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Analyzing the D*D*D* system:
Hexaquark states and the Efimov effect

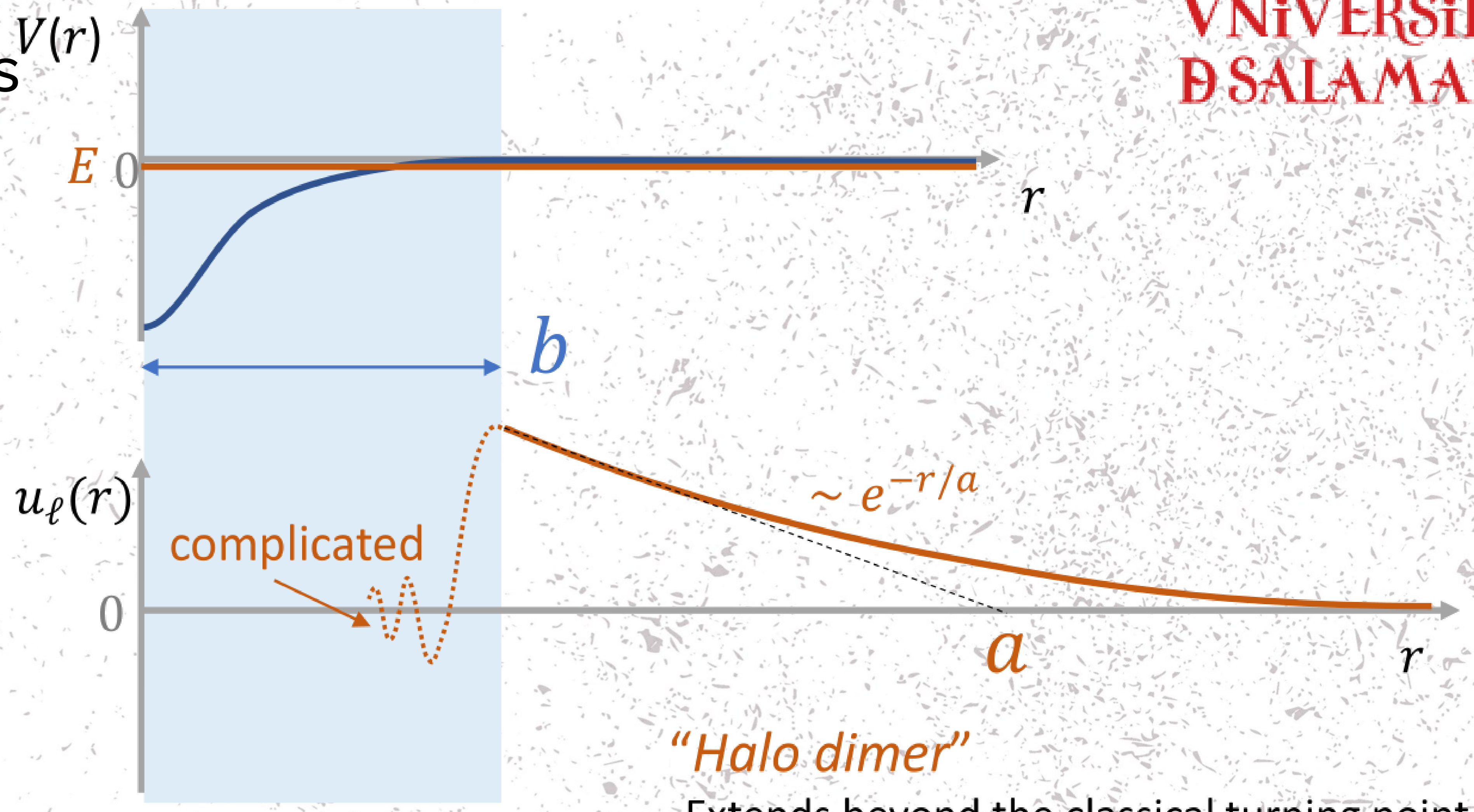
P. G. Ortega

Details in Arxiv: 2403.10344 [hep-ph]

Two body universality

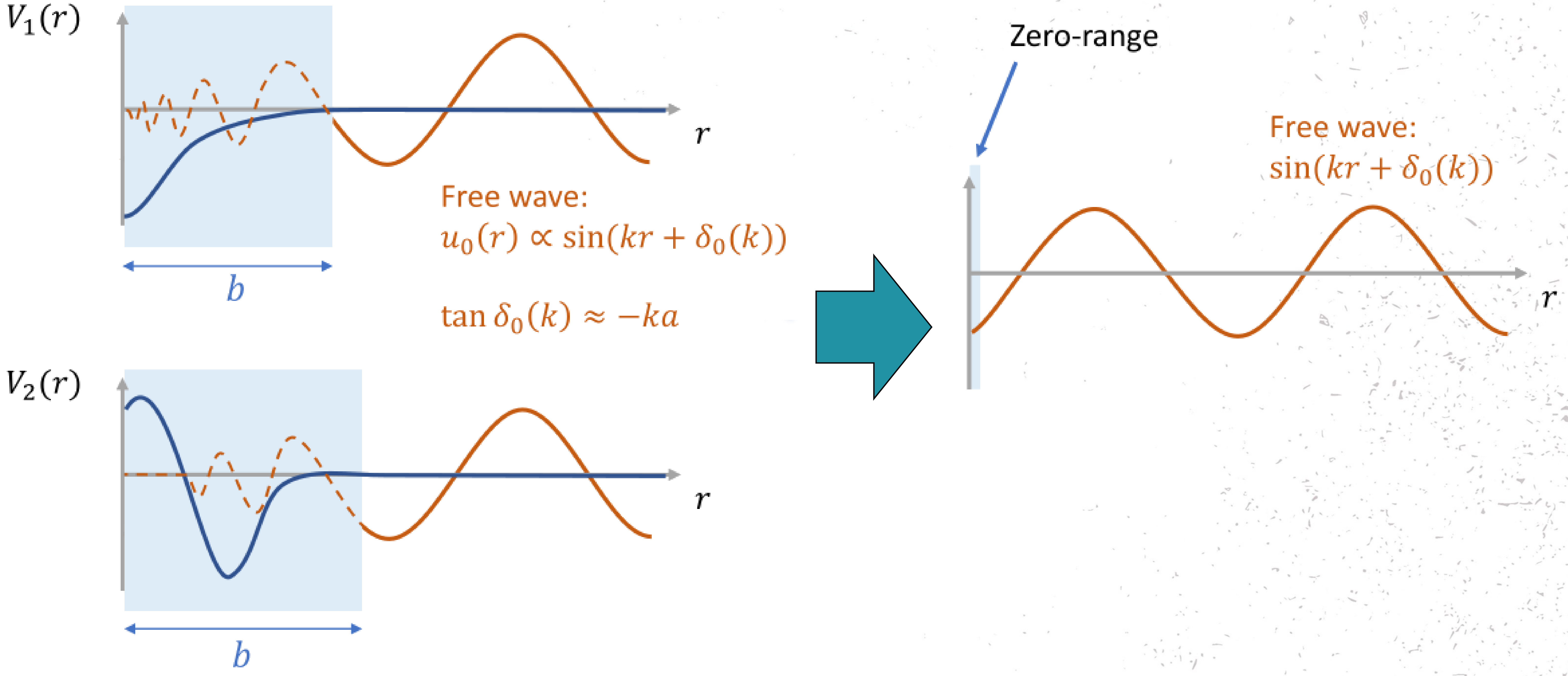
- ✓ Nearly-resonant two-body state due to short-range attractive forces
  → *Characterized by its scattering length*

$$B_2 \longrightarrow \frac{1}{2m_{\text{red}}a^2} \qquad \psi(r) \longrightarrow (2\pi a)^{-1/2}\frac{\exp(-r/a)}{r}$$



"Halo dimer"
Extends beyond the classical turning point

- ✓ System becomes *universal* → Insensitive to short-range details



Free wave:
$u_0(r) \propto \sin(kr + \delta_0(k))$

$\tan\delta_0(k) \approx -ka$

Free wave:
$\sin(kr + \delta_0(k))$

Two body universality

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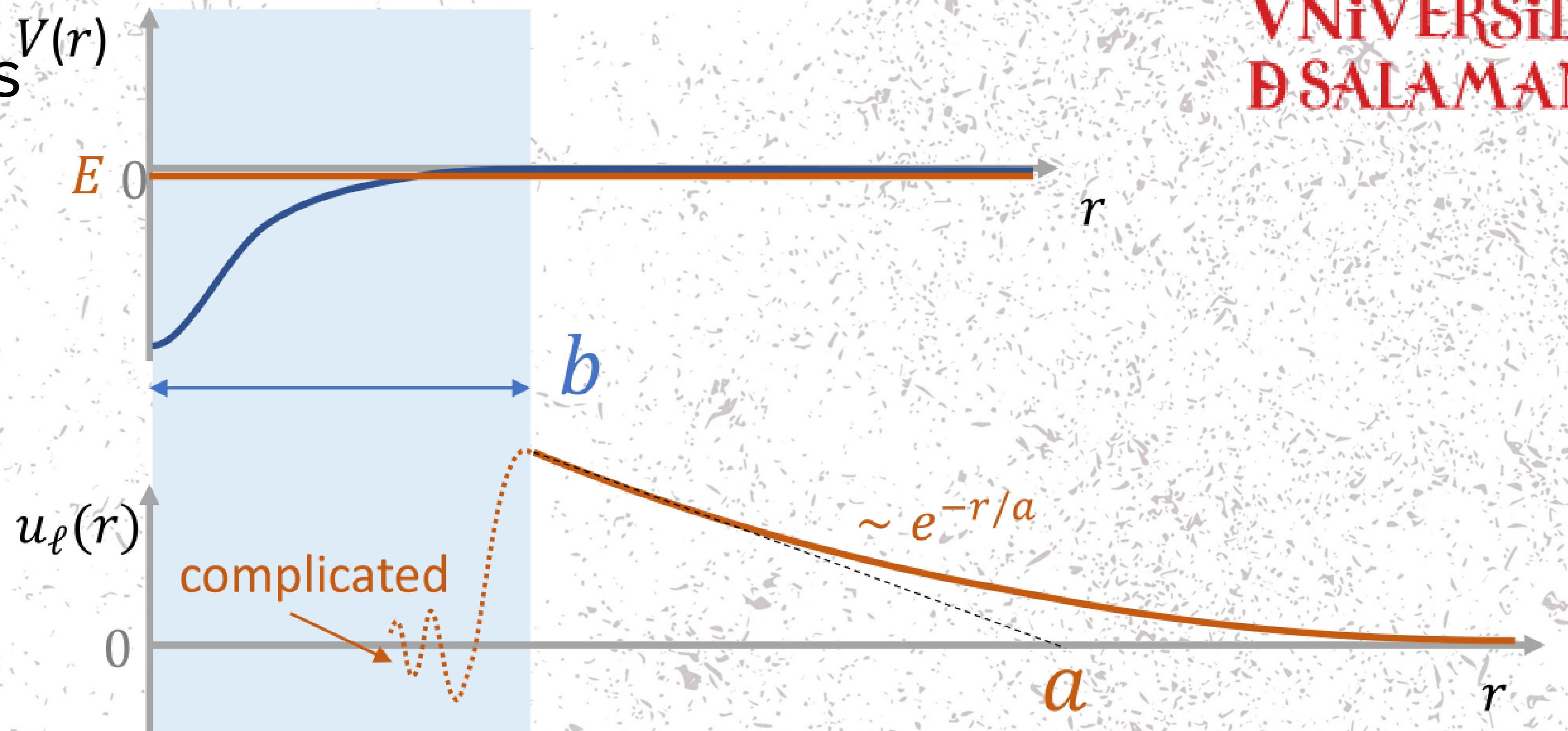
$$B_2 \longrightarrow \frac{1}{2m_{\text{red}}a^2} \qquad \psi(r) \longrightarrow (2\pi a)^{-1/2}\frac{\exp(-r/a)}{r}$$

- ✓ Effective long-range three-body force → ***Efimov attraction***



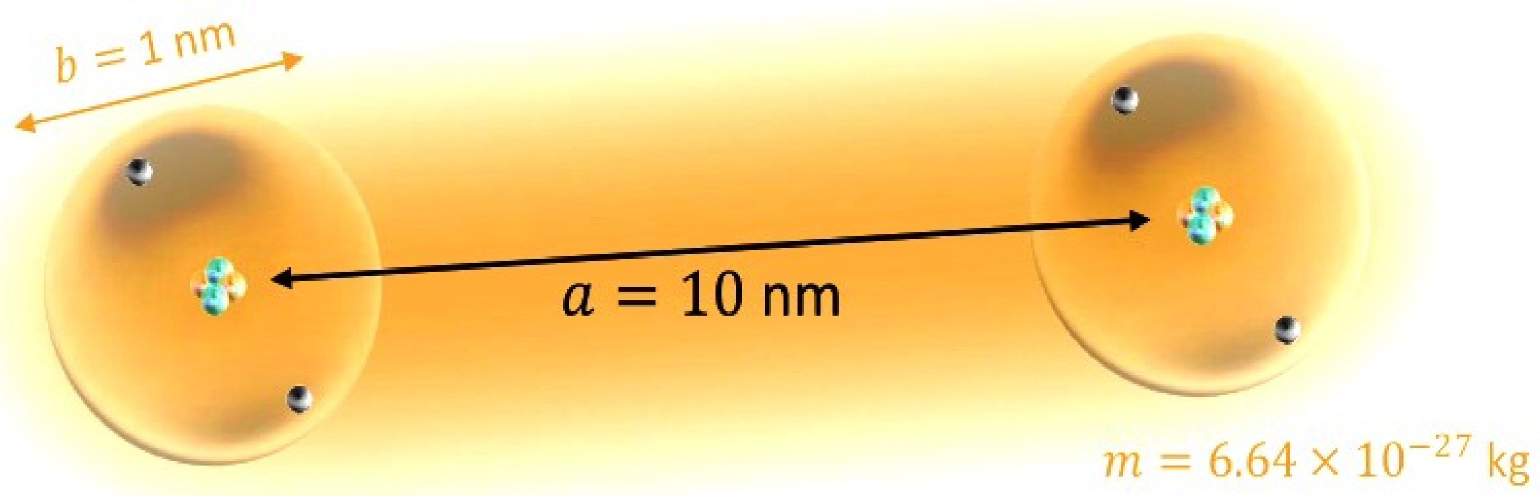
"Halo dimer"

