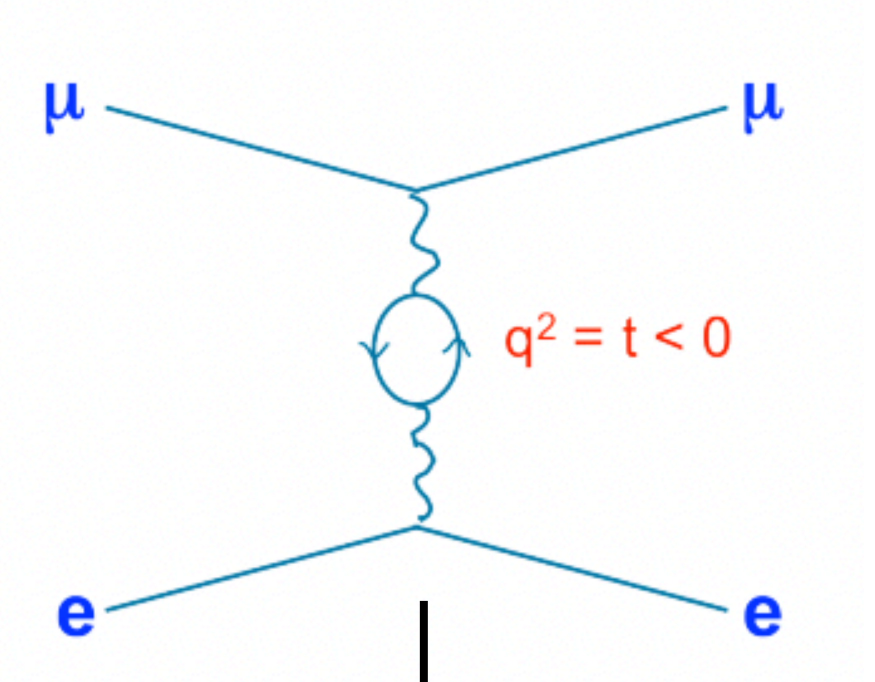
It is a new experimental proposal @ CERN:

- Scattering μ's on e's in a low Z target looks like an ideal process (fixed target experiment)
- It is a pure t-channel process at tree level
- The M2 muon beamline ($E_\mu \simeq 150\,\mathrm{GeV}$) is available at CERN
- Useful C.M. energy to test dominant region of $a_\mu^{LO,HVP}$ $\sqrt{s} \simeq 0.4\,\mathrm{GeV} \;\to\; q^2 < 0.11\,\mathrm{GeV}^2$
- With ~ 3 years of data taking, a statistical accuracy of 0.35% on $a_\mu^{LO,HVP}$ can be achieved
- Easy selection based on the correlation of the electron and muon scattering angles.



$\theta_e < 32\,\mathrm{mrad} \quad (E_e > 1\,\mathrm{GeV})$

$\theta_\mu < 5\,\mathrm{mrad}$



$$\frac{d\sigma}{dt} \approx \frac{d\sigma_0}{dt}\left|\frac{\alpha(t)}{\alpha_0}\right|^2$$

Running of α

$$\left|\frac{1}{1-\Delta\alpha(t)}\right|^2 = \left|\frac{\alpha(t)}{\alpha_0}\right|^2$$

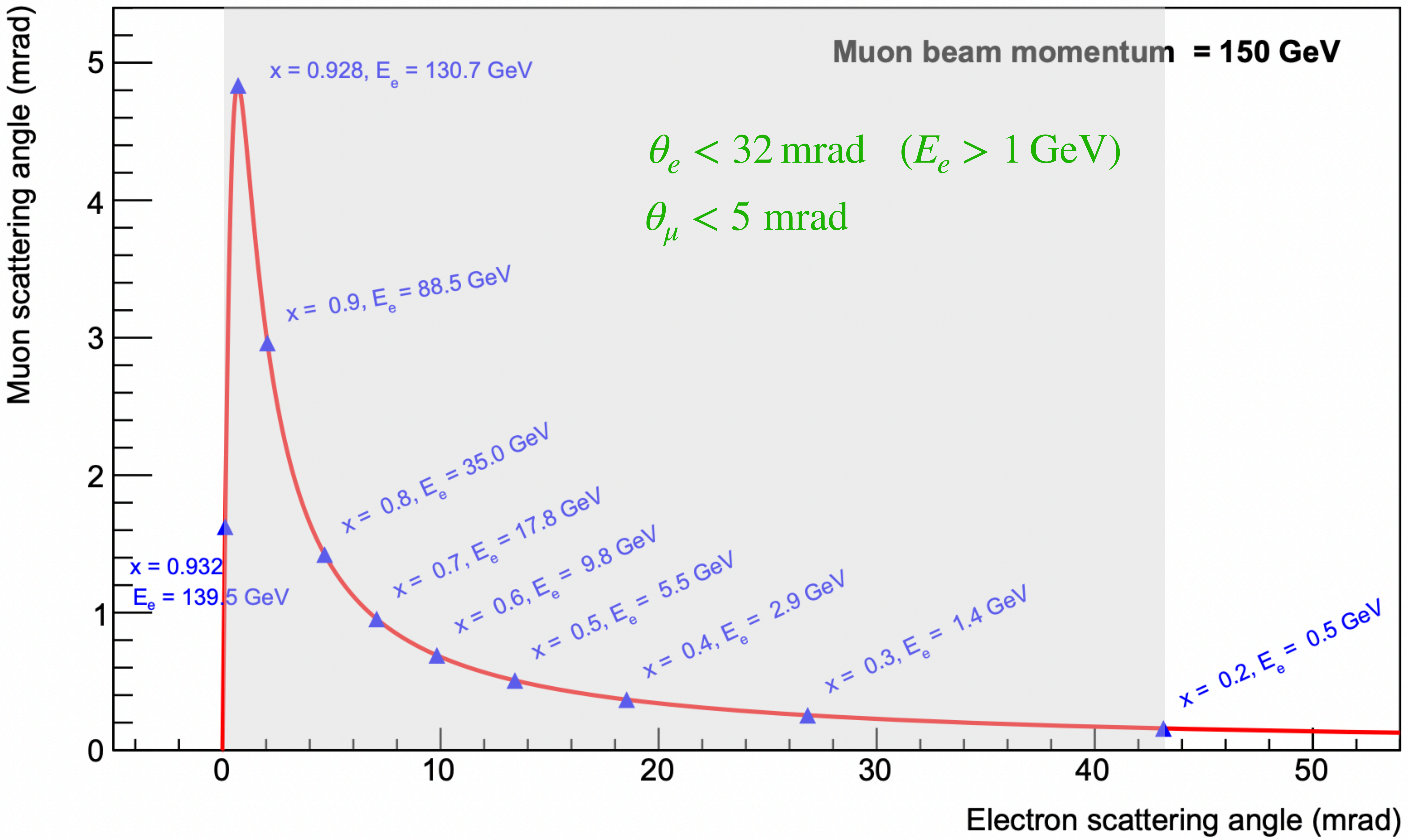
$$\Delta\alpha(t) = \Delta\alpha_{lep}(t) + \Delta\alpha_{had}(t)$$