Extends beyond the classical turning point

$$V(R) \approx -\frac{\hbar^2}{mR^2}$$

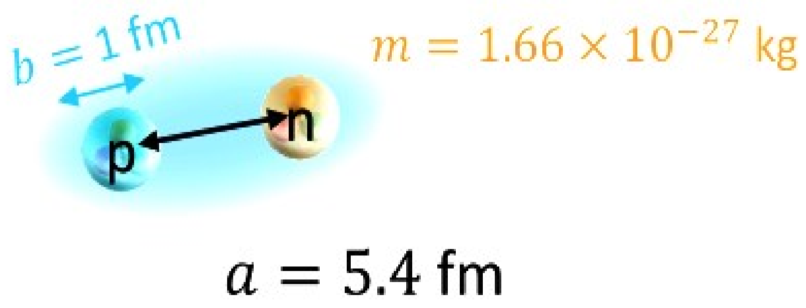
Atomic world Electromagnetic force

Helium-4 dimer



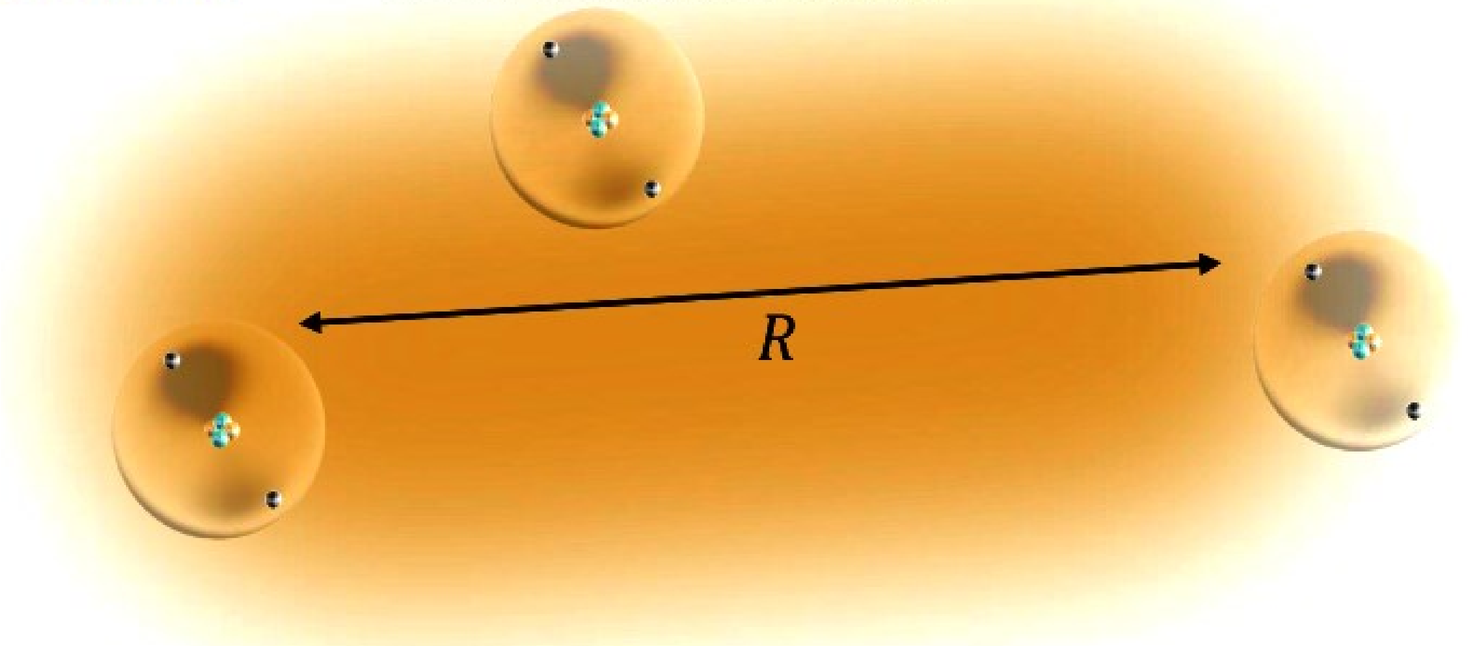
Subatomic world Strong force

Deuteron



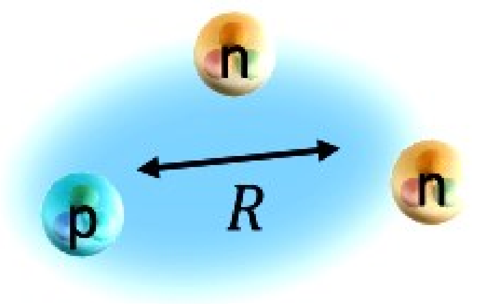
Atomic world Electromagnetic force

Helium-4 trimer



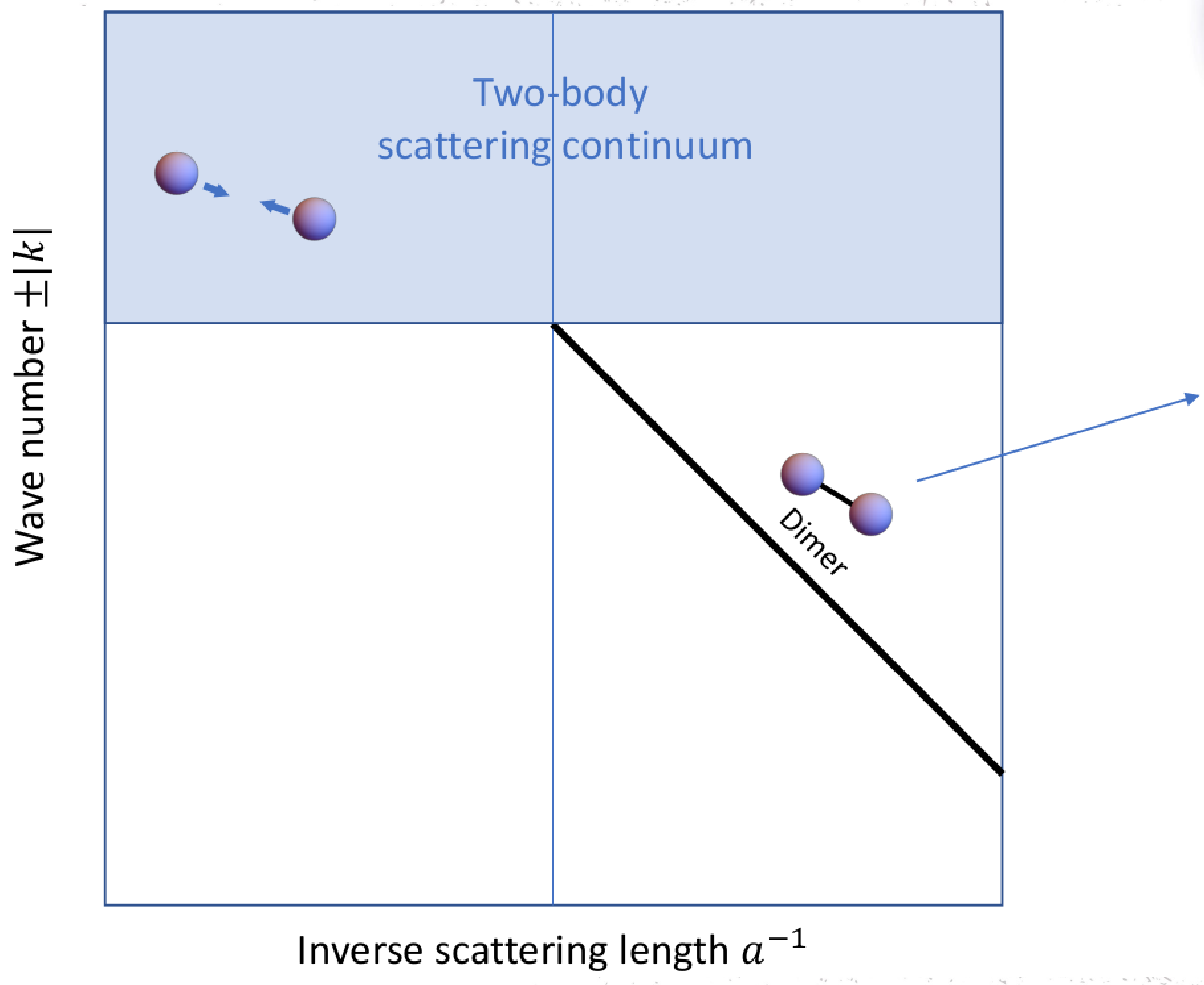
Subatomic world Strong force

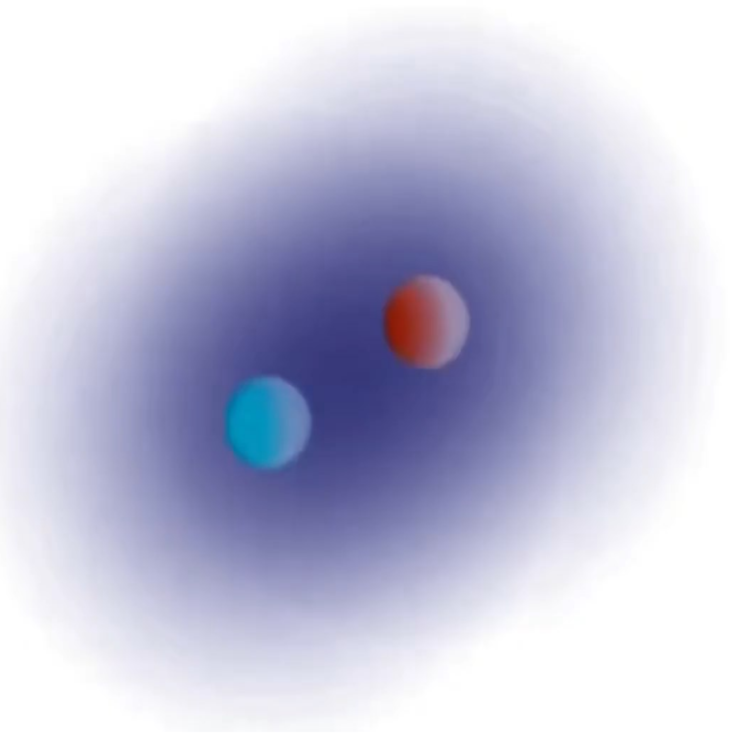
Triton



Efimov effect

- Two-body spectrum in the zero-range theory





$$\psi(\boldsymbol{r}) = \frac{1}{\sqrt{2\pi a r}} e^{-r/a}$$

$$E = -\frac{\hbar^2}{2\mu}\kappa^2$$

With

$$\kappa = a^{-1}$$

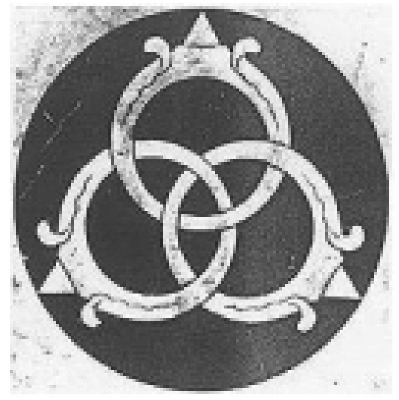
From P. Naidon, RIKEN

Efimov effect

- Three-body spectrum in the zero-range theory

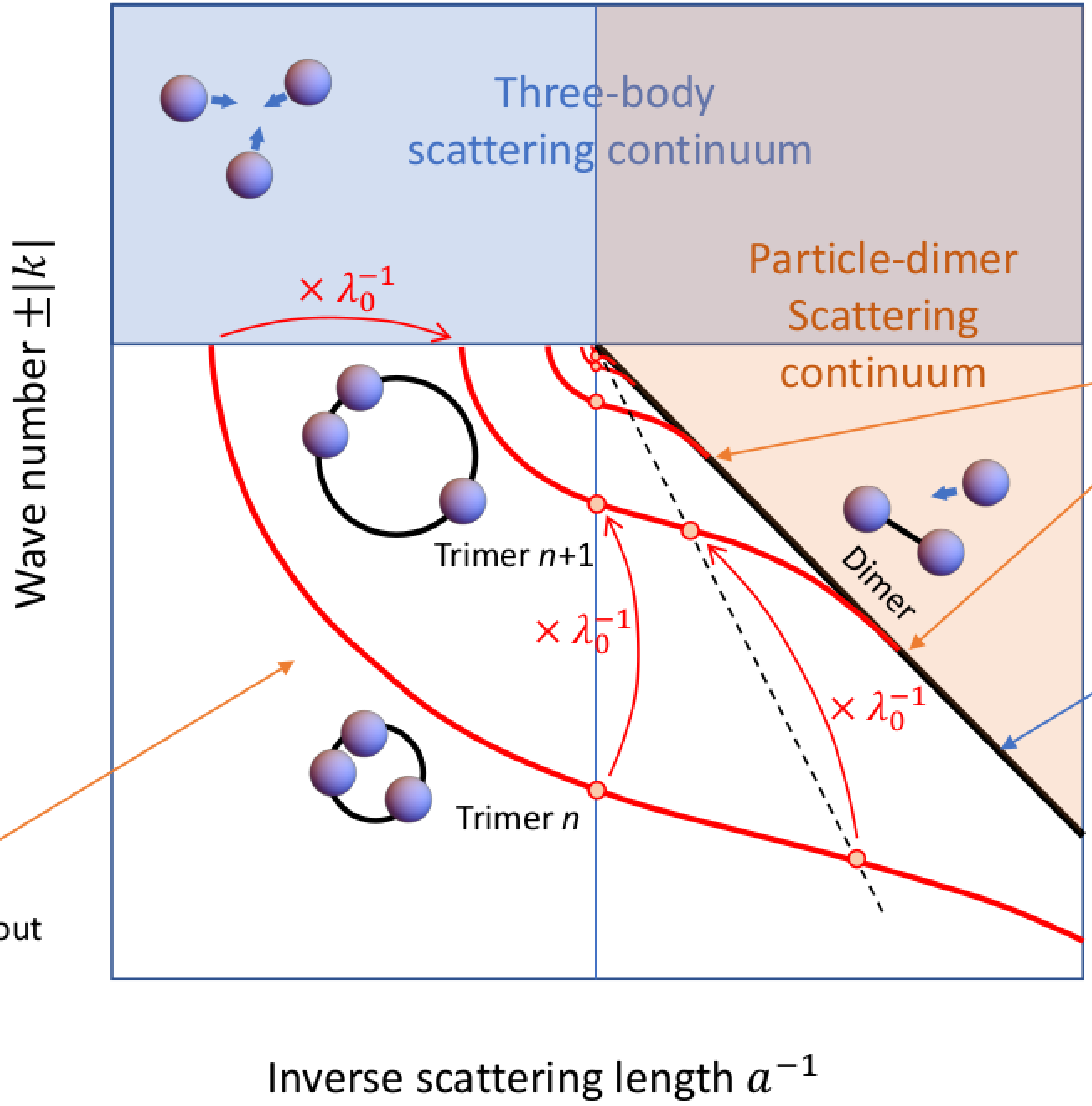
1. Discrete scale invariance

Infinite number of three-body bound states.



2. Borromean states

Three-body bound states without two-body bound states.



3. Trimer dissociation with increasing interaction

$$E_2 = -\frac{\hbar^2 \kappa^2}{m}$$

with $\kappa = a^{-1}$

$$\frac{\Delta E^{(n+1)}}{\Delta E^{(n)}} \to \lambda^{-2} \approx \frac{1}{515}$$

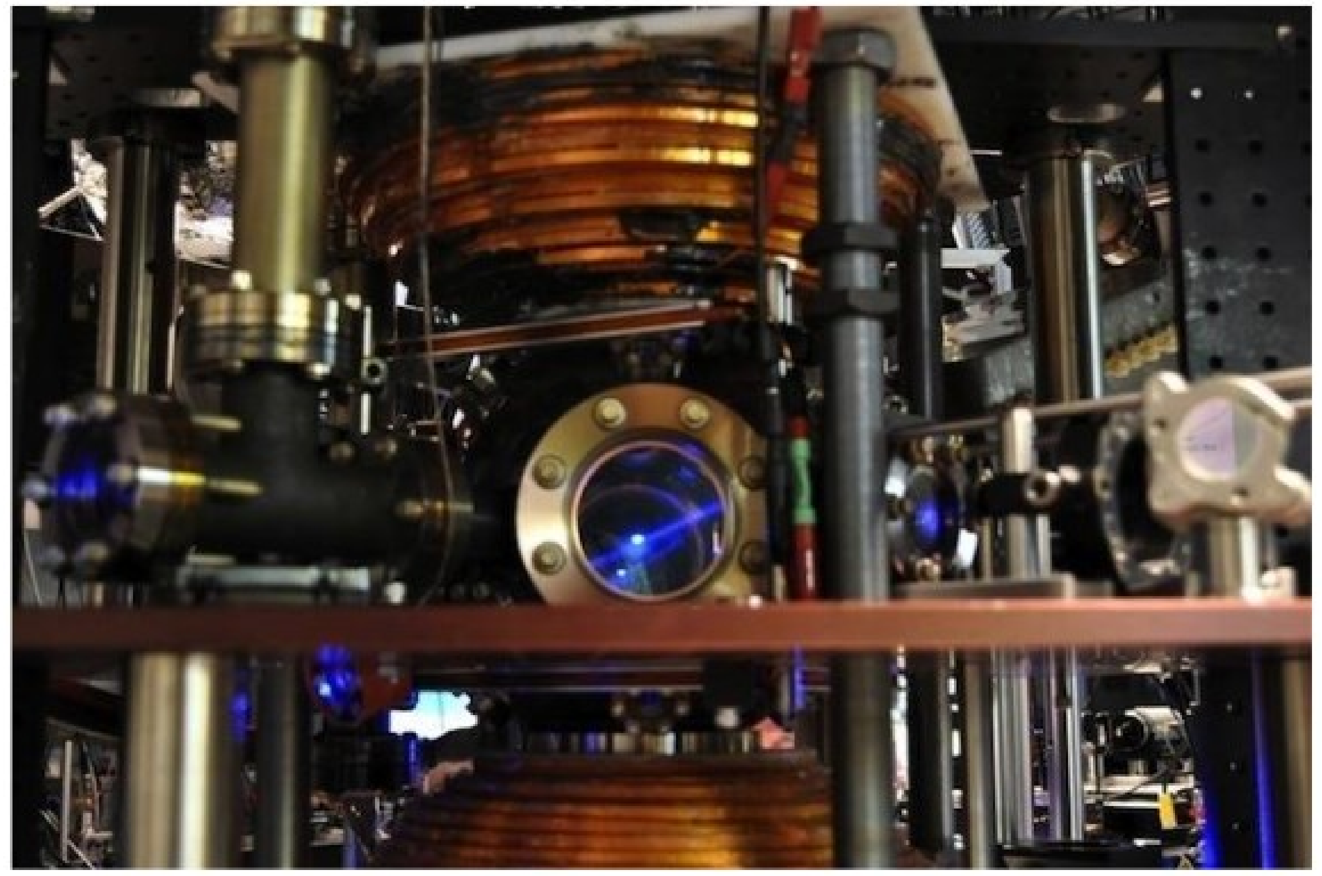
$\lambda_0 \sim 22.7$

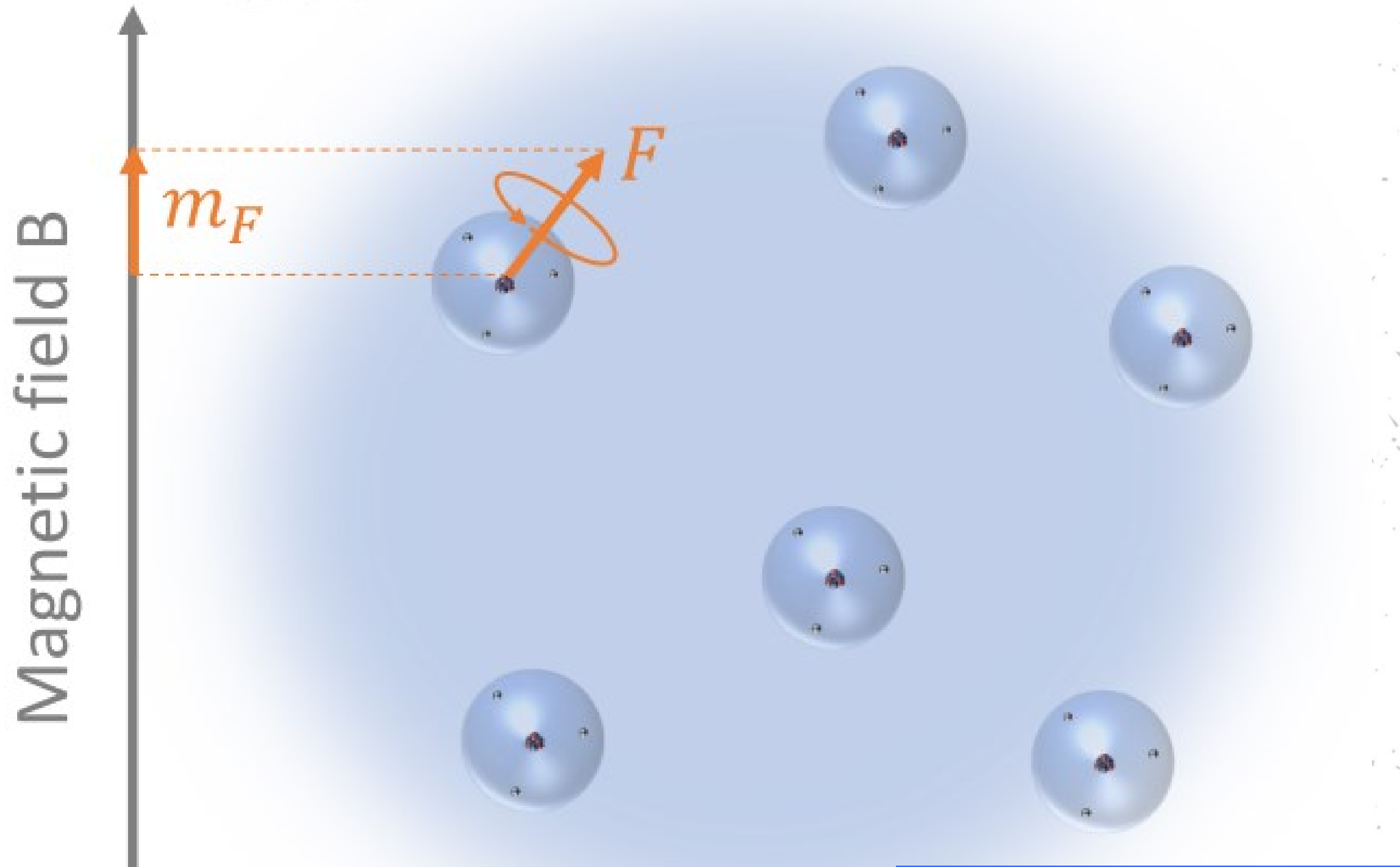
From P. Naidon, RIKEN

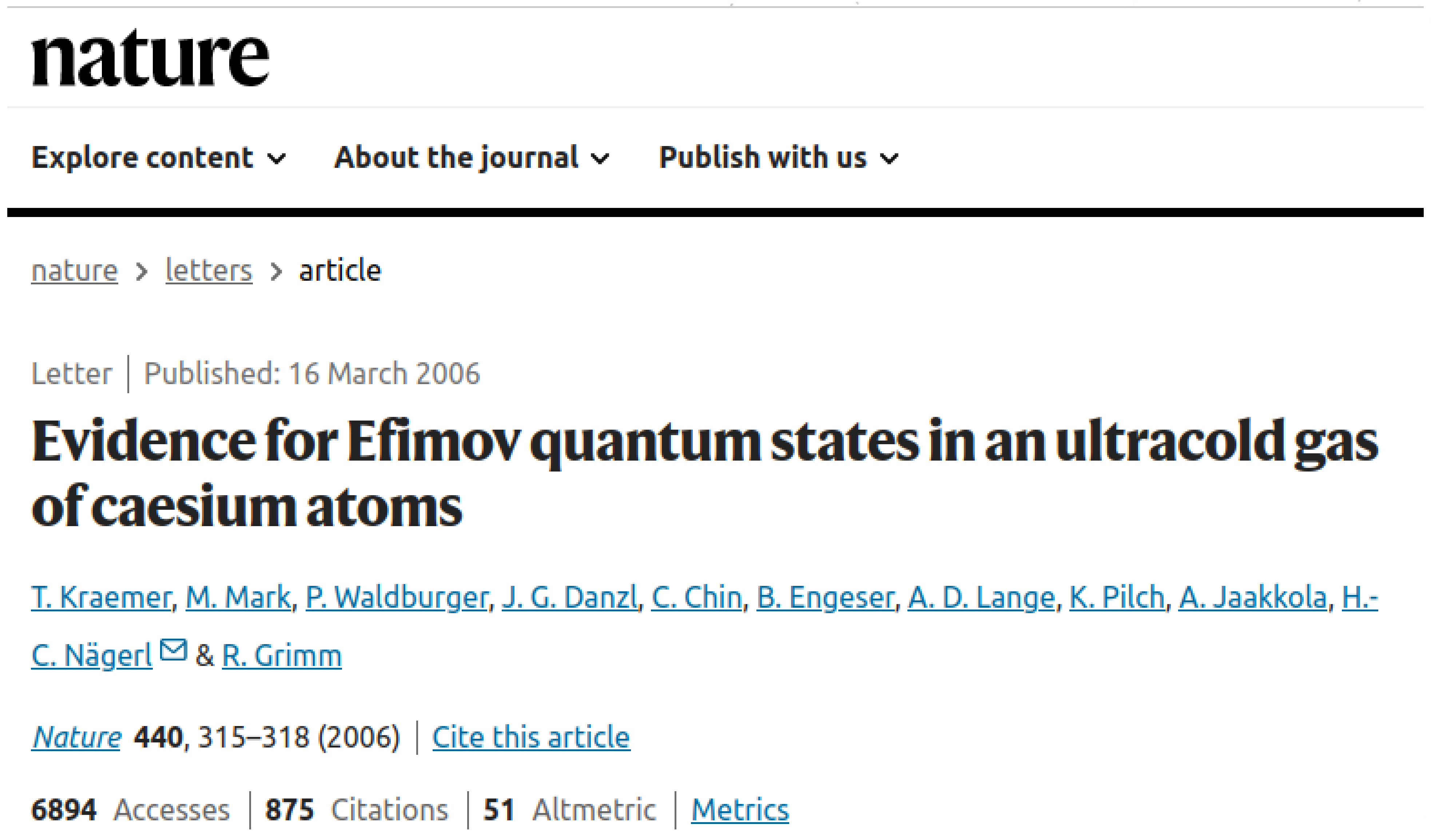
Evidence for Efimov states

- ✓ **Ultra-cold atomic gasses:** Cesium, Helium,...