Known from QED

To be measured

Since it is related with HVP contribution of a_μ

MUonE experiment - event selection



MUonE experiment - measurement in the space-like momentum

Recovering the Master Formula

$\Delta\alpha(q^2) = -\,\mathrm{Re}\,\bar{\Pi}(q^2)$

$\mathrm{Im}\,\bar{\Pi}(q^2 < 0) = 0$

$$a_\mu^{\mathrm{HVP,LO}} = \frac{\alpha^2}{\pi}\int_0^1 \mathrm{d}x\;(1-x)\,\Delta\alpha_{\mathrm{had}}[t(x)]$$

$$t(x) = -\frac{x^2 m_\mu^2}{1-x} \qquad 0 \le -t < \infty \qquad t \leftrightarrow x \qquad x(t) = \frac{t}{2m_\mu^2}\left(1-\sqrt{1-\frac{4m_\mu^2}{t}}\right) \qquad 0 \le x < 1$$

$t < -0.143\,\mathrm{GeV}^2$ <<. Real Experimental Range. >> $0.2 < x < 0.93$

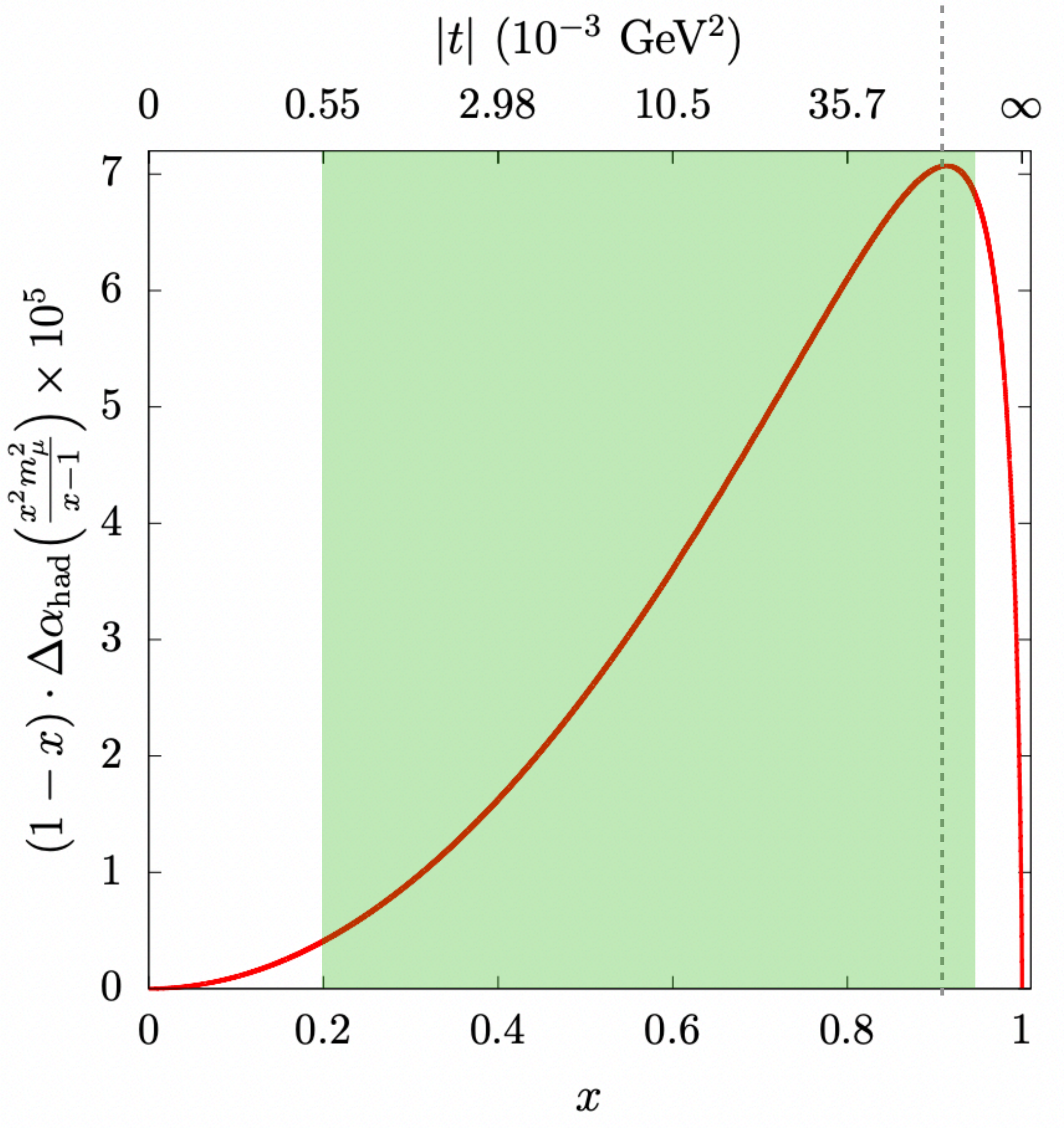
- $a_\mu^{LO,HVP}$ is given by the integral of the curve (smooth behaviour)
- Requires a measurement of $\Delta\alpha_{\mathrm{had}}$ in the space-like region ($t = q^2 < 0$)
- It enhances the contribution from low q^2 region (below 0.11 GeV^2)
- They expect to cover 87% of the $a_\mu^{LO,HVP}$ with the space-like integral
  - But they need to extrapolate: x → 1, (13% missing)
- Its precision is determined by the uncertainty on $\Delta\alpha_{\mathrm{had}}$ in this region

using the output of the routine hadr5n12 (which uses time-like hadron production data and perturbative QCD)