Cloud of atoms cooled to T < 1 μK in a vacuum chamber







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nature > letters > article

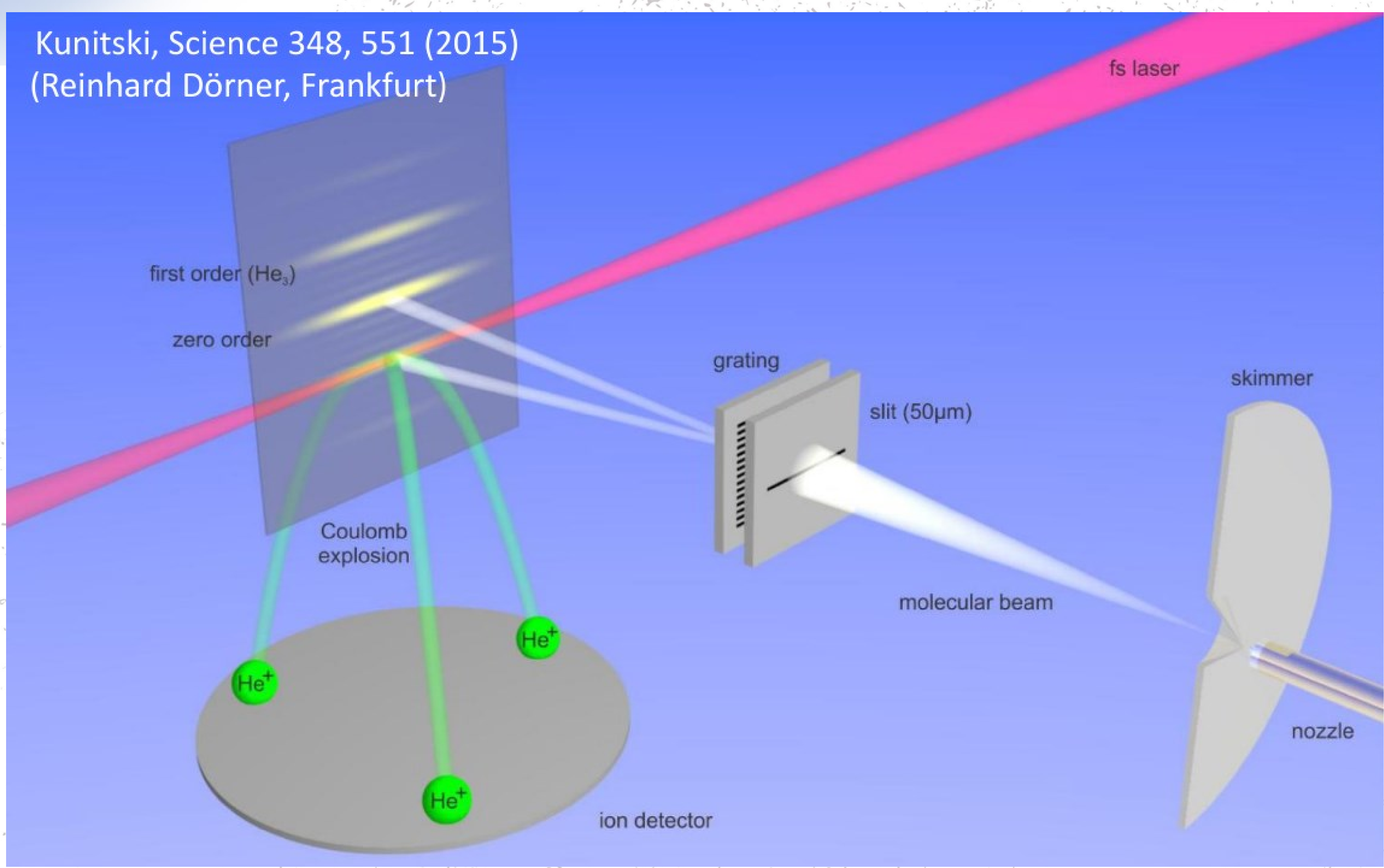
Letter | Published: 16 March 2006

Evidence for Efimov quantum states in an ultracold gas of caesium atoms

T. Kraemer, M. Mark, P. Waldburger, J. G. Danzl, C. Chin, B. Engeser, A. D. Lange, K. Pilch, A. Jaakkola, H.-C. Nägerl & R. Grimm

Nature **440**, 315–318 (2006) | Cite this article

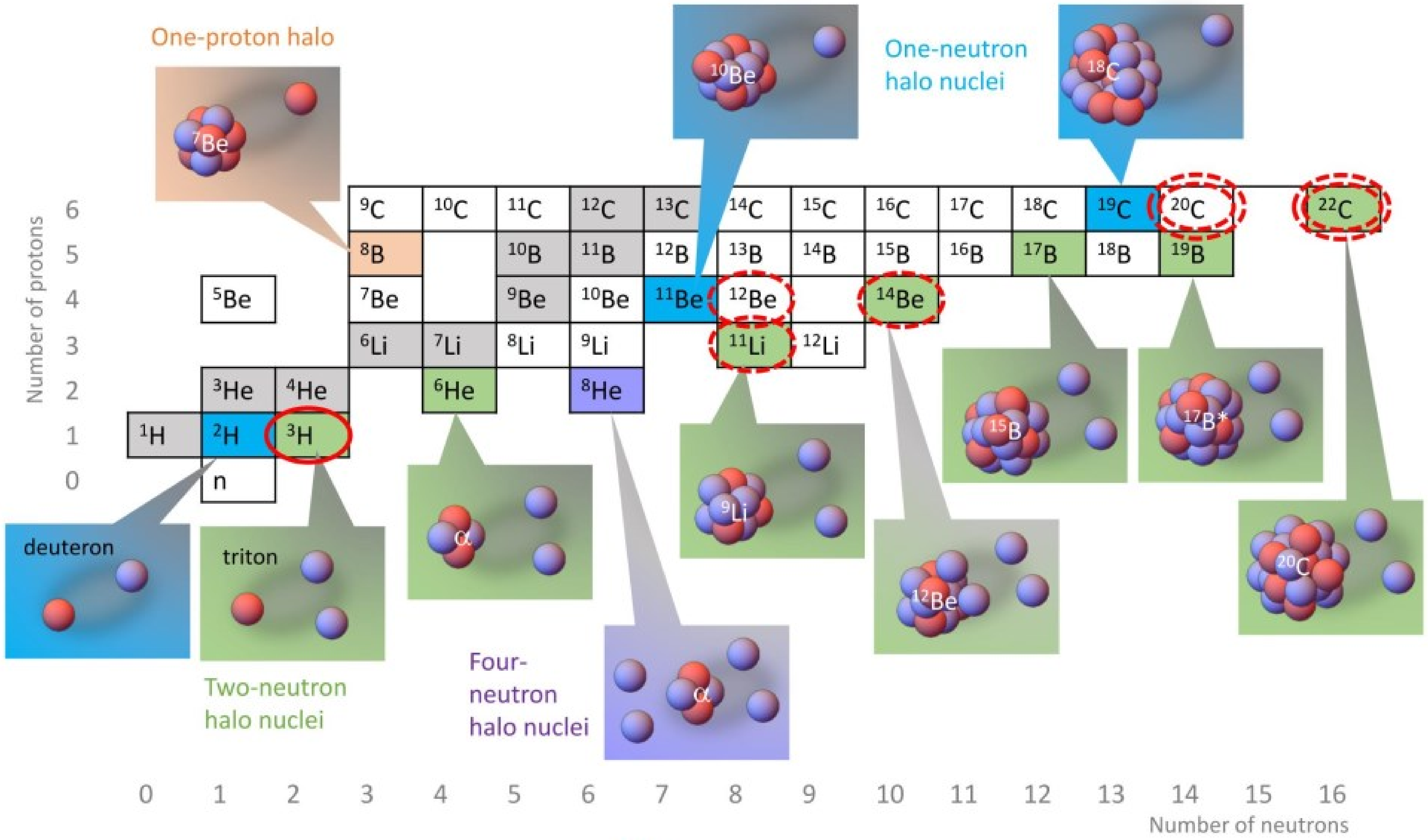
6894 Accesses | **875** Citations | **51** Altmetric | Metrics



Kunitski, Science 348, 551 (2015)
(Reinhard Dörner, Frankfurt)

Evidence for Efimov states

- **Ultra-cold atomic gasses:** Cesium, Helium,...
- **Halo nuclei:** ^{14}Be, ^{22}C, ^{20}C,...

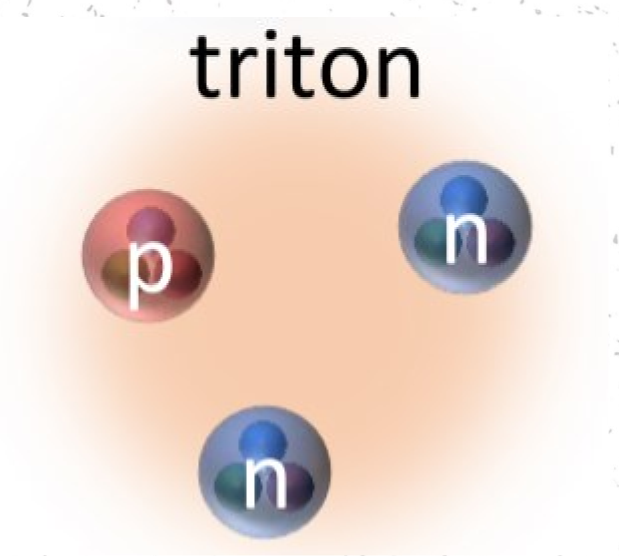


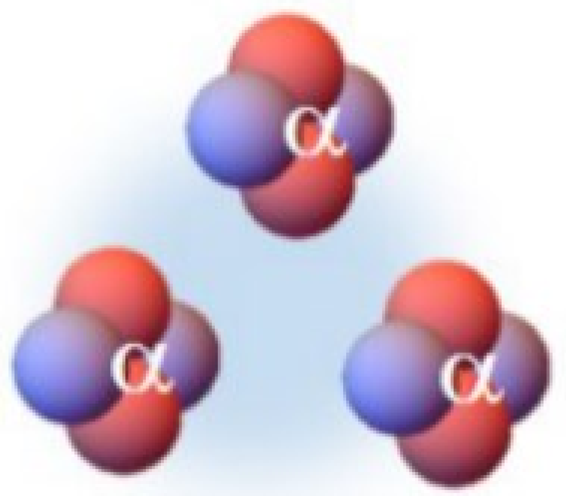
Efimov trimer candidates: ◌ Ground state

◎ First excited state (to be confirmed)

Evidence for Efimov states

- ✓ **Ultra-cold atomic gasses:** Cesium, Helium,...
- ✓ **Halo nuclei:** ^{14}Be, ^{22}C, ^{20}C,...
- ✓ Triton
- ✓ 3 α's Hoyle state of ^{12}C





Evidence for Efimov states

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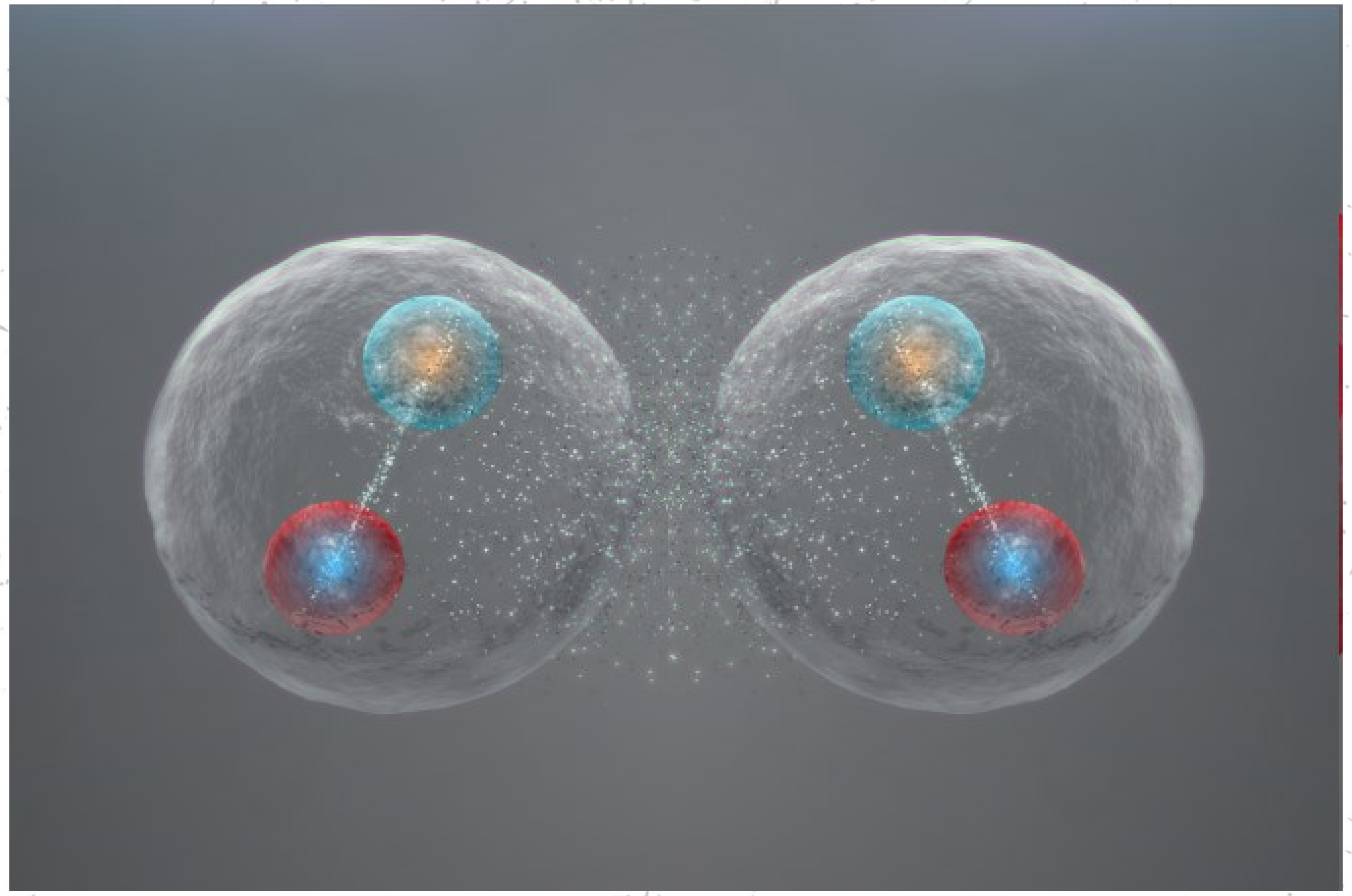


- **But, what about Hadronic Physics?** ⟶ X(3872) best candidate up to now
  - $J^{PC}=1^{++}$ Shallow molecule ⟶ $\mathcal{B}_2$ ~ 0.05 MeV
  - $D^0\bar{D}^{*0}/\bar{D}^0D^{*0}$ interaction in S-wave
  - However, Efimov effect is excluded for the X(3872)
    (E. Braaten, PRD69 (2004) 074005)

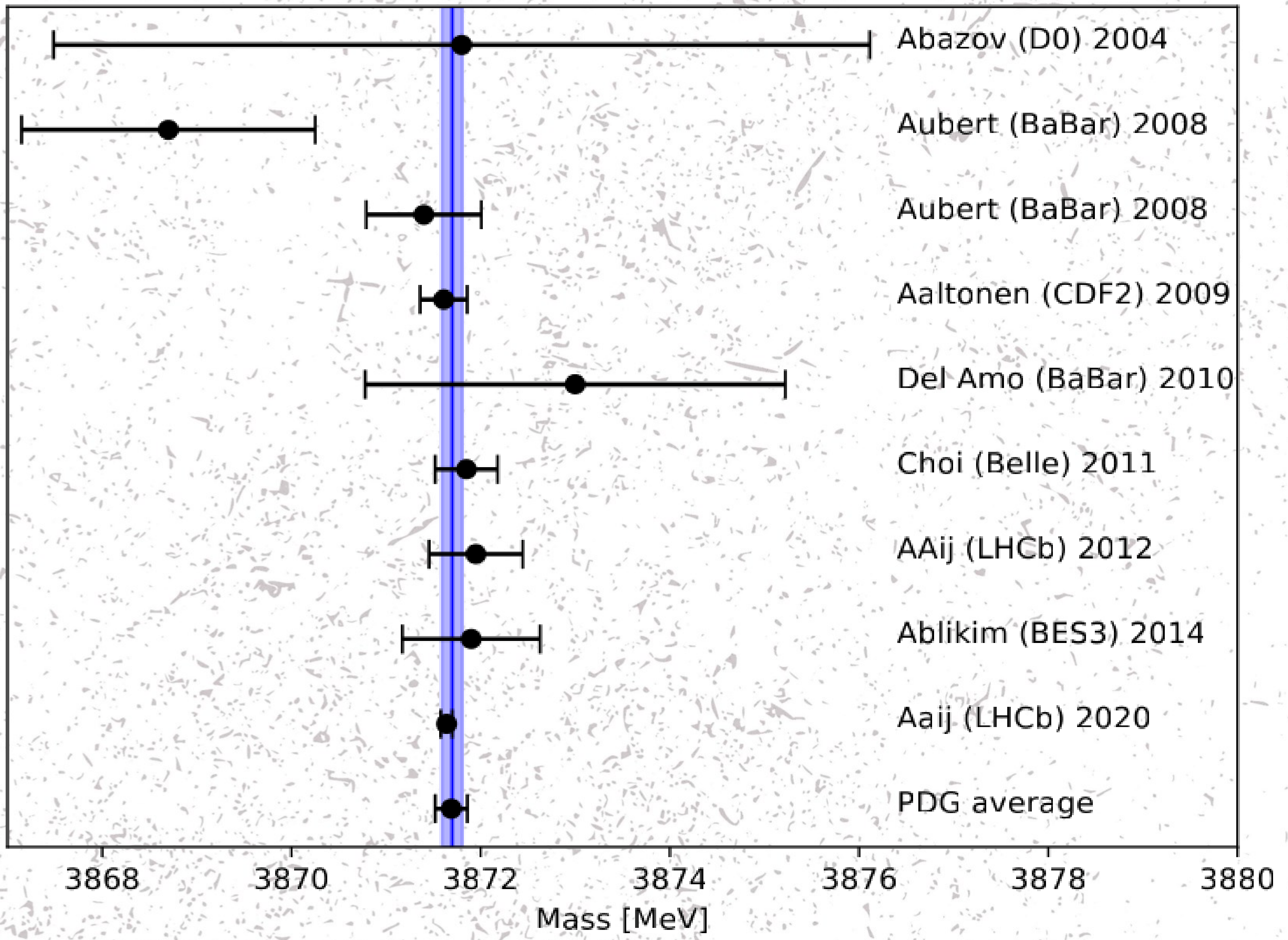


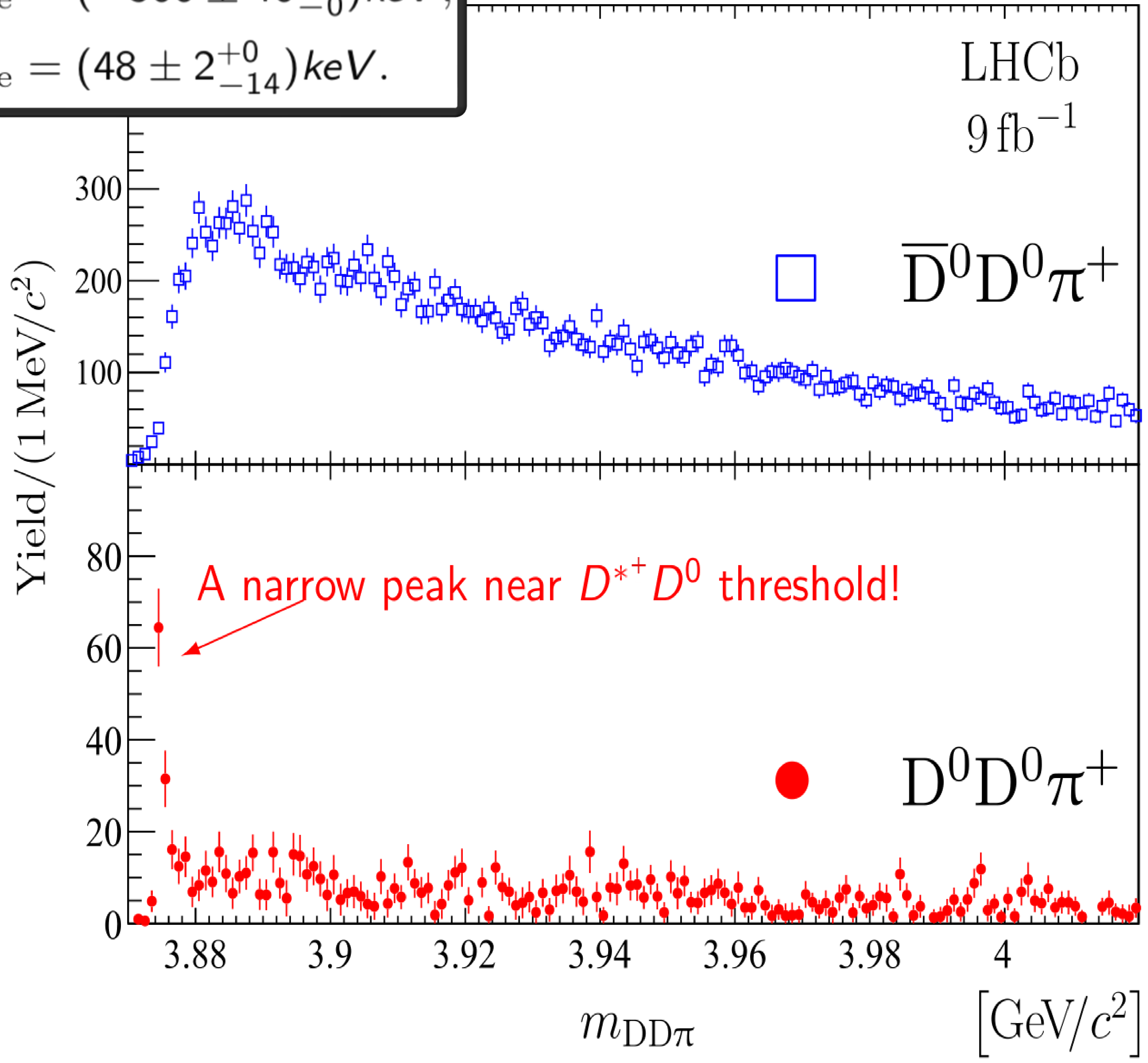
FIG. 1. Different $X(3872)$ masses determinations with standard 68% confidence limits. The band corresponds to the current $\bar{D}D^*$ threshold value with uncertainties.

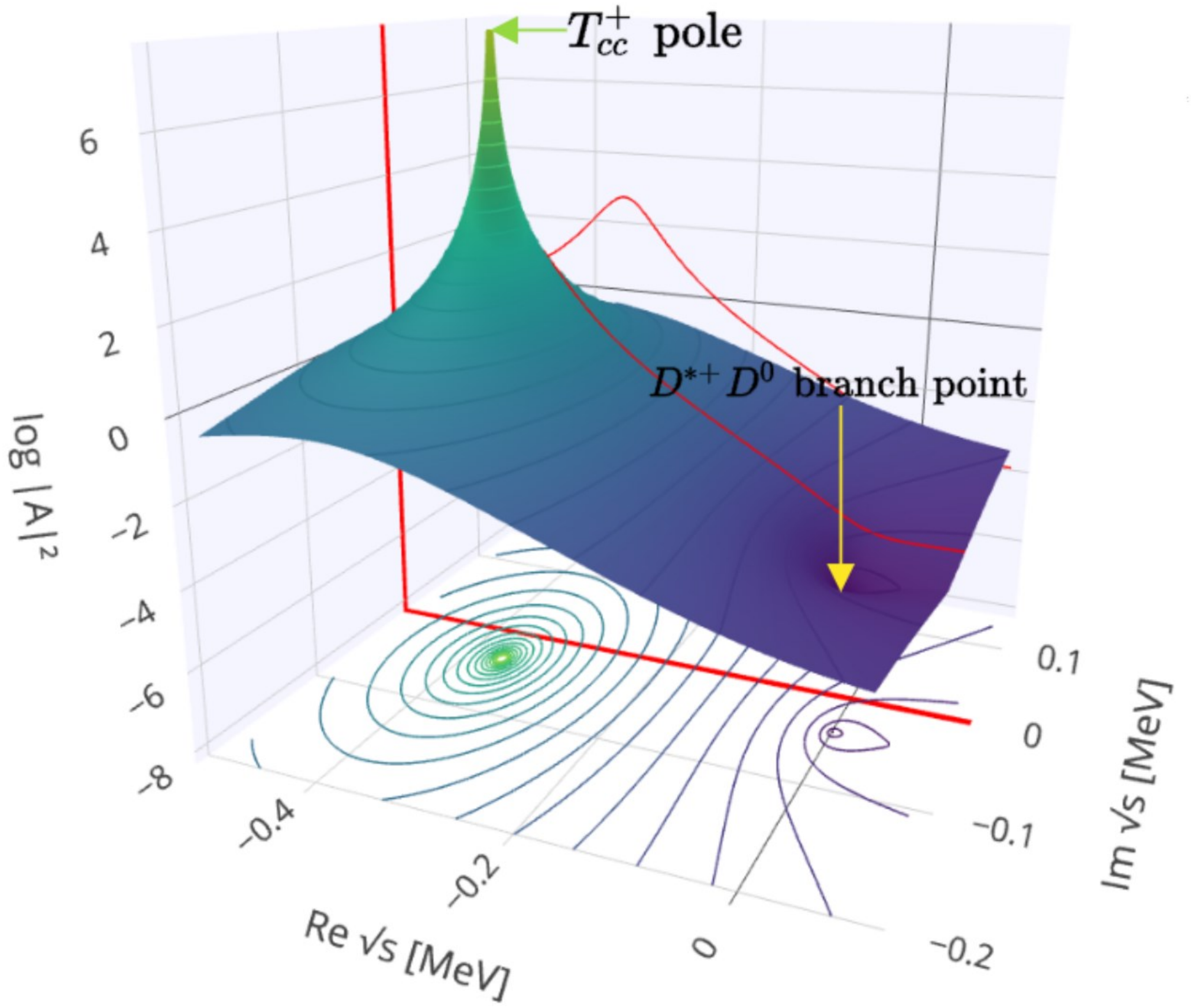
The landmark of 2021: Observation of T_{cc}^{+}

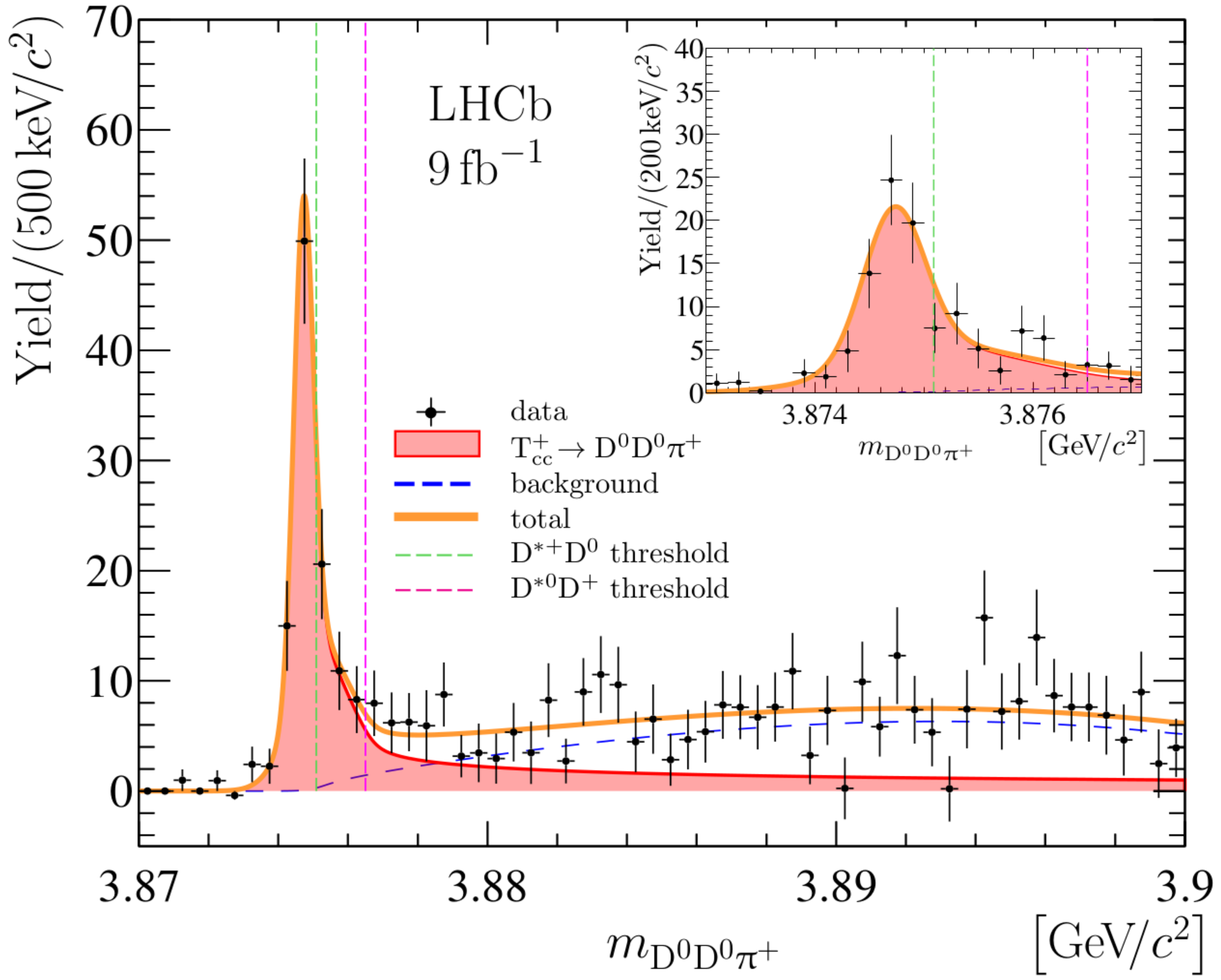
- ✓ $J^P=1^+$ state decaying to DD^* in S-wave
- ✓ Isoscalar due to absence of signal in D^0D^+ and $D^+D^0\pi^+$
- ✓ No isospin violation in couplings to $D^{*+}D^0$ and $D^{*0}D^+$

Extremely narrow state, very close to threshold
Strong candidate for a pure DD* molecular state!

$$\delta m_{\text{pole}} = (-360 \pm 40^{+4}_{-0})\,keV,$$
$$\Gamma_{\text{pole}} = (48 \pm 2^{+0}_{-14})\,keV.$$







LHCb Coll, Nature Phys. 18 (2022), 751;
Nature Commun. 13 (2022) 3351.

Prediction of the T_{cc}^* and D*D*D* system

- ✓ Heavy-Quark Spin Symmetry predicts that the D^*D^* and DD^* interactions in $I(J^P)$ are related:

$$\left\langle D^*D^*,\,0(1^+)\right|\hat{V}\left|D^*D^*,\,0(1^+)\right\rangle = \left\langle D^*D,\,0(1^+)\right|\hat{V}\left|D^*D,\,0(1^+)\right\rangle$$

- ✓ Then, if T_{cc} is a shallow DD^* molecule, a heavy partner should exist in the D^*D^* system ⟶ T_{cc}^*

- ✓ **In this talk**: Explore the universality of the $D^*D^*D^*$ systems in the $J^P=0^-$ sector, with $I=1/2$

  - Identical bosons, simpler calculation
  - In $J^P=0^-$ all D*D* pairs are in relative $(I)J^P=(0)1^+$ ⟶ Nearly-resonant attractive interactions

- ✓ **Conditions for the Efimov effect are met in this system!**

- ✓ Other works that have explored 3-body systems with charmed mesons:

  - DDD* state with ½(1^-): T.-W. Wu et al, PRD105, L031505 (2022)
  - DD*D* and D*D*D* states: S.-Q. Luo et al, PRD105, 074033 (2022)
  - D*D*D* states: M. Bayar et al, Eur. Phys. J. C 83, 46 (2023)

Two-Body Interaction: The T_{cc}^* state

- Two-body amplitude from Bethe-Salpeter equation: $\mathcal{T}_2^{-1}(s) = \mathcal{V}^{-1}(s) - \mathcal{G}(s)$
- Two meson interaction:
(Montesinos:2023qbx)

$$\mathcal{V}^{-1}(s) = C_0 - C_1 \frac{1-\mathcal{P}}{\mathcal{P}}(s - m_*^2) \quad \longrightarrow \quad \left\{ \begin{array}{l} m_* = 2m - \mathcal{B}_2 \\ C_0 \equiv \mathcal{G}(m_*^2) \\ C_1 \equiv \mathcal{G}'(m_*^2) \end{array} \right.$$

- Fixes the existence of a pole below the D*D* threshold, with variable composition:

$\mathcal{V}^{-1}(s) = C_0$ For pure molecular state, $\mathcal{P} = 1$

$\mathcal{V} \sim \frac{1}{s - m_*^2}$ For pure compact state, $\mathcal{P} \to 0$

- Relativistic two-meson loop function. Regularized via sharp cutoff Λ

$$\mathcal{G}(s) = i \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - m_1^2 + i\varepsilon} \frac{1}{(P-q)^2 - m_2^2 + i\varepsilon} \quad \Longrightarrow \quad \mathcal{G}(s) = \frac{1}{(4\pi)^2} \left\{ \sigma \log \frac{\sigma\sqrt{1 + \frac{m^2}{\Lambda^2}} + 1}{\sigma\sqrt{1 + \frac{m^2}{\Lambda^2}} - 1} - 2\log\left[\frac{\Lambda}{m}\left(1 + \sqrt{1 + \frac{m^2}{\Lambda^2}}\right)\right] \right\}$$

$$\sigma = \sqrt{1 - 4m^2/s}$$

- Parameters of the calculation:
  - $\mathcal{B}_2$: Two body binding energy $\mathcal{B}_2 = \{0.01, 0.5, 1.0, 5.0\}$ MeV
  - $\mathcal{P}$: Molecular probability in the Tcc* state $\mathcal{P} = (0, 1]$
  - Λ: Sharp cutoff for loop function $\Lambda = [0.5, 1]$ GeV

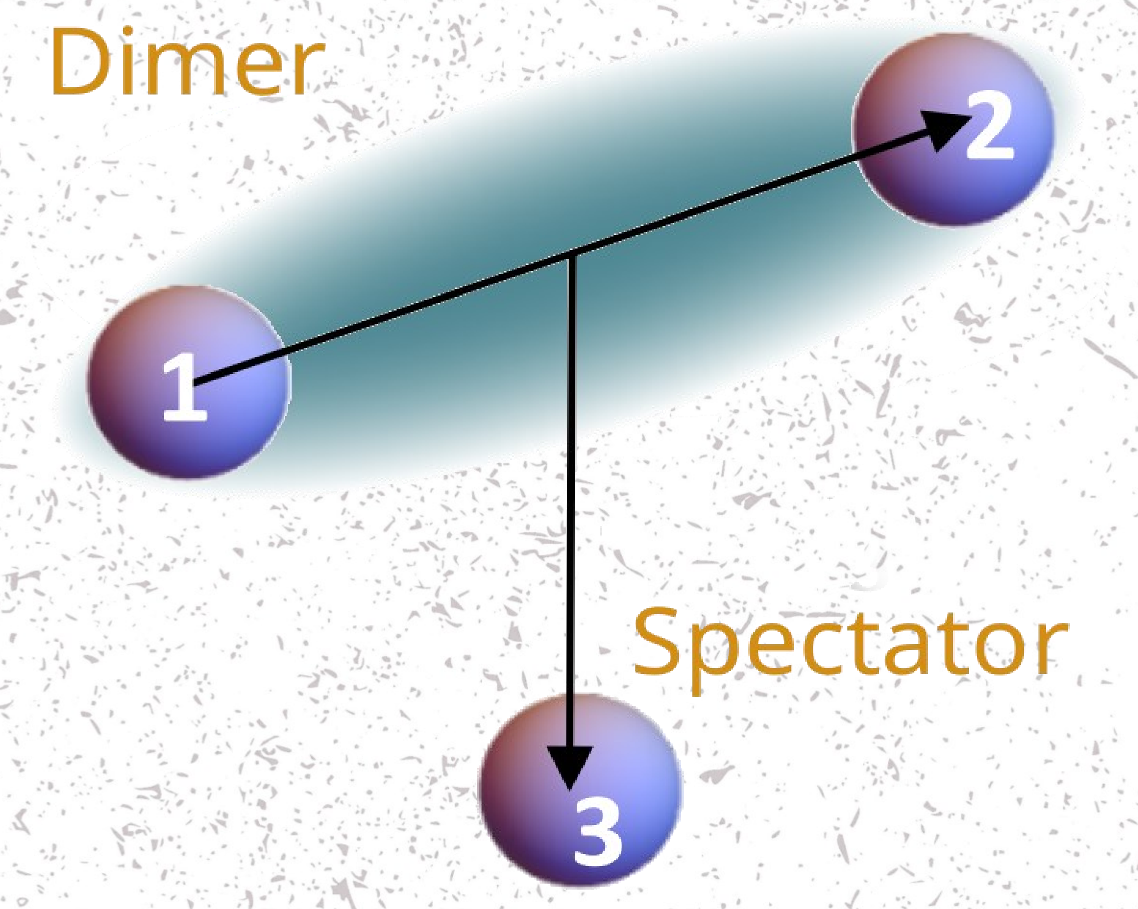
Three-Body Interaction: Ladder amplitude

- Three-body amplitude from Ladder amplitude formalism: $\mathcal{M}_3(\vec{p}_i,\vec{p}_f) = \mathcal{D}(\vec{p}_i,\vec{p}_f) + \cancelto{0}{\mathcal{K}_3(\vec{p}_i,\vec{p}_f)}$

- $\mathcal{K}_3$ ⟶ contributions from short-range three-particle interactions

- $\mathcal{D}$ ⟶ *ladder amplitude*: Sum over all possible pairwise interactions connected through a sequence of one-particle exchanges

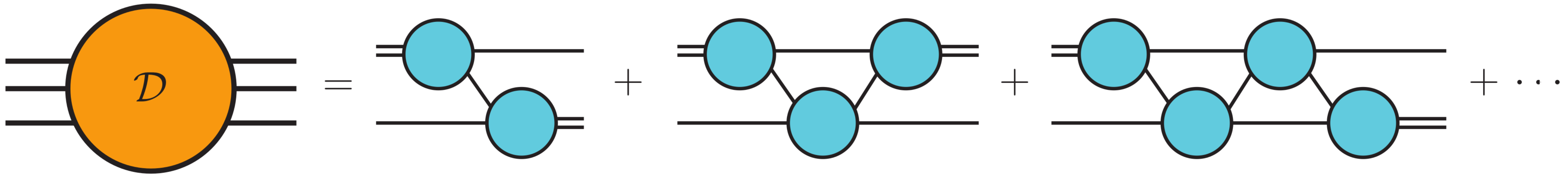
$$\mathcal{D}(\vec{p}_i,\vec{p}_f) = -\mathcal{M}_2(p_i)G(\vec{p}_i,\vec{p}_f)\mathcal{M}_2(p_f) - \mathcal{M}_2(p_i)\int \frac{d^3\vec{q}}{(2\pi)^3 2\omega(q)} G(\vec{p}_i,\vec{q})\mathcal{D}(\vec{q},\vec{p}_f)$$



- **G** ⟶ long-range dimer-spectator interaction mediated by a particle exchange

$$G(\vec{p},\vec{k}) = \frac{1}{(\sqrt{s}-\omega(p)-\omega(k))^2 - (\vec{p}+\vec{k})^2 - m^2 + i\epsilon} \Rightarrow G(p,k) = -\frac{1}{4pk}\log\left(\frac{z(p,k)-2pk+i\epsilon}{z(p,k)+2pk+i\epsilon}\right)$$

with $z(p,k) = (\sqrt{s}-\omega(p)-\omega(k))^2 - k^2 - p^2 - m^2$



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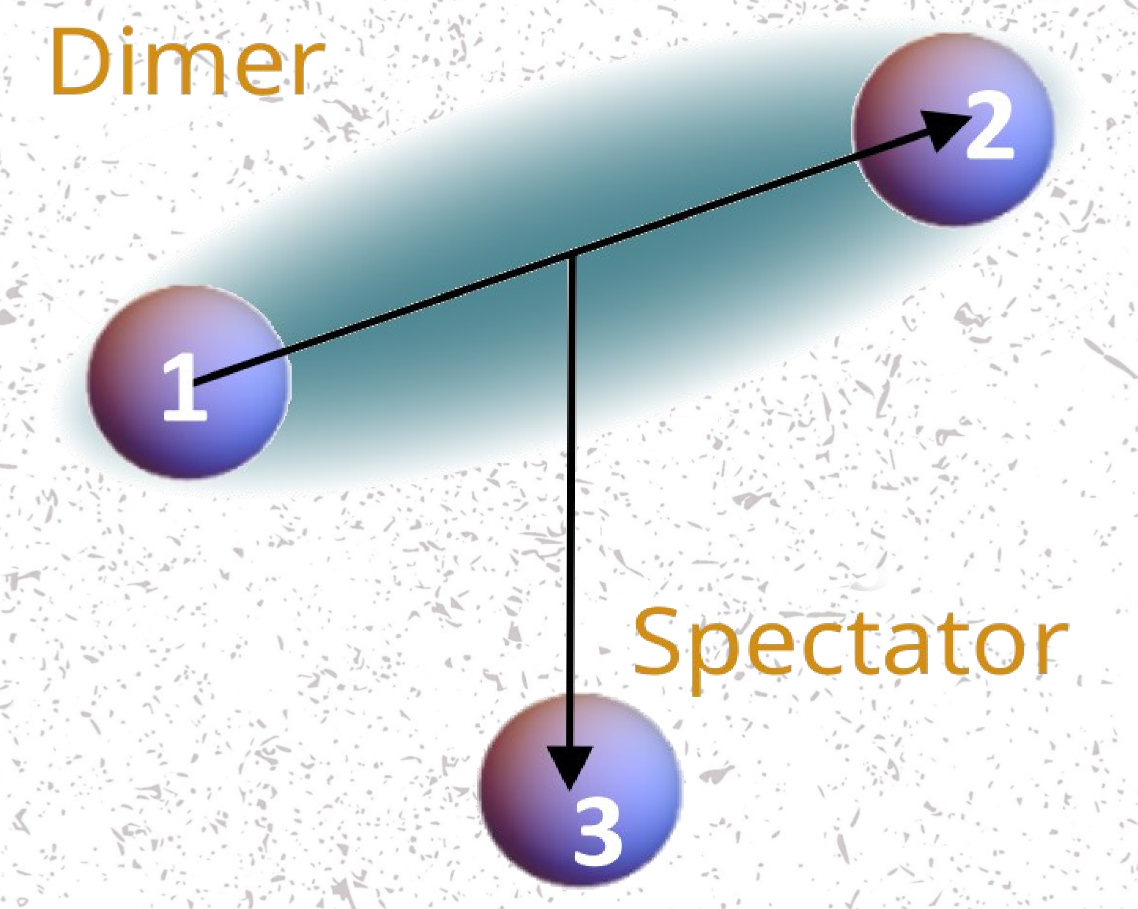

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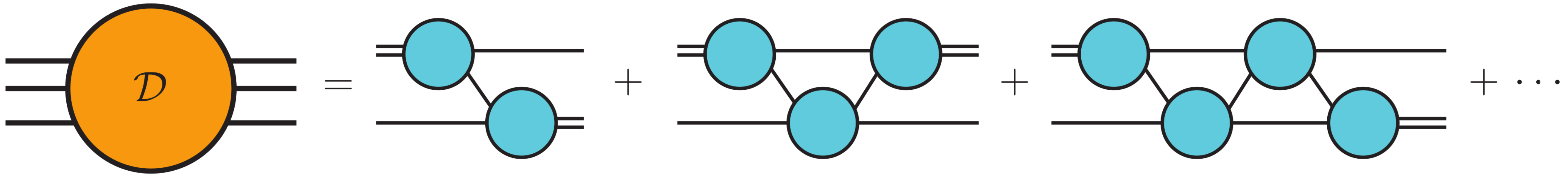
with $z(p,k)$

- Long range despite short-range two-body interaction

→ *Interaction of two particles mediated by a third one*



Results for effective range expansion

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- First simple study: **Leading-order effective range expansion**, that's it,

$$\mathcal{T}_2(p) = \frac{8\pi\sqrt{s_2(p)}}{iq_2(p) + 1/a_{\rm sc}}$$ With $q_2(p) = \sqrt{s_2(p)/4 - m^2}$ momentum of the particles in the dimer.