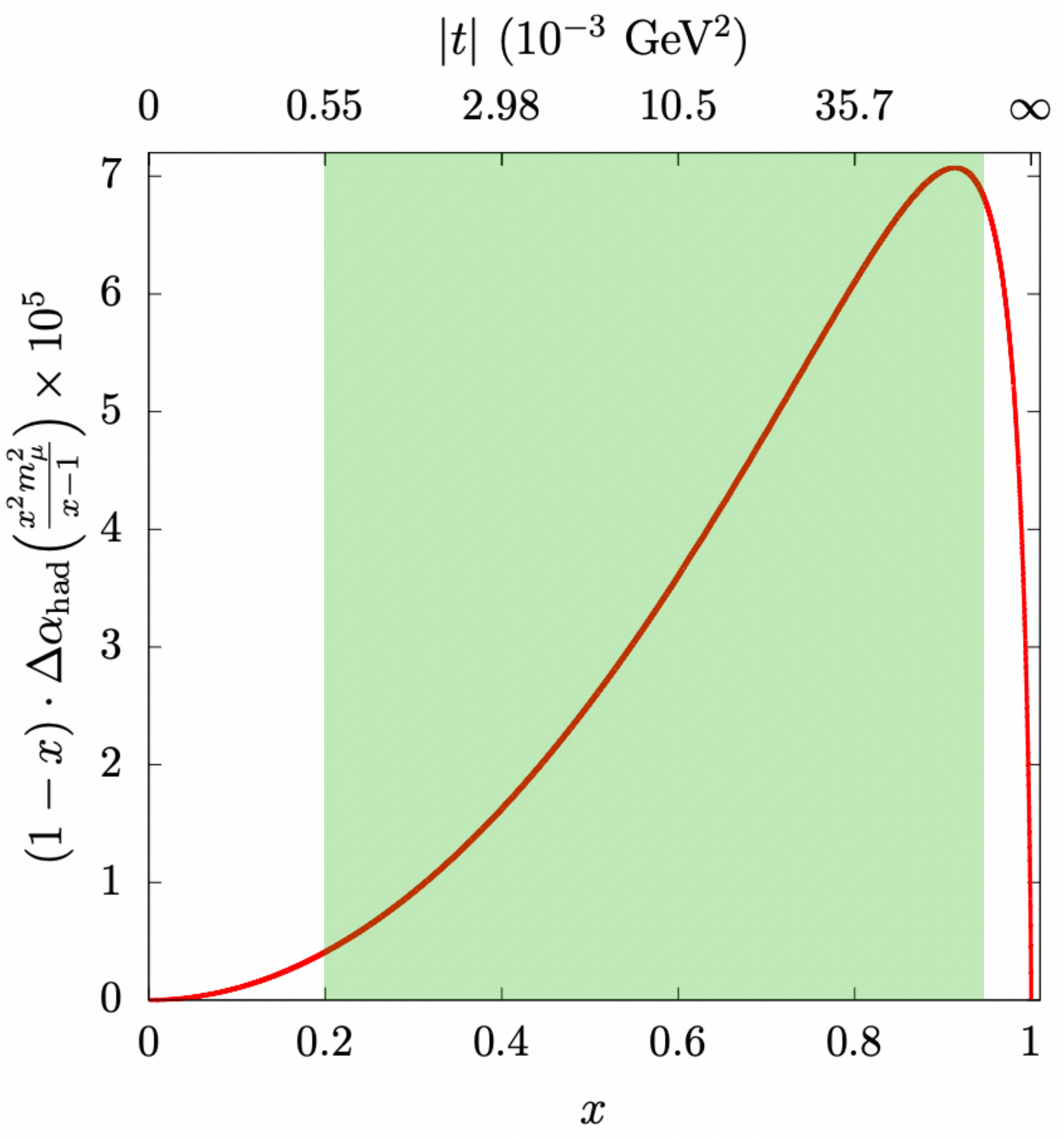
taken from F. Jegerlehner. Nucl Phy proc suppl (2008)

$x_{peak} \simeq 0.914$

$t_{peak} \simeq -0.108\ \mathrm{GeV}^2$



Here is our motivation !!



Finding a reliable method to fit the data + good ***extrapolation*** outside data region

<u>This fitting method must be:</u>

- A very precise fitting
- A fast convergence method to the original function
- It must have the same analytical properties as the original function

Stieltjes function

$$f(z) = \int_0^\infty \frac{\mathrm{d}\phi(u)}{1+zu} \qquad \phi(u) \text{ is a measure in } u \in [0,\infty)$$

$$f(z) = \sum_{i=0}^{\infty} f_i (-z)^i, \qquad f_i = \int_0^\infty u^i \, \mathrm{d}\phi(u)$$

Stieltjes series

$$\begin{vmatrix} f_m & f_{m+1} & \cdots & f_{m+n} \\ f_{m+1} & f_{m+2} & \cdots & f_{m+n+1} \\ \vdots & \vdots & & \vdots \\ f_{m+n} & f_{m+n+1} & \cdots & f_{m+2n} \end{vmatrix} > 0 \qquad m \geq 0 \quad n \geq 0$$

determinant condition

Aubin, Blum, Golterman, Peris (2012) Masjuan, Peris (2009)

- $\Delta\alpha_{\text{had}}(t)$ is a Stieltjes function in $t \in (-\infty, 0]$ since HVP correlator is a Stieltjes function

$$\Delta\alpha_{\text{had}}(t) = \sum_{i=1}^{\infty} a_i \, t^i \longrightarrow 0 < a_i < a_{i+1}, \qquad i \in \mathbb{N}$$

hierarchy

Model of Greynat and de Rafael

$$\Delta\alpha_{\rm had}[q^2] = \bar{\Pi}(q^2) = q^2 \int_{4m_\pi^2}^{\infty} ds \frac{\operatorname{Im}\Pi(s)}{s(s - q^2 + i\epsilon)}$$

$$\operatorname{Im}\Pi_{\rm had}(s) = \frac{1}{4\pi}\left(1 - \frac{4m_\pi^2}{s}\right)^{3/2}\left(\frac{|F(s)|^2}{12} + \sum_{i=u,d,\ldots} Q_i^2\,\Theta(s, s_c, \Delta)\right)\theta(s - 4m_\pi^2)$$