- Phenomenology only depends on the D*D* scattering length

$$a_{\rm sc} = \frac{2\hbar c}{\sqrt{4m^2 - m_*^2}}$$

- At least **two trimers** found for all $\mathcal{B}_2$

→ Efimov effect can emerge in the D*D*D* system

- Second to first trimer ratio approaches the resonant value of

$\lambda^{-2} \sim 1/515 \sim 0.0019$ for $\mathcal{B}_2 \to 0$

TABLE I. Properties of the trimer states in the effective range expansion approach for the 2-body amplitude. *1st column:* Two-body binding energy, in MeV; *2nd column:* Two-body scattering length, in fm; *3rd to 5th columns:* Binding energies of the ith trimer state, $\mathcal{B}_3^{(i)} = 3m - E^{(i)}$, with $E^{(i)}$ the 3-body mass of the ith trimer, in MeV; *6th column:* Ratio of the second to first trimer binding energies. The dagger (†) indicates a virtual state (pole in the second Riemann sheet).

$\mathcal{B}_2$	$a_{\rm sc}$	$\mathcal{B}_3^{(1)}$	$\mathcal{B}_3^{(2)}$	$\mathcal{B}_3^{(3)}$	$\mathcal{B}_3^{(2)}/\mathcal{B}_3^{(1)}$
0.01	44.03	54.592	0.185	0.011	0.003
0.5	6.23	64.158	0.980	0.620†	0.015
1.0	4.40	69.099	1.557	–	0.023
5.0	1.97	91.365	5.521	–	0.060

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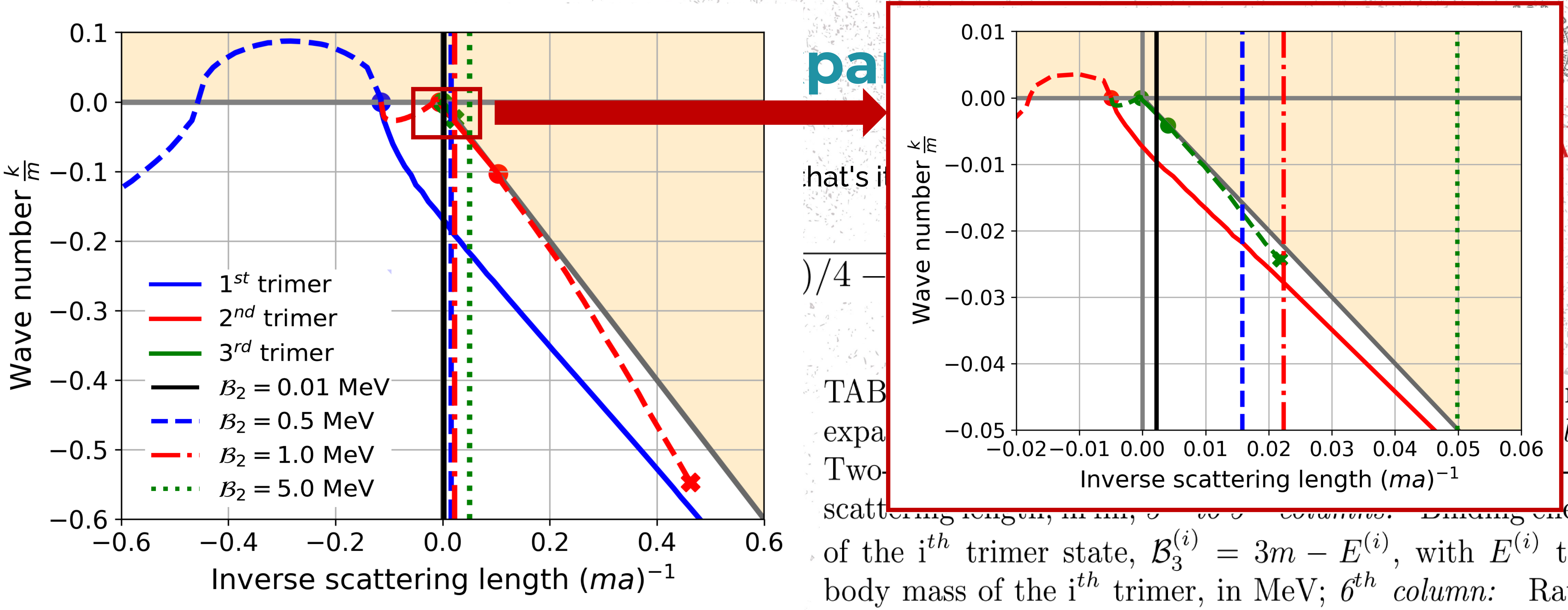

hat's it

)/4 −

TAB range

expa lumn:

Two- -body

scatt ergies

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Trimer masses for full $\mathcal{T}_2$

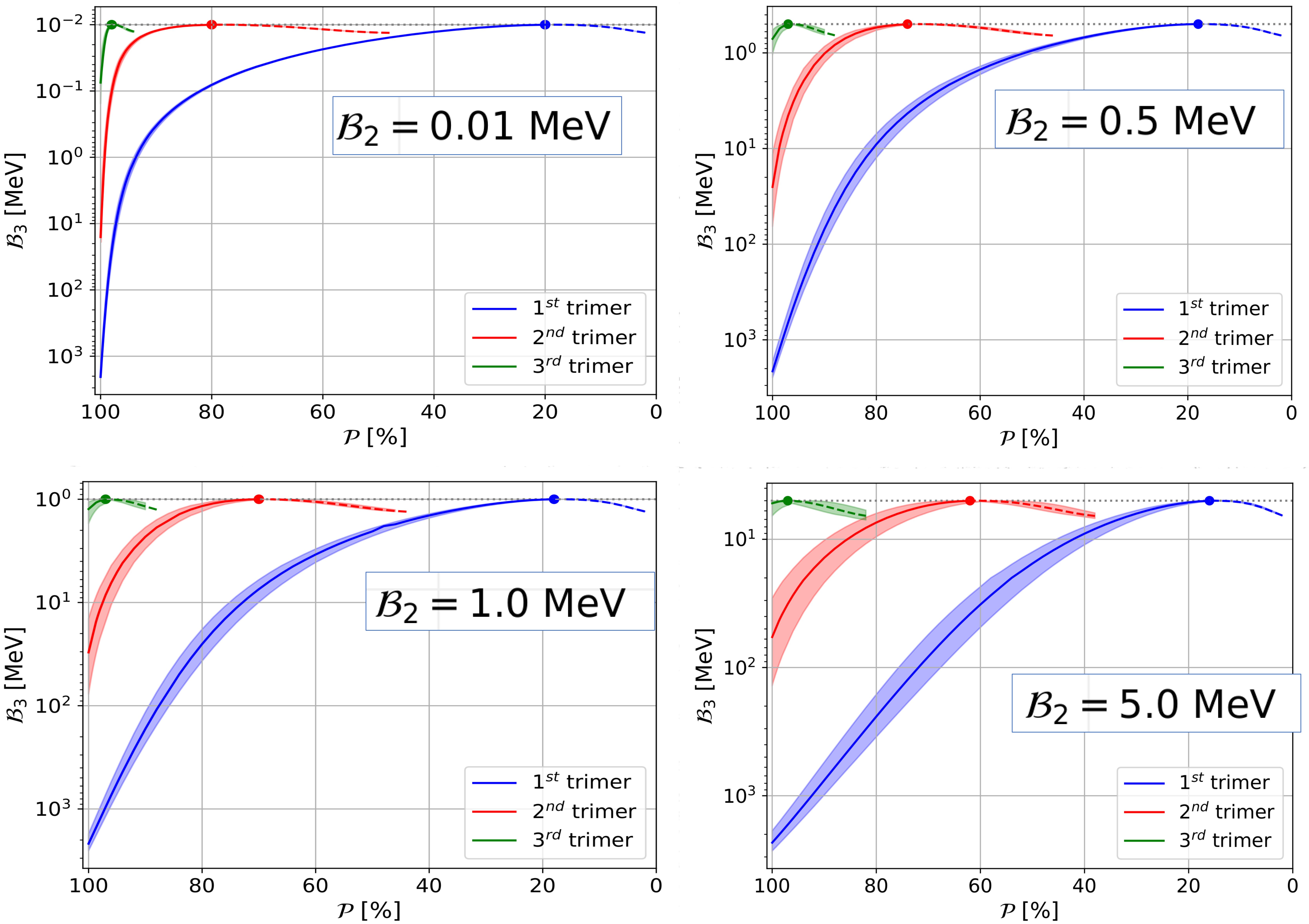
- $D^*D^*D^*$ binding energies ($\mathcal{B}_3$=3m-E$_3$) as a function of $\mathcal{P}$.
- $\mathcal{P}$=100%: Pure molecular Tcc* state

  $\mathcal{P}$=0%: Pure compact Tcc* state
- Central line: Λ = **0.7 GeV.**

  Shadow band: Λ = **[0.5,1] GeV**
- Solid lines: Bound states.

  Dashed lines: Virtual states

  Dots: Riemann sheet change



Second to first trimer ratio

- ✓ The ratio increases as T_{cc}^* becomes more compact.
- ✓ It shows a small valley that gets shallower as $\mathcal{B}_2$ increases.
- ✓ The bottom of this valley approaches the *Efimov scaling law* the smaller $\mathcal{B}_2$

$\lambda^{-2} \sim 1/515 \sim 0.0019$ for $\mathcal{B}_2 \to 0$

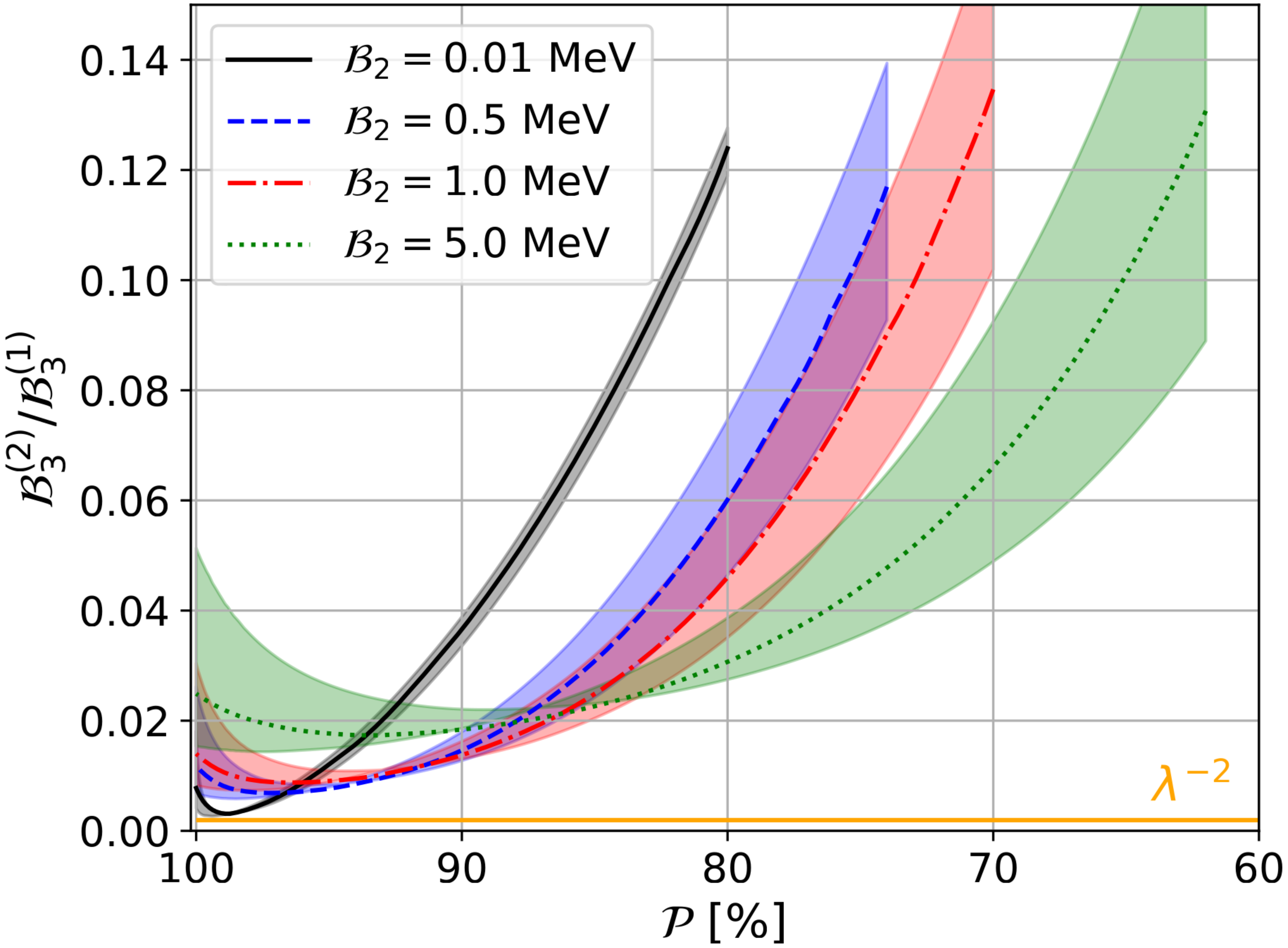


FIG. 3. Ratio of the second to first trimer binding energies for different $\mathcal{B}_2$ as a function of the T_{cc}^* composition, where $\mathcal{P} = 100\%$ denotes a pure two-body T_{cc}^* molecule and $\mathcal{P} = 0\%$ a pure compact state. The horizontal orange line represents the Efimov scaling factor $\lambda^{-2} \sim 1/515$ at the unitary limit $a_{\rm sc} \to \infty$. A cutoff of $\Lambda = 0.7$ GeV has been used in Eq. (4) for the central line, while the color error bands indicate the results for the cutoff range $\Lambda = [0.5, 1]$ GeV.

T_{cc}*-D* scattering

- ✓ $\mathcal{D}$ has poles at the values of the dimer's energies equal to the T_{cc}^* bound state.

- ✓ The residue of the three-body amplitude is proportional to the dimer-spectator amplitude $\mathcal{M}_{TD}$

$$\mathcal{D}(p_i, p_f) = \frac{g^2 \mathcal{M}_{\text{TD}}(s)}{(s_2(p_i) - m_*^2)(s_2(p_f) - m_*^2)} + \ldots$$

- ✓ The T_{cc}^*-D^* scattering length a_{TD} can help to discover the shallowest trimers

$$-\frac{1}{a_{\text{TD}}} = \lim_{s \to m_{\text{TD}}^2} 8\pi\sqrt{s} Re(\mathcal{M}_{\text{TD}}^{-1}(s))$$

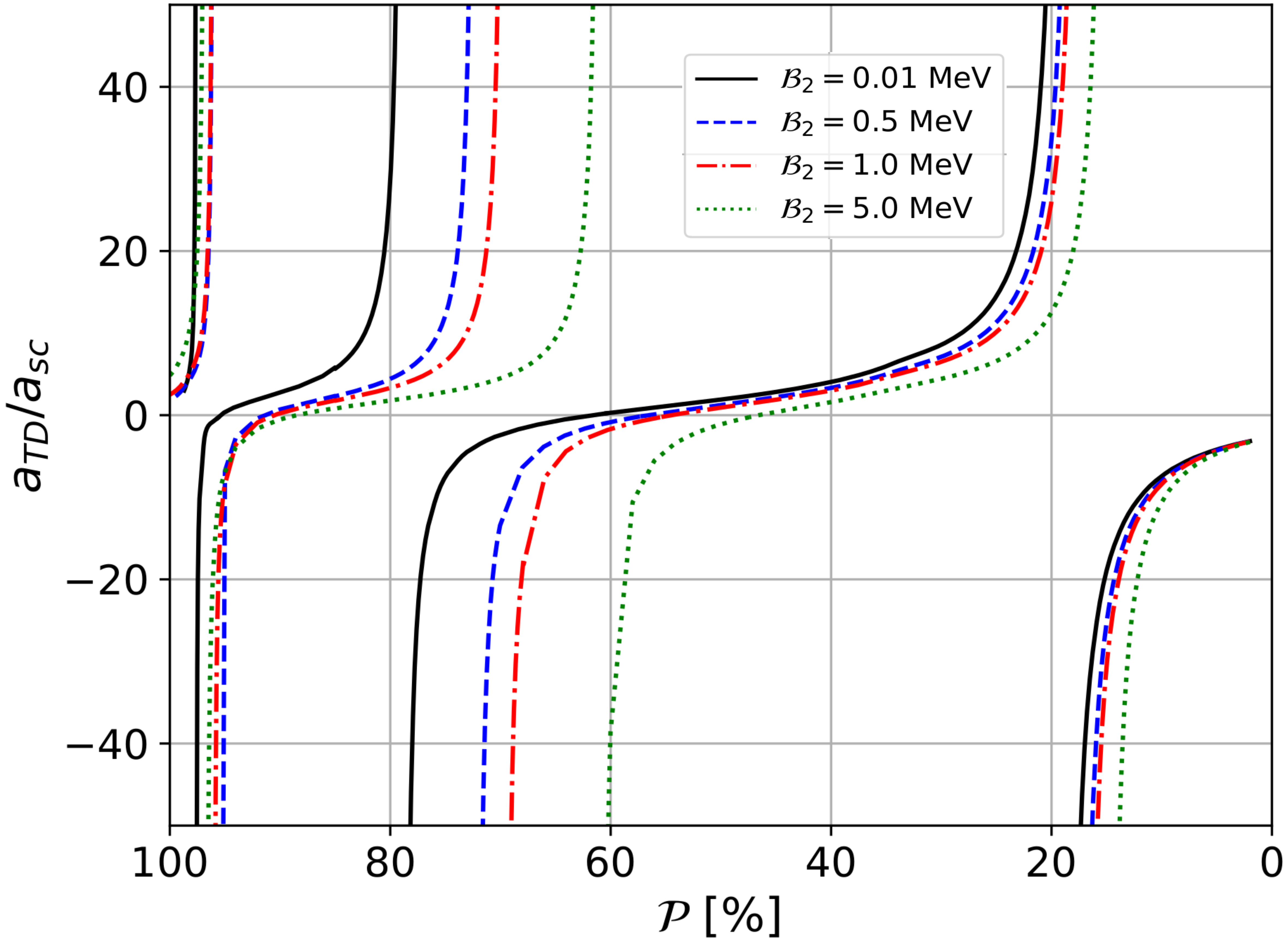
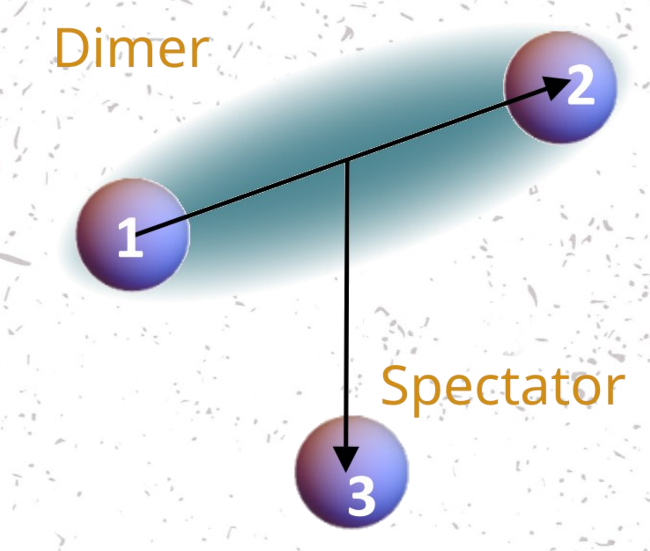