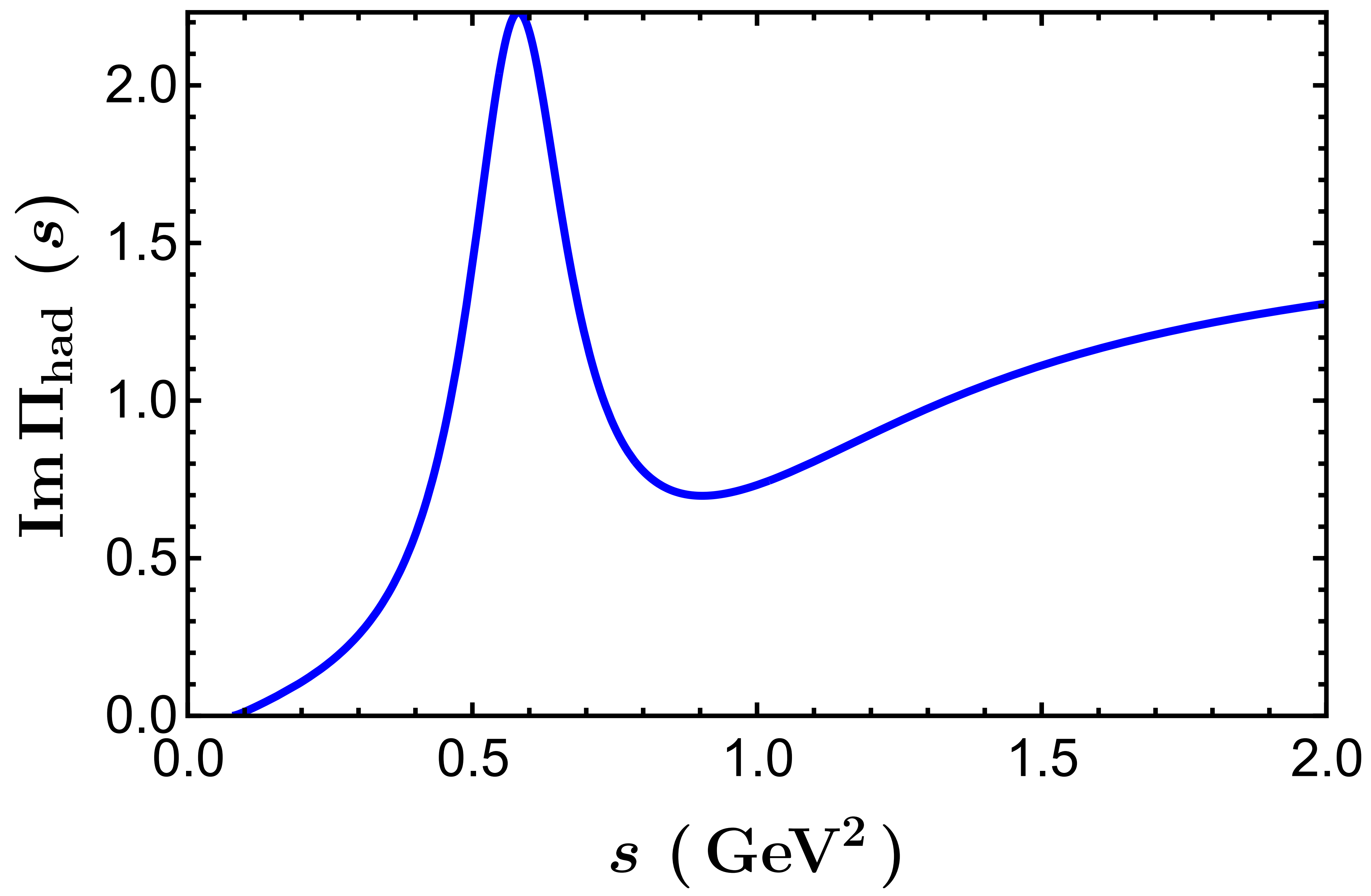
Greynat, de Rafael (2022)

model used to generate toy data

$$|F(s)|^2 = \frac{m_\rho^4}{(m_\rho^2 - s)^2 + m_\rho^2\,\Gamma(s)^2}$$

$$\Gamma(s) = \frac{m_\rho\, s}{96\pi f_\pi^2}\left[\left(1 - \frac{4m_\pi^2}{s}\right)^{3/2}\theta(s - 4m_\pi^2) + \frac{1}{2}\left(1 - \frac{4m_k^2}{s}\right)^{3/2}\theta(s - 4m_k^2)\right]$$

$$\Theta(s) = \frac{2}{\pi}\left[\frac{\arctan\left(\frac{s - s_c}{\Delta}\right) - \arctan\left(\frac{4m_\pi^2 - s_c}{\Delta}\right)}{\frac{\pi}{2} - \arctan\left(\frac{4m_\pi^2 - s_c}{\Delta}\right)}\right]$$



Fitting Method

- Fitting function from PAs — example

matching coefficients

$$\Delta\alpha_{\text{had}}(t) = a_1\, t + a_2\, t^2 + \ldots$$ unknown Taylor series coefficients

$$P_1^1(t) = \frac{q_0 + q_1\, t}{1 + r_1\, t} \approx q_0 + (q_1 - q_0 r_1)t + (q_0 r_1^2 - q_1 r_1)t^2 + \ldots$$

$$P_1^1(t) = \frac{a_1^2\, t}{1 - a_2\, t}$$

$$t = -\frac{x^2\, m_\mu^2}{1 - x}$$

$$P_1^1(x) = -\frac{a_1^2\, m_\mu^2\, x^2}{a_1 - a_1\, x + a_2\, m_\mu^2\, x^2} = -\frac{b_1\, m_\mu^2\, x^2}{1 - x + b_2\, m_\mu^2\, x^2}$$

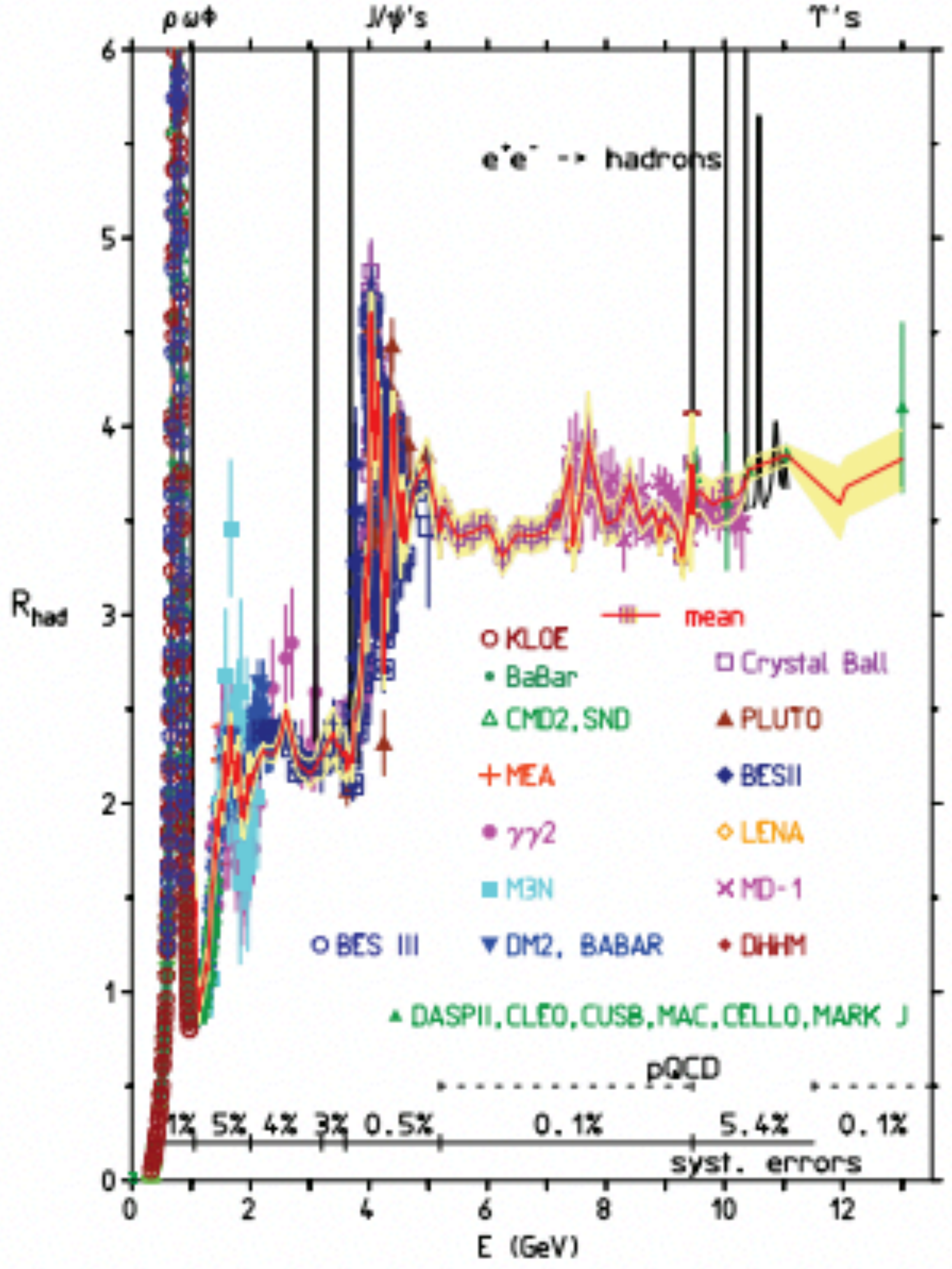
$$b_1 = a_1 < 0 \qquad b_2 = \frac{a_2}{a_1} > 1$$