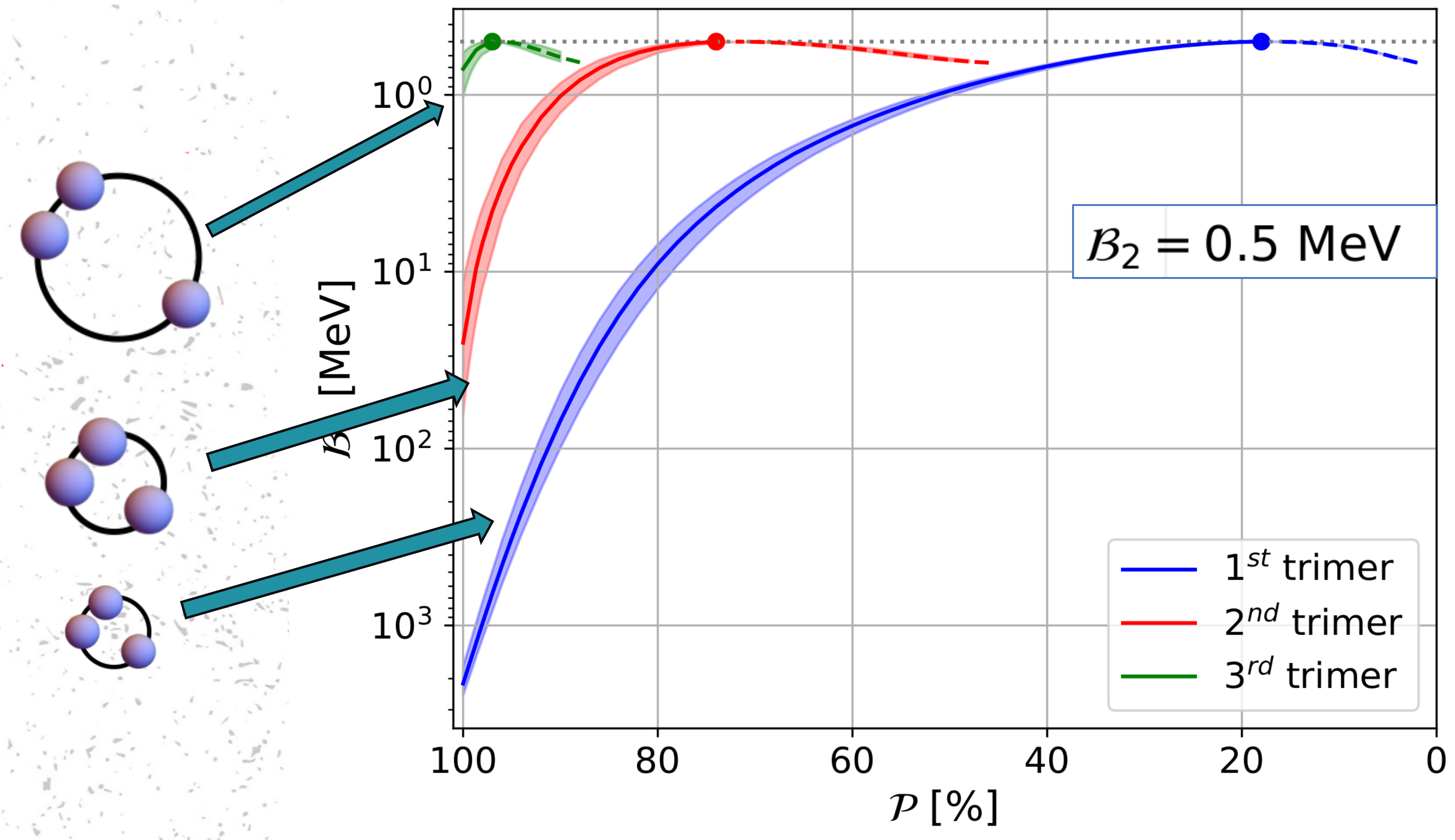
FIG. 4. $T_{cc}^* D^*$ scattering length normalized over the $D^* D^*$ scattering length as a function of the T_{cc}^* composition for different binding energies $\mathcal{B}_2$, using $\Lambda = 0.7$ GeV. Same legend as in Fig. 3.

Summary

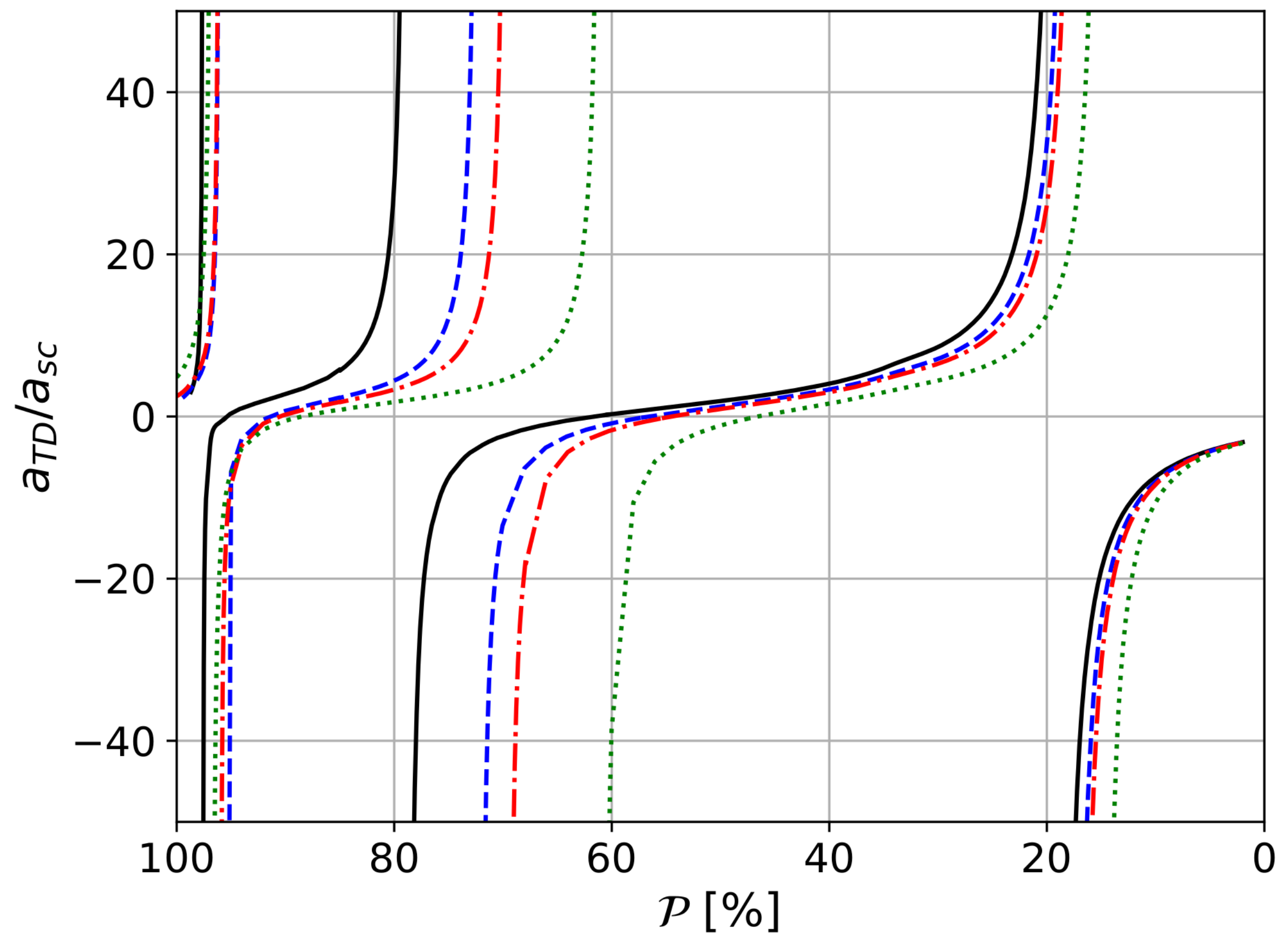


- Analysis of the D*D*D* system in the $(I)J^{\mathcal{P}}=(\frac{1}{2})0^-$ sector, assuming the T$_{cc}$* exists

- Efimov effect can emerge ⟶ Up to three trimers can be formed, depending on $\mathcal{P}$ and $\mathcal{B}_2$



- The shallowest state can be seen in the T$_{cc}$*D* scattering



THANKS FOR YOUR ATTENTION!

VNiVERSiDAD
D SALAMANCA

Backup slides

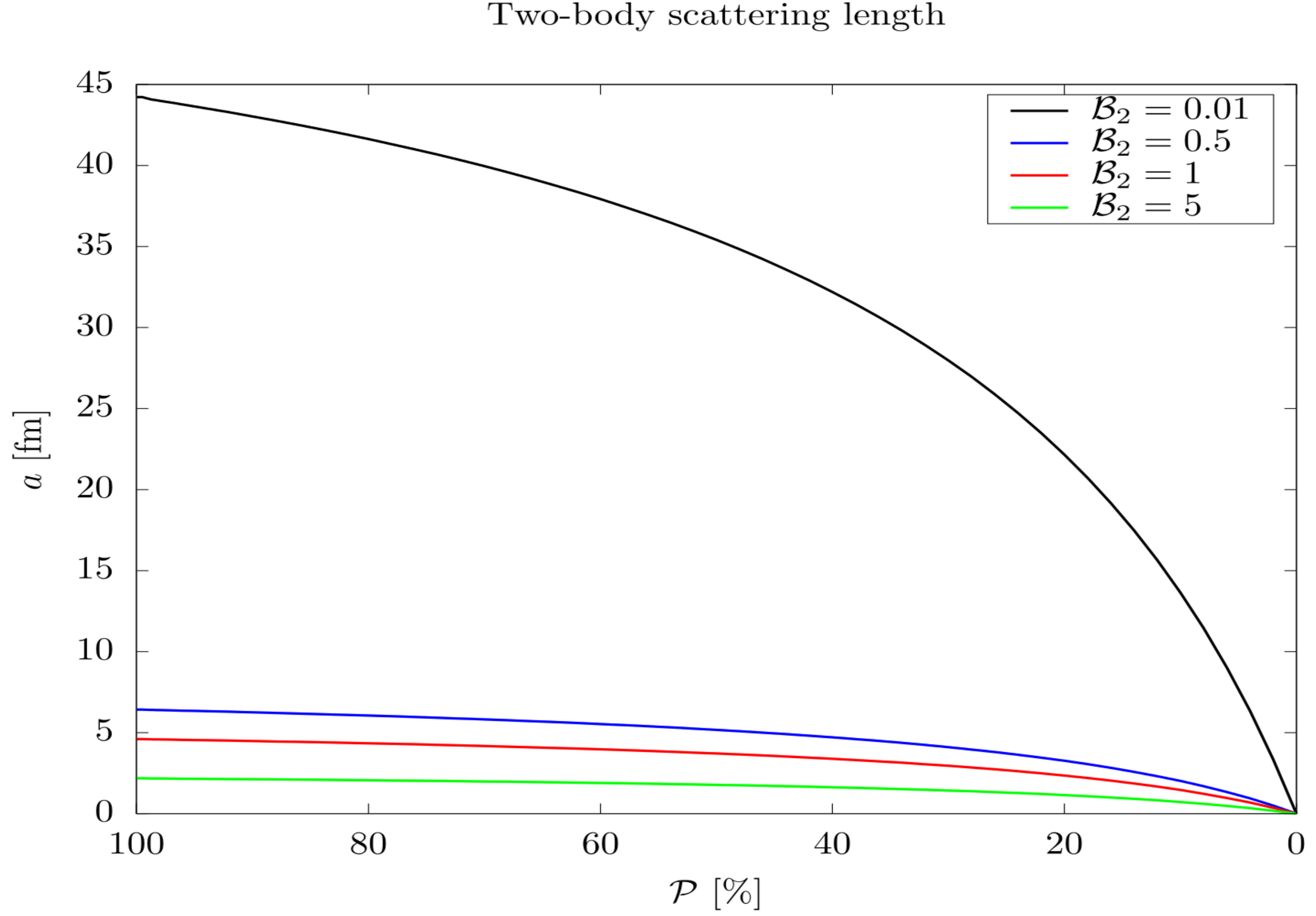
Two-body state properties

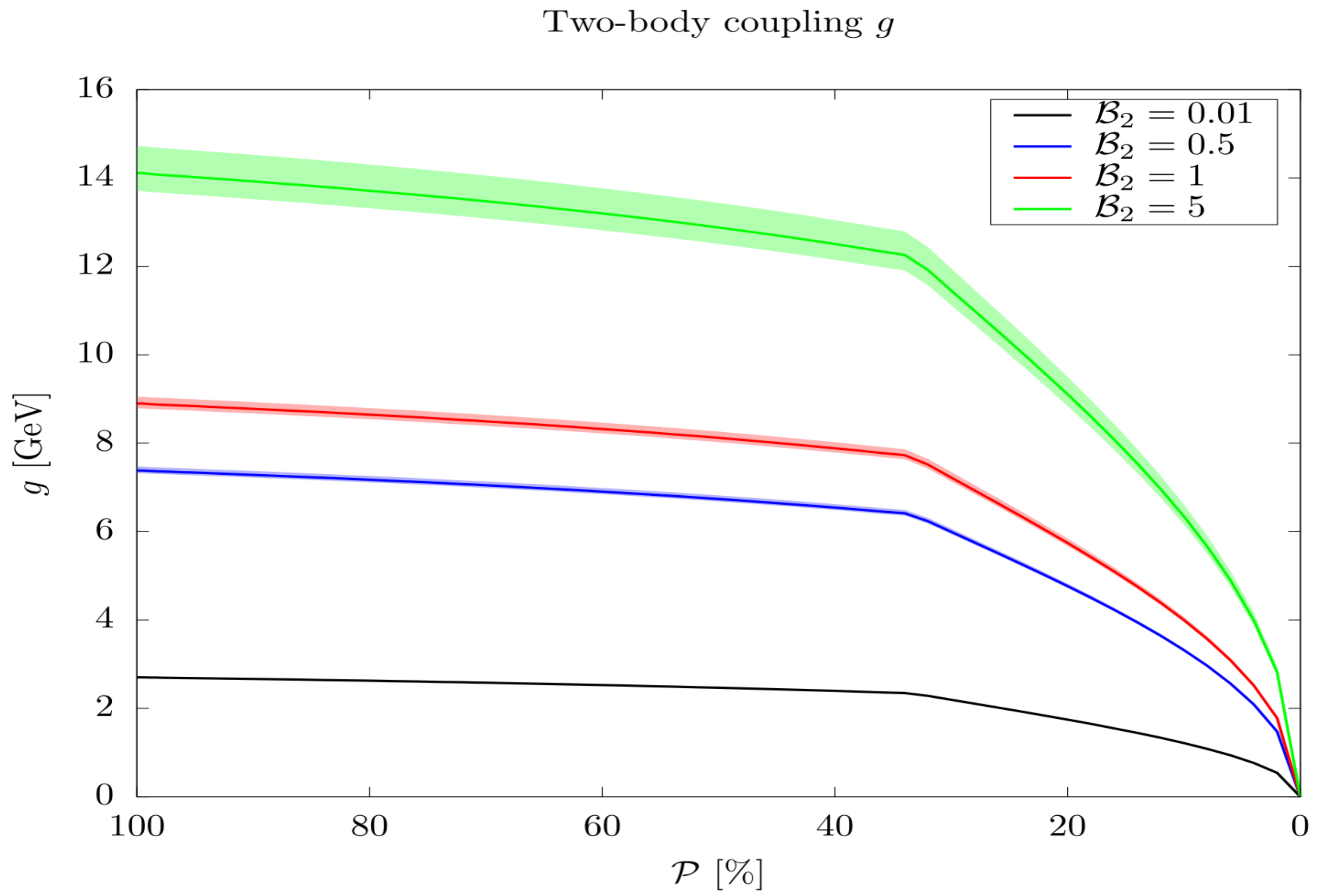
- Weinberg's compositeness: $a_{\rm sc} \approx \frac{2\hbar c}{\sqrt{m\mathcal{B}_2}}\frac{\mathcal{P}}{1+\mathcal{P}}$ $\qquad r_{\rm eff} \approx \frac{\mathcal{P}-1}{\mathcal{P}}\frac{\hbar c}{\sqrt{m\mathcal{B}_2}}$
- $\mathrm{T_{cc}}^*$ coupling to D^*D^* (*g*) and scattering length (a) drop to cero for $\mathcal{P} \longrightarrow 0$
- Shallow bound states with purely attractive potential gives $r_{\rm eff} > 0$
- The effective range is mostly negative, but for $\mathcal{P} \longrightarrow 1$
  $\Longrightarrow$ Usual when a compact component is present
- LHCb measured a non-positive *r* for $\mathrm{T_{cc}}$ with an upper limit (LHCb:2021auc):

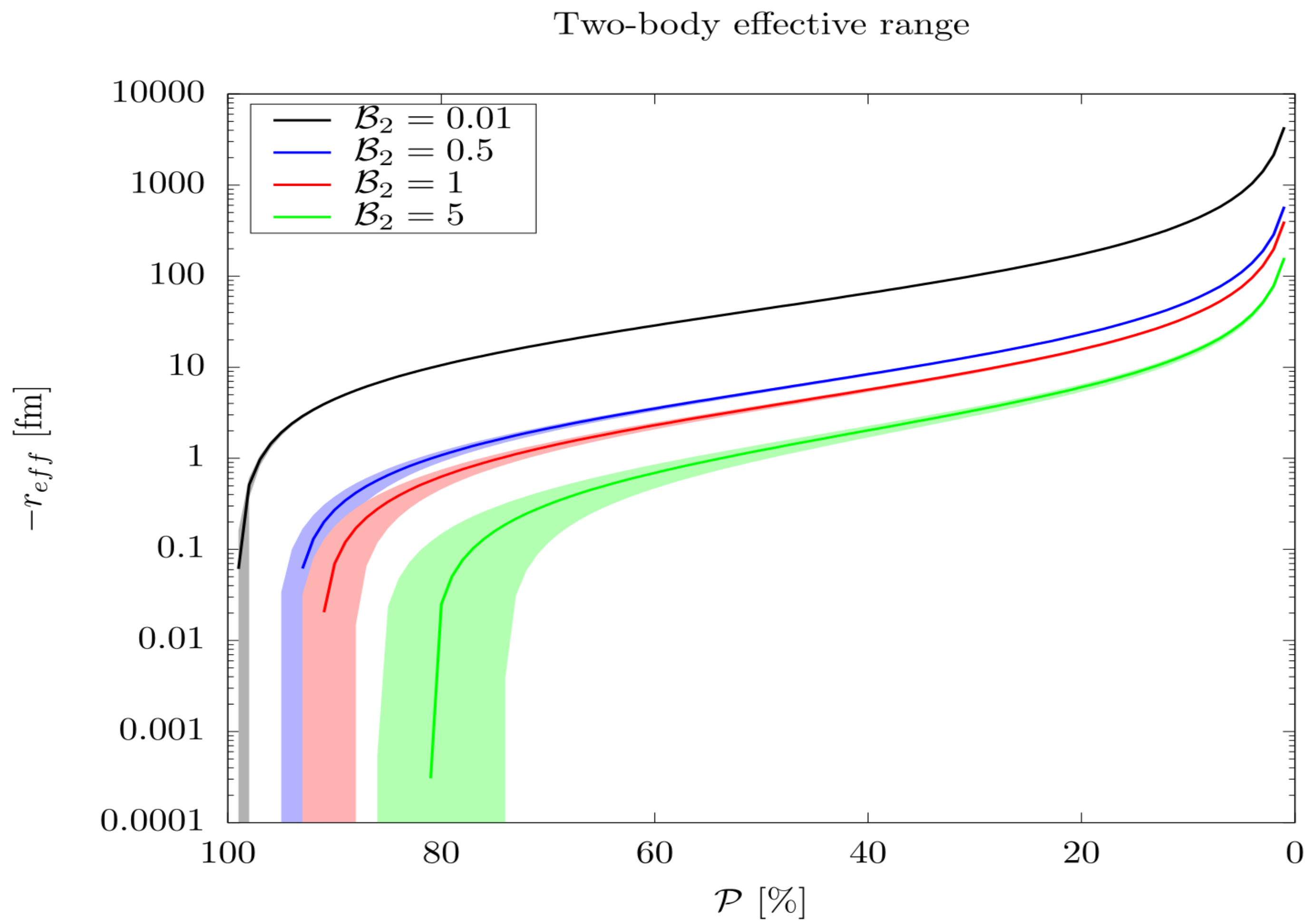
$$0 \;\leq\; -r < 11.9\,(16.9)\ \mathrm{fm\ at\ 90\ (95)\%\ CL.}$$