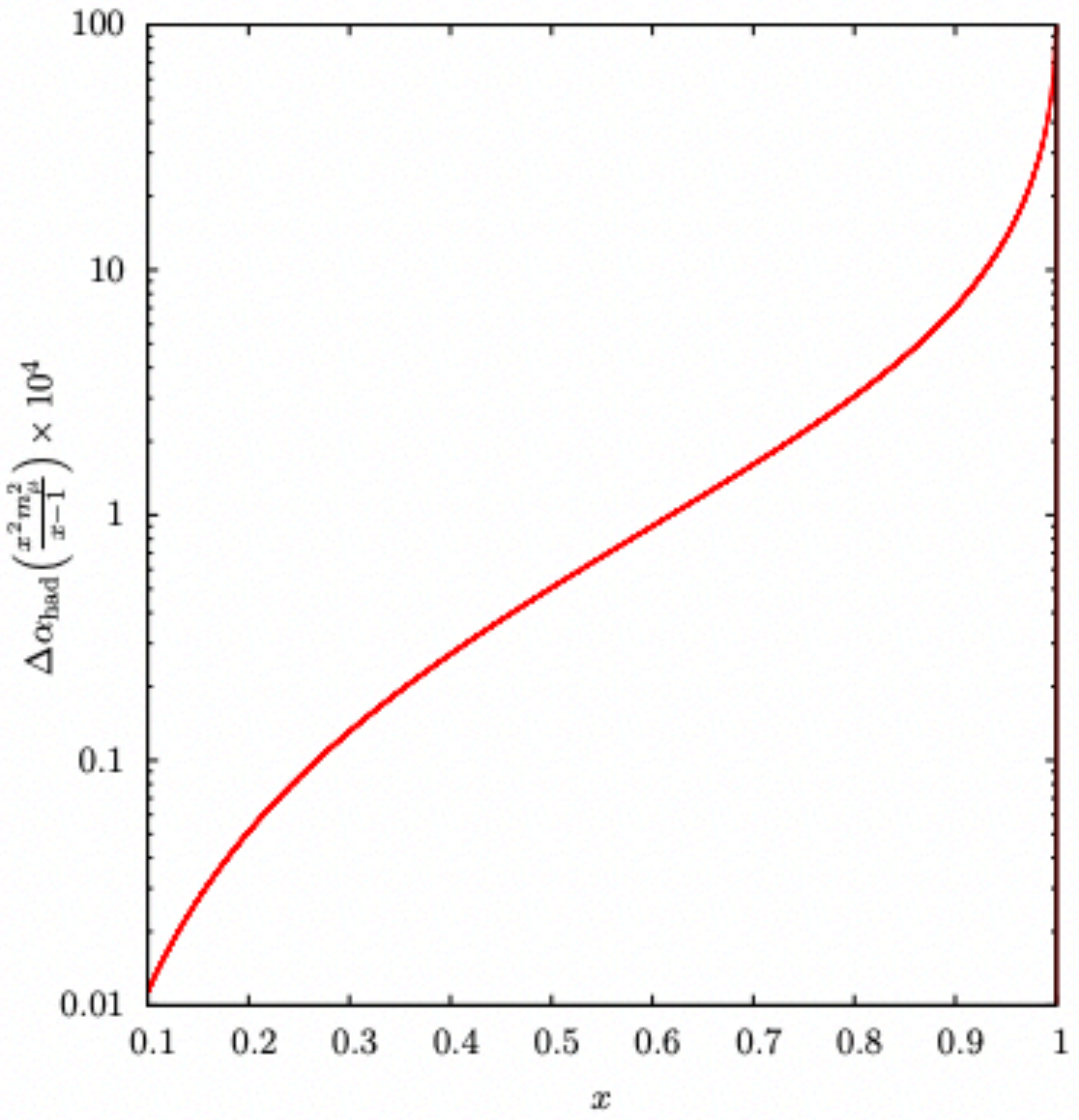
model-independent constraints

- Modified χ^2 function with penalties

$$\chi^2 = \sum_{i,j=1}^{30} \left[\alpha\, \Delta\alpha_{\text{had}}(x_i) \times 10^5 - P_N^M(x_i)\right] (C^{-1})_{ij} \left[\alpha\, \Delta\alpha_{\text{had}}(x_j) \times 10^5 - P_N^M(x_j)\right] + n_{\text{dof}} \sum_{i=2}^{N+M} \theta(a_i - a_{i-1})$$

Time-like $\mapsto$ Space-like





Model parametrisation used by MUonE team

Physics-inspired from the calculable contribution of lepton-pairs and top quarks at t<0

$q^2 = t < 0$

$$\Delta\alpha_{had}(t) = k\left\{-\frac{5}{9} - \frac{4M}{3t} + \left(\frac{4M^2}{3t^2} + \frac{M}{3t} - \frac{1}{6}\right)\frac{2}{\sqrt{1-\frac{4M}{t}}}\log\left|\frac{1-\sqrt{1-\frac{4M}{t}}}{1+\sqrt{1-\frac{4M}{t}}}\right|\right\}$$

M with dimension of mass squared, related to the mass of the fermion in the vacuum polarization loop
k depending on the coupling $\alpha(0)$, the electric charge and the colour charge of the fermion

Ref: *Giovanni Abbiendi* *arXiv:2201.13177v1* ***[physics.ins-det]***

D-Log Padé Approximants

How is our D-Log type function ?

Make a Taylor series from a model in t-variable → Construct a PA sequence in the D-log space → Get the D-log Padé approximants to the original function in t → Analysing the structure of approximants and propose a general function to fit

$f(t)$ → $\tilde{P}_N^M[F(t)]$ → $\mathrm{D}_N^M(t)$

$$f(t) \cong \mathrm{D}_N^M(t) = f(0)\frac{t\, e^{\sum_{i=1}^{M-N+1} a_i t^i}}{(\mu_1 - t)^{\gamma_1}\dots(\mu_{N-1} - t)^{\gamma_{N-1}}}$$

Making change of variable

$t \to x$

	$D_M^N(t)$	$D_M^N(x)$
D_2^1	$\frac{-f_0\, t}{(r_1 - t)^{\gamma_1}}$	$\frac{f_0\, m_\mu^2\, x^2 (1-x)^{-1+\gamma_1}}{(r_1 - r_1 x + m_\mu^2 x^2)^{\gamma_1}}$
D_2^2	$\frac{-f_0\, t\, e^{\beta t}}{(r_1 - t)^{\gamma_1}}$	$\frac{f_0\, m_\mu^2\, x^2 (1-x)^{-1+\gamma_1}}{(r_1 - r_1 x + m_\mu^2 x^2)^{\gamma_1}} e^{\beta \frac{m_\mu^2 x^2}{(x-1)}}$
D_3^2	$\frac{-f_0\, t}{(r_1 - t)^{\gamma_1} (r_2 - t)^{\gamma_2}}$	$\frac{f_0\, m_\mu^2\, x^2 (1-x)^{-1+\gamma_1+\gamma_2}}{(r_1 - r_1 x + m_\mu^2 x^2)^{\gamma_1} (r_2 - r_2 x + m_\mu^2 x^2)^{\gamma_2}}$
D_3^3	$\frac{-f_0\, t\, e^{\beta t}}{(r_1 - t)^{\gamma_1} (r_2 - t)^{\gamma_2}}$	$\frac{f_0\, m_\mu^2\, x^2 (1-x)^{-1+\gamma_1+\gamma_2}}{(r_1 - r_1 x + m_\mu^2 x^2)^{\gamma_1} (r_2 - r_2 x + m_\mu^2 x^2)^{\gamma_2}} e^{\beta \frac{m_\mu^2 x^2}{(x-1)}}$

Fitting function

An example of Padé's approximant (dataset without errors)

First we can consider a data point set without uncertainties just to see the behaviour of the Padé approximant

Example: Padé Approximant (2 dofs)