- And a Weinberg factor ($Z=1-\mathcal{P}$) and coupling:

$$Z < 0.52\,(0.58)\ \mathrm{at}\ 90\,(95)\%\ \mathrm{CL.} \qquad |g| > 5.1\,(4.3)\ \mathrm{GeV\ at}\ 90\,(95)\ \%\ \mathrm{CL.}$$

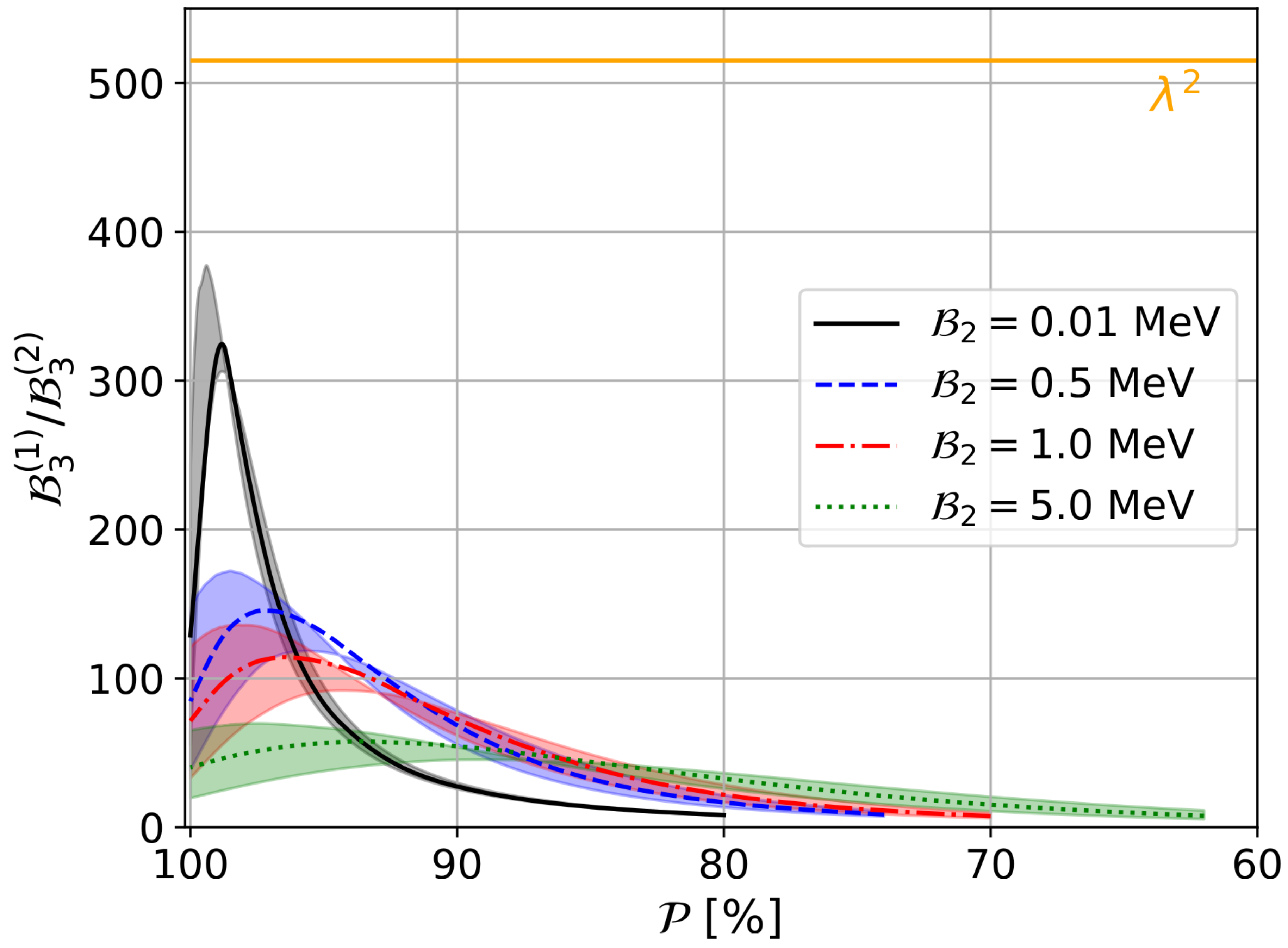


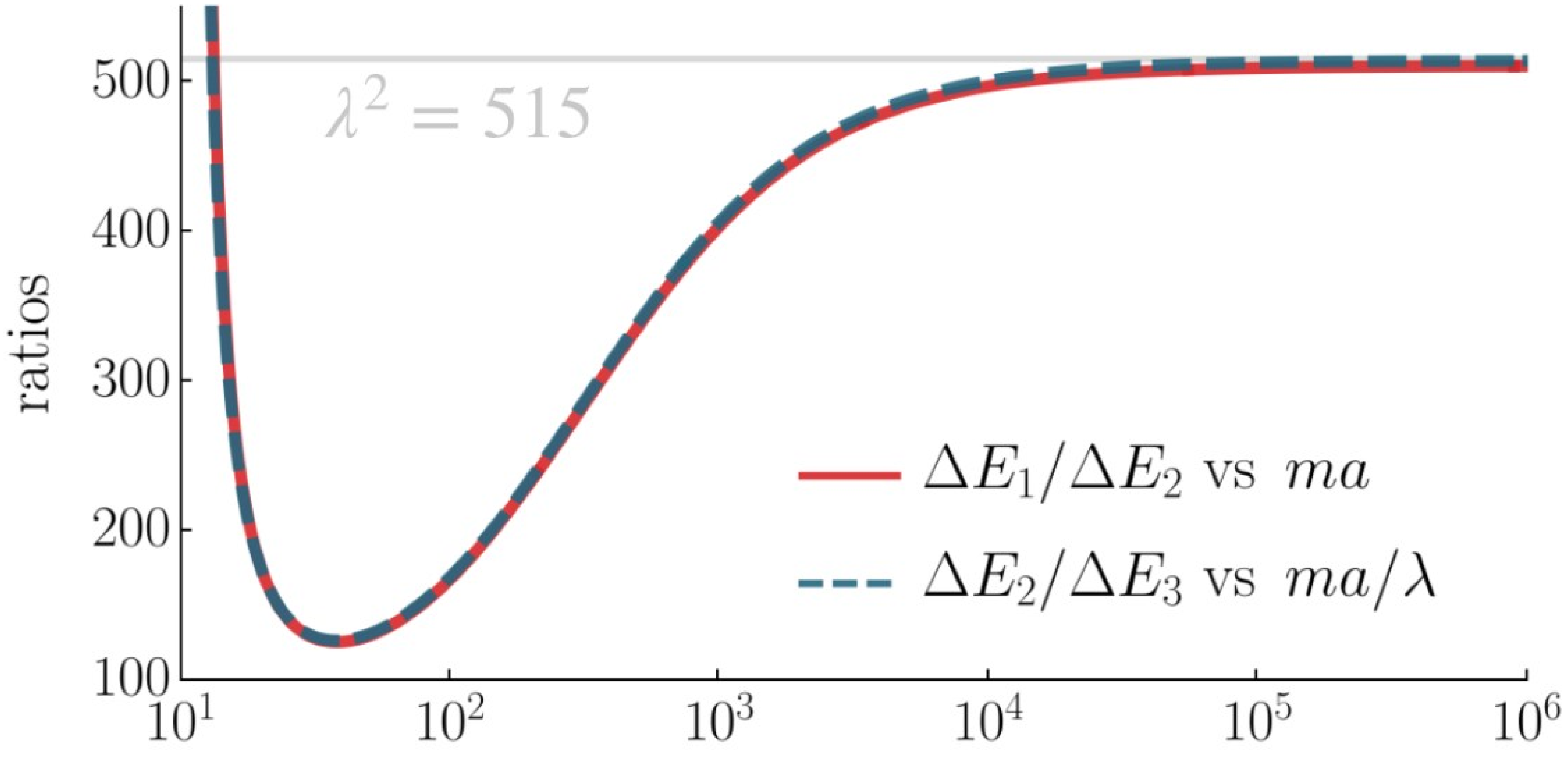




Discrete scale invariance

- ✓ Scaling law predicts $\frac{\Delta E^{(n+1)}}{\Delta E^{(n)}} \to \lambda^{-2} \approx \frac{1}{515}$ But only in the resonant limit
- ✓ For finite binding energies the scaling is reduced.
- ✓ This also happens in the zero-range theory





Ratios of binding energies vs ma (x-axis). Dawid:2023kxu

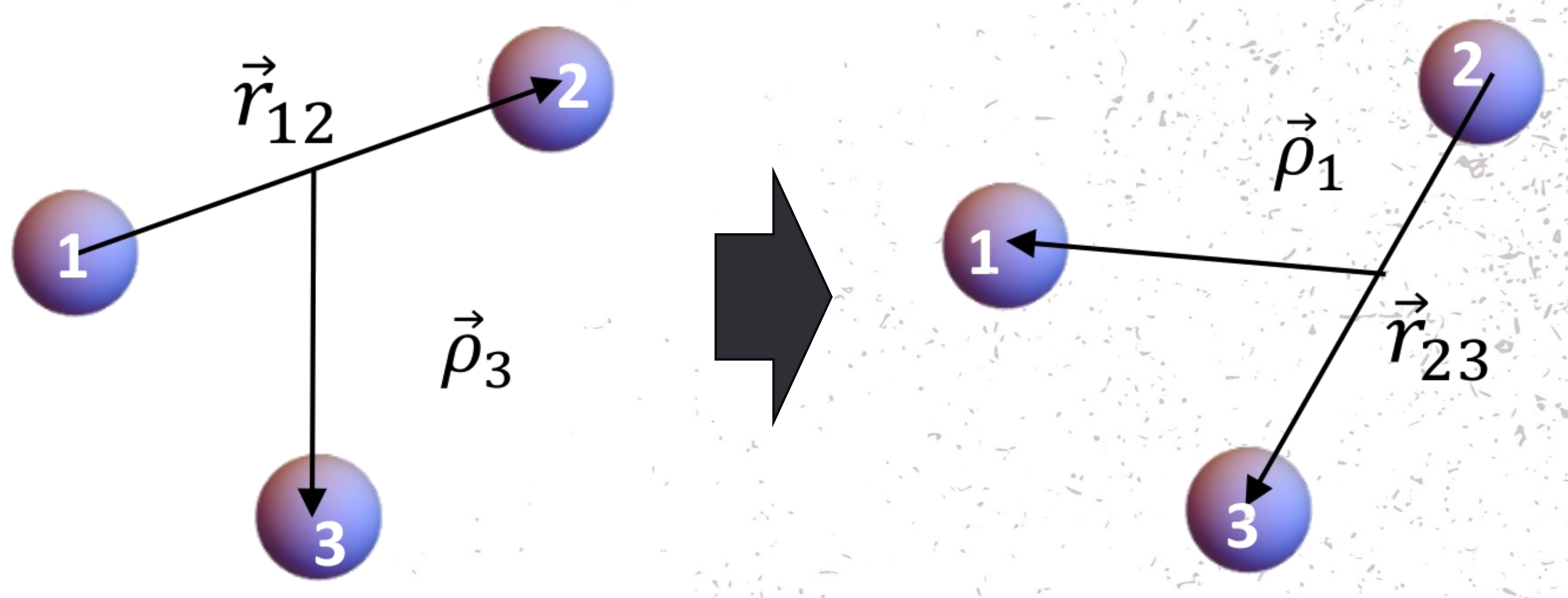
Why D*D*D* in $(I)J^P=(\frac{1}{2})0^-$ sector?

- Three possible resonant pairs: Three identical bosons

- D*D* wave function totally symmetric
- $J=0$ implies $s_{12}=1_{\mathcal{A}}$ necessarily → Needs $t_{12}=0_{\mathcal{A}}$, so $I=\frac{1}{2}$.

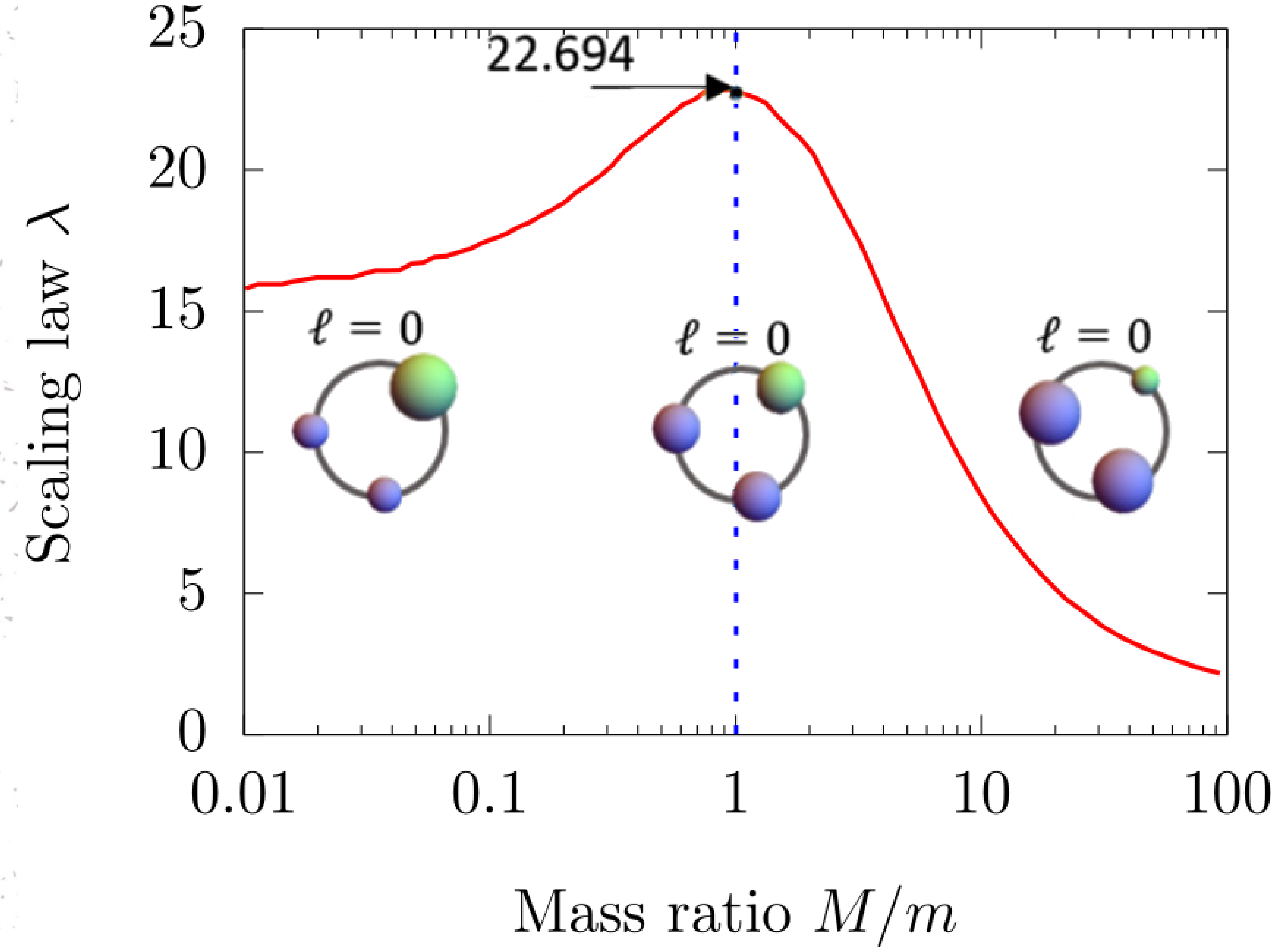
- If we rotate the coordinate system (12)3 → (23)1 , then, the $|I,J>_{23}$ state is

$$|0,1\rangle_{12} \rightarrow \frac{1}{2}|0,1\rangle_{23} - \cancelto{0}{\frac{\sqrt{3}}{2}|1,1\rangle_{23}}$$



Possible in DD*D* in $(I)J^P=(½)1^-$ sector?

- Three possible resonant pairs: Two identical bosons plus one.



- M/m ~1.08 ⟶ Scaling law ~22.7 again

  Possible Efimov states in $T_{cc}D^*$ - T_{cc}^*D system?

- $J=1$ implies $s_{12}=1_{\mathcal{A}}$ for D^*D^* ⟶ Needs $t_{12}=0_{\mathcal{A}}$, so $I=½$.

- If we rotate the coordinate system (12)3 ⟶ (23)1 , then, the $|I,J>_{23}$ state is $$|0,1\rangle_{12} \to \frac{1}{2}|0,1\rangle_{23} - \boxed{\frac{\sqrt{3}}{2}|1,1\rangle_{23}}$$

- It can mix with a non-resonant repulsive pair… Similar to the triton case? Dependence on the scattering length of DD^* for I=1

DD*-D*D* coupling

- The DD^*-D^*D^* coupling is ignored → It can be modeled as a source of width for the T_{cc}^* and the $D^*D^*D^*$ trimers (Dai:2021vgf, Bayar:2022bnc, Luo:2021ggs)

- This coupling reduces as $\mathcal{B}_2$ tends to zero → D^*D^* component dominates at threshold.

- Small effect expected → Estimated total T_{cc}^* width including DD^* and $D^*\pi$ rescatterings: (Jia:2022qwr)

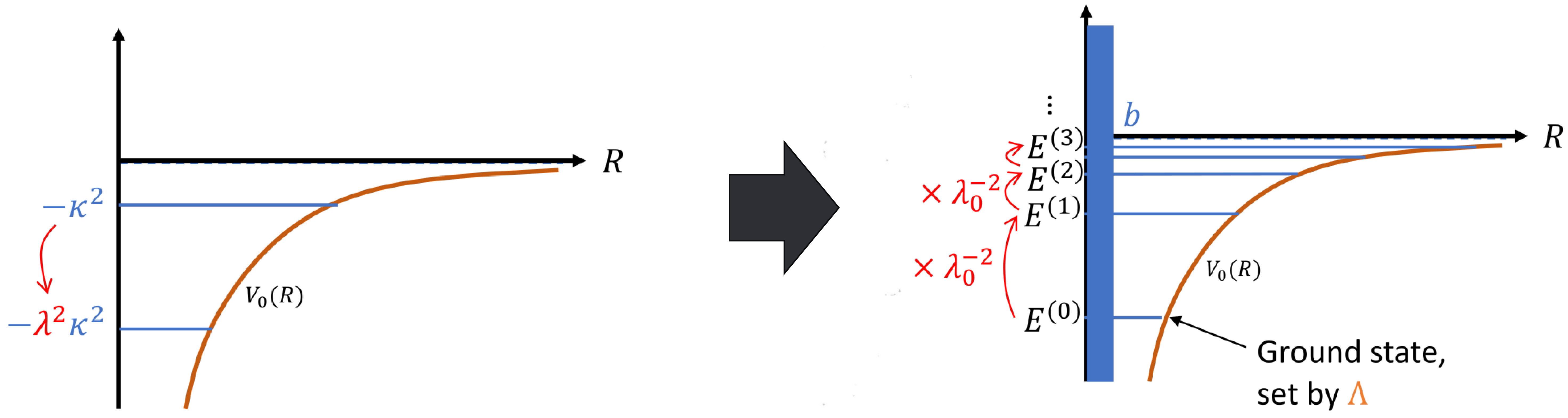
$$\Gamma = 41 \pm 2 \text{ keV for a } \mathcal{B}_2 = 503 \pm 40 \text{ keV}$$

- For the same reason, the D^* width is neglected → Small contribution to T_{cc}^* width and small correction to OPE

$$m \rightarrow m - \frac{\Gamma_{D*}}{2}\, i \rightarrow \left(1 - 1.7 \cdot 10^{-5}\, i\right) m$$

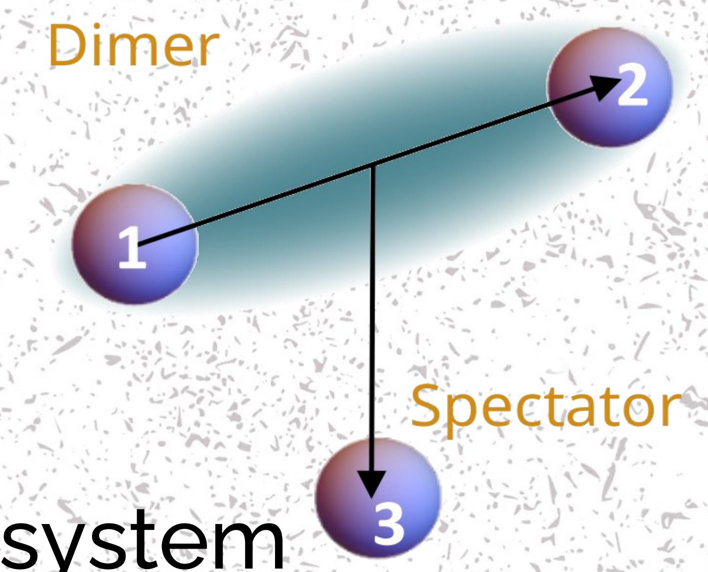
Three-body forces

- We make explicit use of $\mathcal{K}_3=0$ → No fundamental short-range 3b forces included
- Clean cancellation between the off-shell parts of the $\mathcal{T}_2$ and three-body forces in chiral Lagrangians (MartinezTorres:2008gy,Khemchandani:2008rk)
- **But** there is a three-body parameter induced from the two-body interaction → **Λ and $\mathcal{P}$**
- Provides a non-zero range for the two-body interaction and modifies the energy of the first trimer.
- Prevents the Thomas collapse → Infinitely close 3 particles with infinite binding energy in zero-range theory



Higher partial waves

- Only S-wave interactions are considered for the two-body and particle-dimer system
- The proximity of the T_{cc}^* to the D^*D^* threshold suppresses partial waves L>0 in the dimer.