$P_1^1(t)$

$a_\mu^{PA} = 6933 \times 10^{-11} \quad (0.9\%)$

In terms of Taylor Coeffs. (a_i)

$$\frac{a_1\, t}{1 - a_2\, t}$$

$a_1 = -912\ \text{GeV}^{-2} \quad (0.7\%)^\dagger$

$a_2 = -1489\ \text{GeV}^{-4} \quad (15\%)^\dagger$

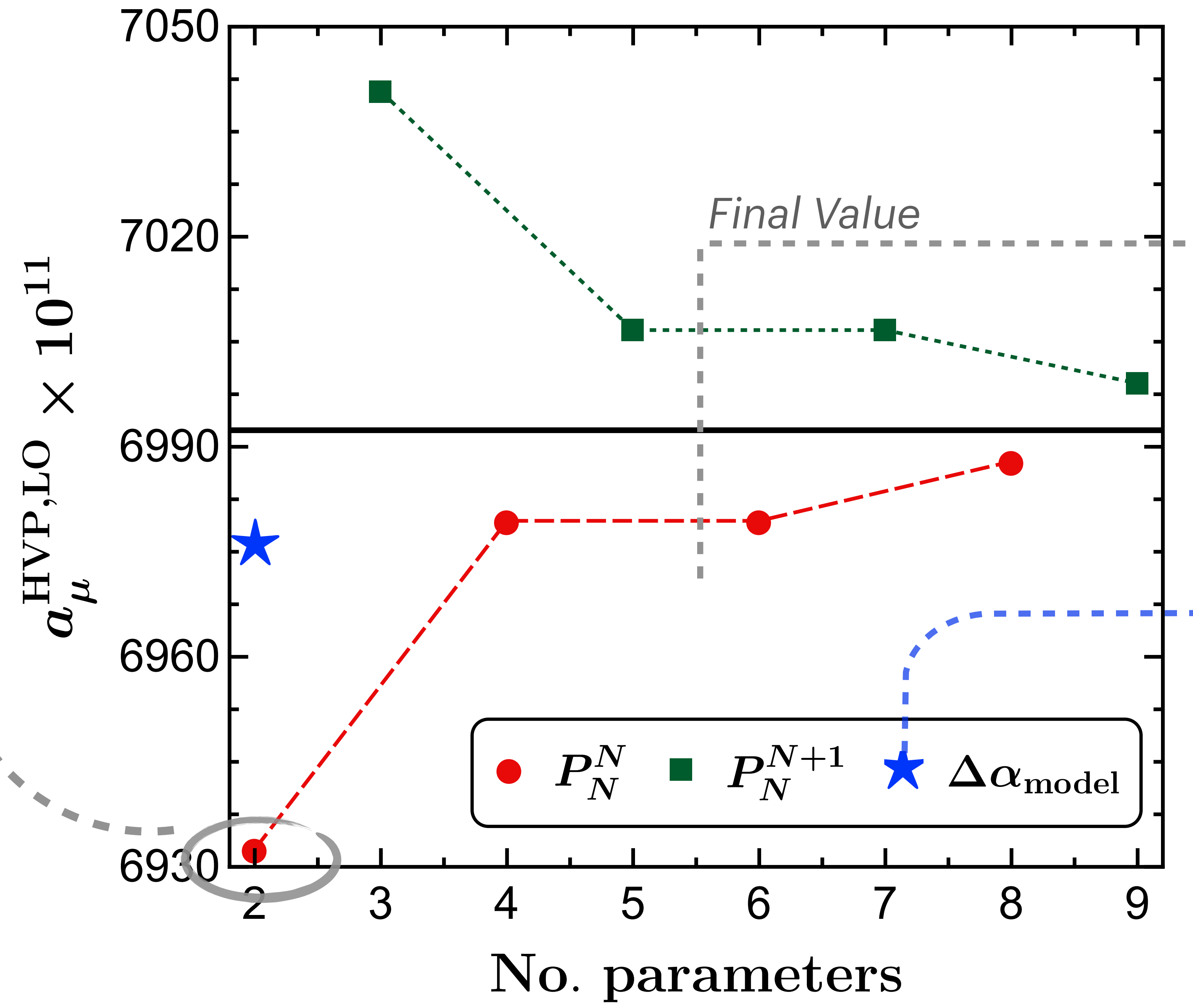
$t \rightarrow x$

$$\frac{b_1\, m_\mu^2\, x^2}{1 - x + b_2\, m_\mu^2\, x^2}$$

$\chi^2_{min}/\text{dof} = 1.18 \times 10^{-3}$

p-value = 1

† Relative error from Taylor coefficients of the model



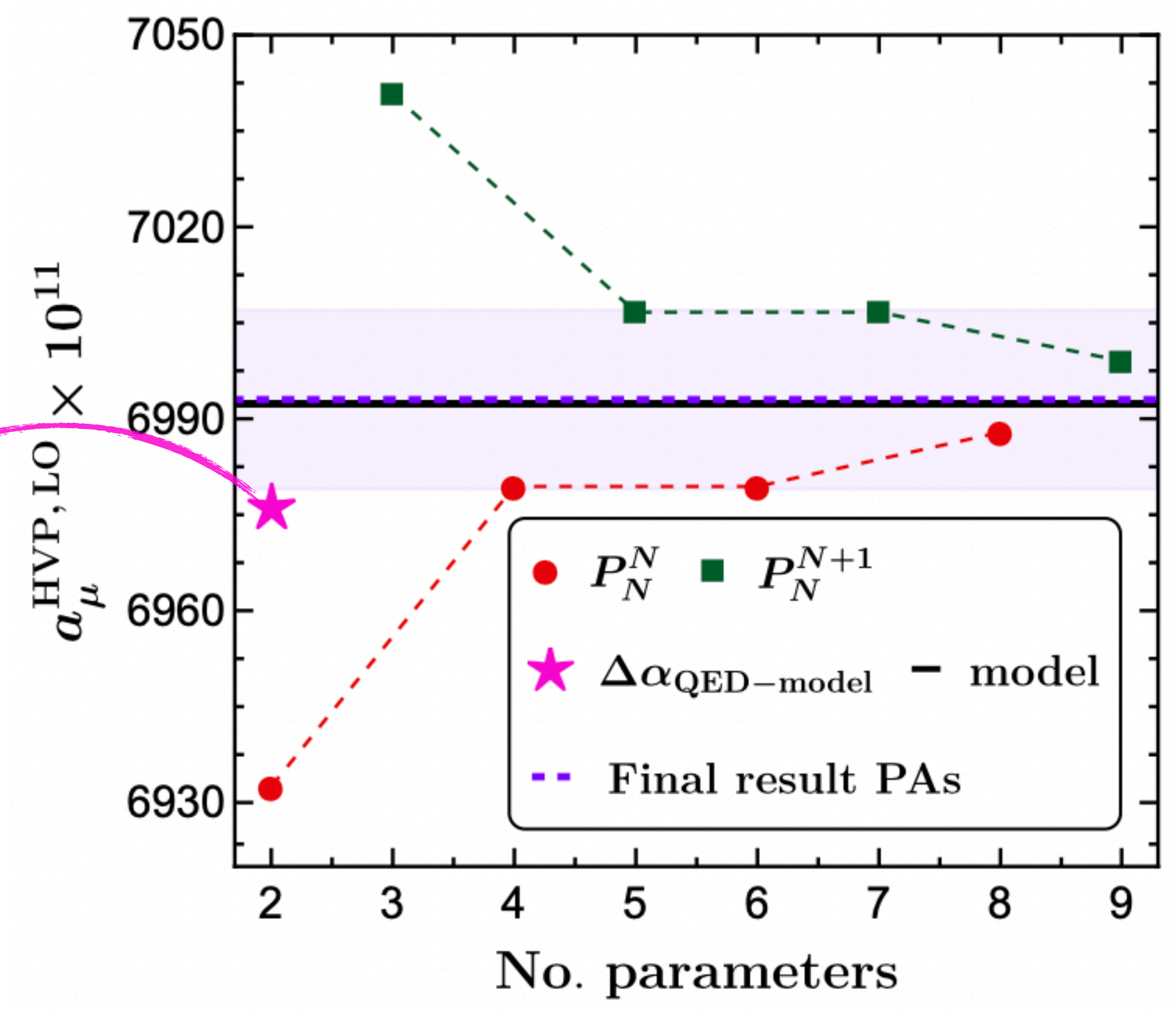
$\bar{a}_\mu^{PAs} = 6991 \times 10^{-11}$, (0.02%)