- The dominance of S-wave in the dimer also suppress higher partial waves in the particle-dimer system
- Same approach taken in, e.g., Bayar:2022bnc and Luo:2021ggs for the study of $D^*D^*D^*$
- Luo:2021ggs studied $D^*D^*D^*$ for all L≤2 configurations

ℓ_{12}	L_3	L	t_{12}	s_{12}	S
0	0	0	0	1	0
0	2	2	0	1	2
0	2	2	1	2	2
2	0	2	0	1	2
2	0	2	1	2	2

> *"Turning on the S-D mixing only has a small effect but, in general, slightly increases the binding energy of the system of interest (for the same cutoff)."*

Derivation for three identical bosons

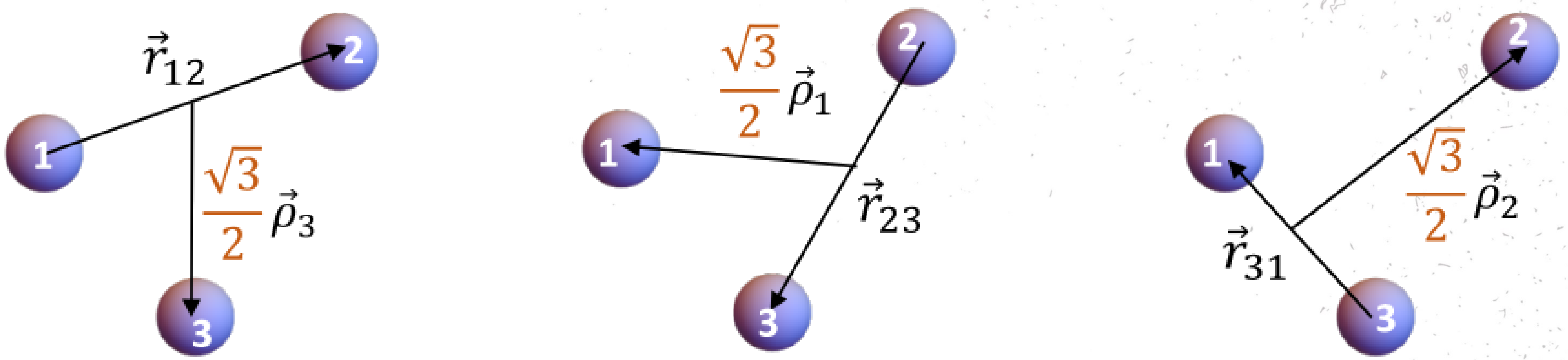
Hamiltonian in coordinate representation:

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 - \frac{\hbar^2}{2m}\nabla_3^2$$

with the two-body condition $\Psi \xrightarrow[r_{ij}\to 0]{} \propto \frac{1}{r_{ij}} - \frac{1}{a}$

1. Eliminate the centre of mass $\vec{R} = \vec{x}_1 + \vec{x}_2 + \vec{x}_3$

2. Express the remaining coordinates in terms of Jacobi coordinates:



$$\vec{r}_{ij} = \vec{x}_j - \vec{x}_i$$

$$\frac{\sqrt{3}}{2}\vec{\rho}_k = \vec{x}_k - \frac{\vec{x}_i + \vec{x}_j}{2}$$

Schrödinger equation:

$$\left(-\nabla^2_{r_{12}} - \nabla^2_{\rho_3} - k^2\right)\Psi = 0$$

For a total energy
$E = \hbar^2 k^2/m$

From P. Naidon, RIKEN

Derivation for three identical bosons

3. Make the Faddeev decomposition:

$$\Psi = \chi(\vec{r}_{12},\vec{\rho}_3) + \chi(\vec{r}_{23},\vec{\rho}_1) + \chi(\vec{r}_{31},\vec{\rho}_2)$$

$$= \chi(\vec{r},\vec{\rho}) + \chi\left(-\tfrac{1}{2}\vec{r} + \tfrac{\sqrt{3}}{2}\vec{\rho}, -\tfrac{\sqrt{3}}{2}\vec{r} - \tfrac{1}{2}\vec{\rho}\right) + \chi\left(-\tfrac{1}{2}\vec{r} - \tfrac{\sqrt{3}}{2}\vec{\rho}, \tfrac{\sqrt{3}}{2}\vec{r} - \tfrac{1}{2}\vec{\rho}\right)$$

Where χ satisfies

$$\left(-\nabla_r^2 - \nabla_\rho^2 - k^2\right)\chi(\vec{r},\vec{\rho}) = 0$$

4. Apply the two-body condition $\Psi \xrightarrow[r_{ij}\to 0]{} \propto \frac{1}{r_{ij}} - \frac{1}{a}$ $\Leftrightarrow$ $\frac{\partial}{\partial r}(r\Psi) = -\frac{1}{a}(r\Psi)$ for $r \to 0$

$$\left[\frac{\partial}{\partial r}\left(r\chi(\vec{r},\vec{\rho})\right)\right]_{r\to 0} + \left[\frac{\partial}{\partial r}\left(r\chi\left(-\cancel{\tfrac{1}{2}\vec{r}} + \tfrac{\sqrt{3}}{2}\vec{\rho}, -\cancel{\tfrac{\sqrt{3}}{2}\vec{r}} - \tfrac{1}{2}\vec{\rho}\right)\right)\right]_{r\to 0} + \left[\frac{\partial}{\partial r}\left(r\chi\left(-\cancel{\tfrac{1}{2}\vec{r}} - \tfrac{\sqrt{3}}{2}\vec{\rho}, \cancel{\tfrac{\sqrt{3}}{2}\vec{r}} - \tfrac{1}{2}\vec{\rho}\right)\right)\right]_{r\to 0}$$

$$= -\frac{1}{a}\left[r\left(\chi(\vec{r},\vec{\rho}) + \chi\left(-\cancel{\tfrac{1}{2}\vec{r}} + \tfrac{\sqrt{3}}{2}\vec{\rho}, -\cancel{\tfrac{\sqrt{3}}{2}\vec{r}} - \tfrac{1}{2}\vec{\rho}\right) + \chi\left(-\cancel{\tfrac{1}{2}\vec{r}} - \tfrac{\sqrt{3}}{2}\vec{\rho}, \cancel{\tfrac{\sqrt{3}}{2}\vec{r}} - \tfrac{1}{2}\vec{\rho}\right)\right)\right]_{r\to 0}$$

$$\left[\frac{\partial}{\partial r}\left(r\chi(\vec{r},\vec{\rho})\right)\right]_{r\to 0} + \chi\left(\tfrac{\sqrt{3}}{2}\vec{\rho}, -\tfrac{1}{2}\vec{\rho}\right) + \chi\left(-\tfrac{\sqrt{3}}{2}\vec{\rho}, -\tfrac{1}{2}\vec{\rho}\right)$$

$$= -\frac{1}{a}\left[r\left(\chi(\vec{r},\vec{\rho}) + \chi\cancel{\left(\tfrac{\sqrt{3}}{2}\vec{\rho}, -\tfrac{1}{2}\vec{\rho}\right)} + \chi\cancel{\left(-\tfrac{\sqrt{3}}{2}\vec{\rho}, -\tfrac{1}{2}\vec{\rho}\right)}\right)\right]_{r\to 0}$$

Derivation for three identical bosons

Equation:

$$\left(-\nabla_r^2 - \nabla_\rho^2 - k^2\right)\chi(\vec{r},\vec{\rho}) = 0$$

Boundary condition $r \to 0$:

$$\left[\frac{\partial}{\partial r}\left(r\chi(\vec{r},\vec{\rho})\right)\right]_{r\to 0} + \chi\left(\tfrac{\sqrt{3}}{2}\vec{\rho}, -\tfrac{1}{2}\vec{\rho}\right) + \chi\left(-\tfrac{\sqrt{3}}{2}\vec{\rho}, -\tfrac{1}{2}\vec{\rho}\right) = -\frac{1}{a}\left[r\chi(\vec{r},\vec{\rho})\right]_{r\to 0}$$

5. Expand χ in partial waves. For a total angular momentum $L = 0$,

$$\chi(\vec{r},\vec{\rho}) = \frac{\chi_0(r,\rho)}{r\rho}$$

➡ $\left(-\frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{1}{\rho}\frac{\partial^2}{\partial \rho^2}\rho - k^2\right)\frac{\chi_0(r,\rho)}{r\rho} = 0$ with $\left[\frac{\partial}{\partial r}\frac{\chi_0(r,\rho)}{\rho}\right]_{r\to 0} + 2\times\frac{\chi_0\left(\frac{\sqrt{3}}{2}\rho,\frac{1}{2}\rho\right)}{\frac{\sqrt{3}}{2}\rho\cdot\frac{1}{2}\rho} = -\frac{1}{a}\frac{\chi_0(0,\rho)}{\rho}$

➡ $\left(-\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial \rho^2} - k^2\right)\chi_0(r,\rho) = 0$ with $\left[\frac{\partial}{\partial r}\chi_0(r,\rho)\right]_{r\to 0} + \frac{8}{\sqrt{3}\rho}\chi_0\left(\frac{\sqrt{3}}{2}\rho,\frac{1}{2}\rho\right) = -\frac{1}{a}\chi_0(0,\rho)$

Derivation for three identical bosons

Equation:

$$\left(-\frac{\partial^2}{\partial r^2}-\frac{\partial^2}{\partial \rho^2}-k^2\right)\chi_0(r,\rho)=0$$

Boundary condition $r \to 0$:

$$\left[\frac{\partial}{\partial r}\chi_0(r,\rho)\right]_{r\to 0}+\frac{8}{\sqrt{3}\rho}\chi_0\left(\frac{\sqrt{3}}{2}\rho,\frac{1}{2}\rho\right)=-\frac{1}{a}\chi_0(0,\rho)$$

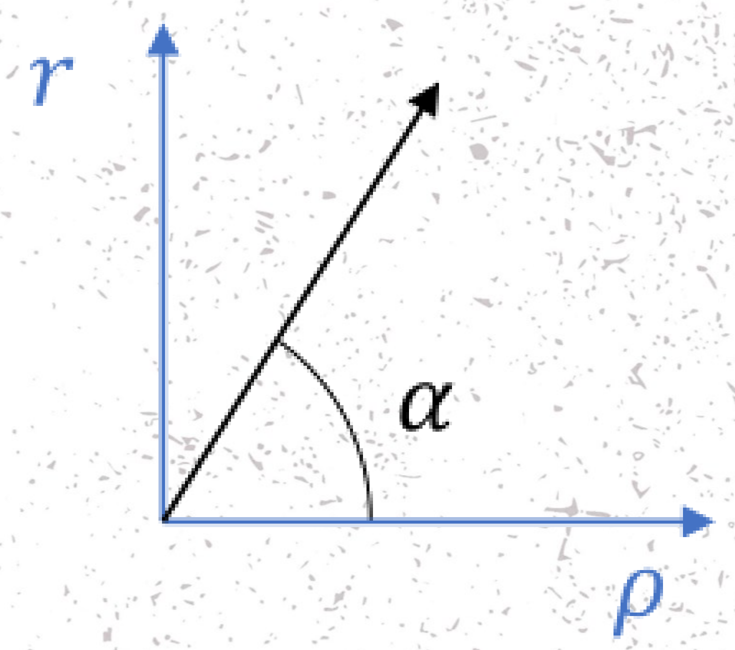
6. Change the coordinates (r,ρ) to polar coordinates (R,α)

$r = R\sin\alpha$

$\rho = R\cos\alpha$

$R = \sqrt{r^2+\rho^2}$ (hyper-radius)

$\alpha = \arctan r/\rho$ (hyper-angle)



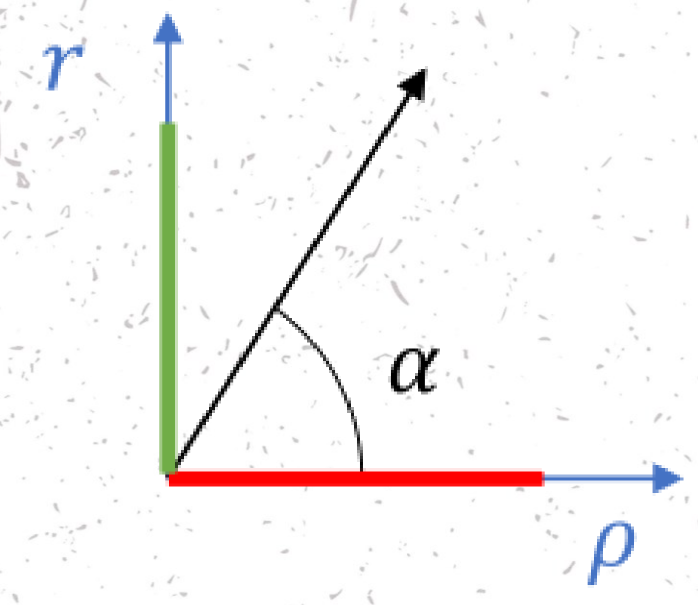
$$\left(-\frac{\partial^2}{\partial R^2}-\frac{1}{R}\frac{\partial}{\partial R}-\frac{1}{R^2}\frac{\partial^2}{\partial \alpha^2}-k^2\right)\chi_0(R,\alpha)=0$$

with $\left[\frac{\partial}{\partial \alpha}\chi_0(R,\alpha)\right]_{\alpha\to 0}+\frac{8}{\sqrt{3}}\chi_0\left(R,\frac{\pi}{3}\right)=-\frac{R}{a}\chi_0(R,0)$

Derivation for three identical bosons

Equation:

$$\left(-\frac{\partial^2}{\partial R^2}-\frac{1}{R}\frac{\partial}{\partial R}-\frac{1}{R^2}\frac{\partial^2}{\partial \alpha^2}-k^2\right)\chi_0(R,\alpha)=0$$



Solutions of the form: $\chi_0(R,\alpha)=F_n(R)\phi_n(\alpha)$

Eigenfunctions of $-\partial^2/\partial\alpha^2$:

$$-\frac{\partial^2}{\partial \alpha^2}\phi_n(\alpha)=s_n^2\phi_n(\alpha) \quad \Rightarrow \quad \phi_n(\alpha)=\sin\left(s_n\left(\frac{\pi}{2}-\alpha\right)\right)$$

$$s_n\cos\left(\frac{s_n\pi}{2}\right)+\frac{8}{\sqrt{3}}\sin\left(\frac{s_n\pi}{6}\right)=0$$

$$\left(-\frac{\partial^2}{\partial R^2}-\frac{1}{R}\frac{\partial}{\partial R}+\frac{s_n^2}{R^2}-k^2\right)F_n(R)=0$$

$$\left(-\frac{\partial^2}{\partial R^2}+\underbrace{\boxed{\frac{s_n^2-\frac{1}{4}}{R^2}}}_{V_n(R)}-k^2\right)\sqrt{R}F_n(R)=0$$

Derivation for three identical bosons

Effective Schrödinger equation for the hyper-radius R

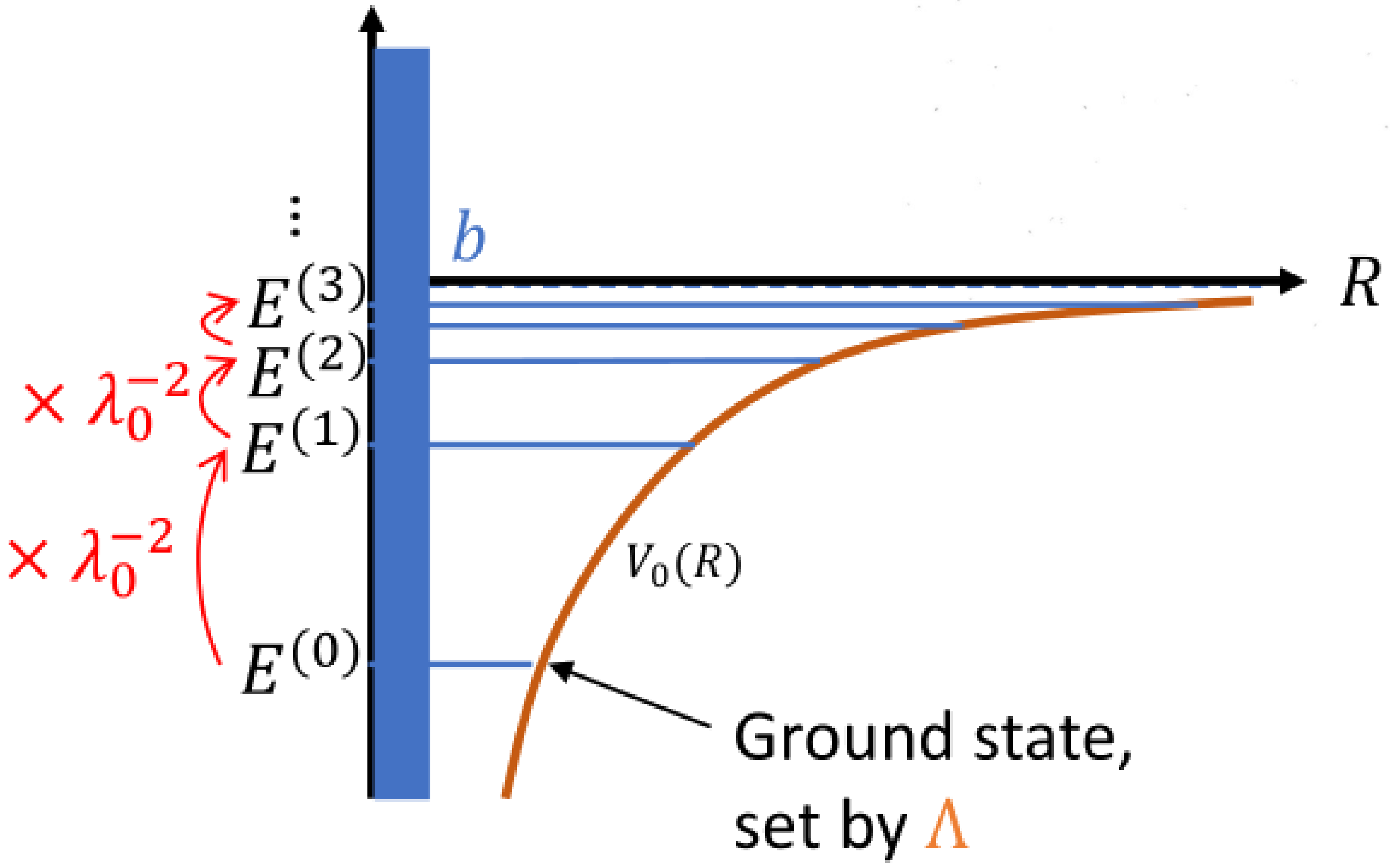
$$\left(-\frac{\partial^2}{\partial R^2}+\underbrace{\boxed{\frac{s_n^2-\frac{1}{4}}{R^2}}}_{V_n(R)}-k^2\right)\sqrt{R}F_n(R)=0$$

$$s_n\cos\left(\frac{s_n\pi}{2}\right)+\frac{8}{\sqrt{3}}\sin\left(\frac{s_n\pi}{6}\right)=0$$

All s_n are real, except one: $s_0=\pm i1.00624$

For $n=0$, one gets the Efimov attractive potential $\boxed{V_0(R)=-\frac{|s_0|^2+\frac{1}{4}}{R^2}}$

Scale invariance
$R\rightarrow\lambda R$



Ground state,
set by Λ

Solution F at energy $-\kappa^2$ ➡ Solution F at energy $-\lambda^2\kappa^2$

$$\left(-\frac{\partial^2}{\partial R^2}+\frac{s_n^2-\frac{1}{4}}{R^2}+\kappa^2\right)\sqrt{R}F_n(R)=0$$

$$\left(-\frac{\partial^2}{\partial R^2}+\frac{s_n^2-\frac{1}{4}}{R^2}+\lambda^2\kappa^2\right)\sqrt{\lambda R}F_n(\lambda R)=0$$

For small R, $F_0(R)=\alpha R^{i|s_0|}+\beta R^{-i|s_0|}\propto\cos(|s_0|\ln\Lambda R)$

Three-body parameter

$$F_0(\lambda R)\propto\cos(|s_0|\ln\Lambda\lambda R)=\cos(|s_0|\ln\Lambda R+\underbrace{|s_0|\ln\lambda}_{\pi})\propto F_0(R)$$

$$\boxed{\lambda_0=e^{\pi/|s_0|}\approx 22.7}\Rightarrow\boxed{E^{(n)}=E^{(0)}\lambda_0^{-2n}}$$