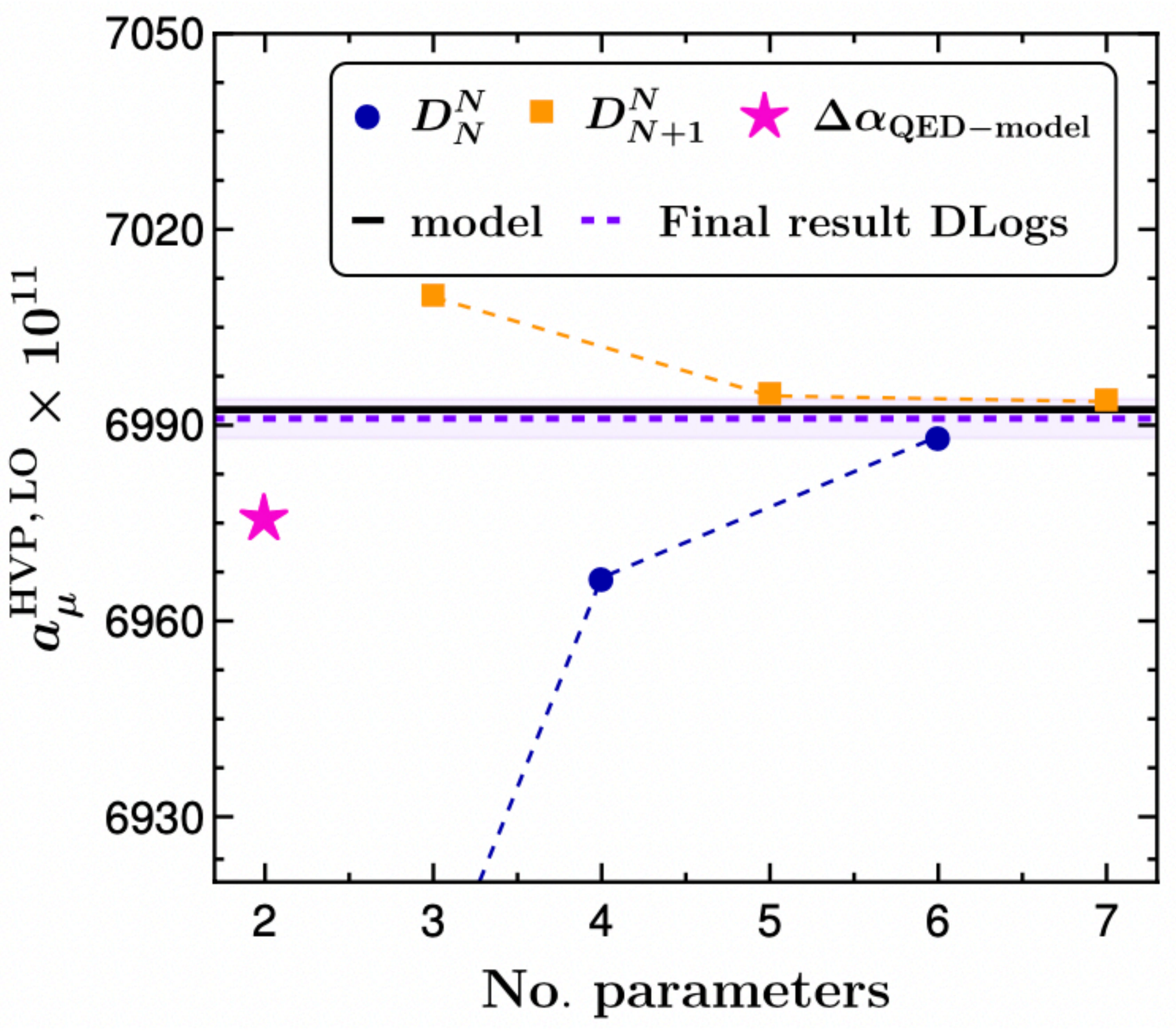
We compute a weighted average for the final value of a_μ^{HLO} (*weights = N*)

$a_\mu^{LO,HVP}$ computed using the parametrisation used by the MUonE team

- The fitting functions for the two sequences proposed (P_N^N and P_N^{N+1}) can be obtained in the same way (results in the right plot)

Fitting to data with no errors





$$\Delta\alpha_{\mathrm{QED-model}}(t) = KM\left[-\frac{5}{9} - \frac{4M}{3t} + \frac{2\left(\frac{4M^2}{3t^2} + \frac{M}{3t} - \frac{1}{6}\right)}{\sqrt{1-\frac{4M}{t}}}\log\left|\frac{1-\sqrt{1-\frac{4M}{t}}}{1+\sqrt{1-\frac{4M}{t}}}\right|\right]$$

[Giovanni Abbiendi ´22] (arXiv:2201.13177v1)

- We confirmed the pattern of convergence
- Some defect effects at order 6 and 7 in PAs
- The systematic uncertainty that we can achieve at each order