Study of forbidden $\beta$ decays within the realistic shell model
Outline

• The problem of the renormalization of $\sigma\tau$ operator
• The impact on $0\nu\beta\beta$ decay matrix element
• The $\beta$ spectrum, the shape function
• The Realistic Shell Model
• Details of the Calculation
• Conclusions and perspectives
• The problem of the renormalization of $\sigma\tau$ operator
Renormalization of $\sigma T$ matrix elements

Gamow-Teller transitions (β-decay, EC, 2νββ, charge-exchange) are hindered from expected values based on sum rules derived by nuclear structure models.
Gamow-Teller transitions (β-decay, EC, 2νββ, charge-exchange) are hindered from expected values based on sum rules derived by nuclear structure models.

Quenching of $\sigma\tau$ matrix elements is quite a general phenomenon in nuclear-structure physics.

$$g_A = g_A^{\text{eff}} = q \; g_A$$
Renormalization of $\sigma T$ matrix elements

$g_A = g_A^{\text{eff}} = q \ g_A$

Martinez-Pinedo et al. PRC53 2602(1996)

Suonen, ACTA PHYSICA POLONICA B, 3 (2018)

<table>
<thead>
<tr>
<th>Mass range</th>
<th>$g_A^{\text{eff}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Full 0$p$ shell</td>
<td>1.03$^{+0.03}_{-0.02}$</td>
</tr>
<tr>
<td>0$p$–low 1$s0d$ shell</td>
<td>1.18 ± 0.05</td>
</tr>
<tr>
<td>Full 1$s0d$ shell</td>
<td>0.96$^{+0.03}_{-0.02}$</td>
</tr>
<tr>
<td>$A = 41$–50 (1$p0f$ shell)</td>
<td>1.0</td>
</tr>
<tr>
<td>1$p0f$ shell</td>
<td>0.937$^{+0.019}_{-0.018}$</td>
</tr>
<tr>
<td>$^{56}$Ni</td>
<td>0.98</td>
</tr>
<tr>
<td>$A = 52$–67 (1$p0f$ shell)</td>
<td>0.71</td>
</tr>
<tr>
<td>$A = 67$–80 (0$f_{5/2}1p0g_{9/2}$ shell)</td>
<td>0.838$^{+0.021}_{-0.020}$</td>
</tr>
<tr>
<td>$A = 63$–96 (1$p0f0g_{1d2s}$ shell)</td>
<td>0.869 ± 0.019</td>
</tr>
<tr>
<td>$A = 76$–82 (1$p0f0g_{9/2}$ shell)</td>
<td>0.8</td>
</tr>
<tr>
<td>$A = 90$–97 (1$p0f0g_{1d2s}$ shell)</td>
<td>0.76</td>
</tr>
<tr>
<td>$^{100}$Sn</td>
<td>0.6</td>
</tr>
<tr>
<td>$A = 128$–130 (0$g_{7/2}1d2s0h_{11/2}$ shell)</td>
<td>0.52</td>
</tr>
<tr>
<td>$A = 130$–136 (0$g_{7/2}1d2s0h_{11/2}$ shell)</td>
<td>0.72</td>
</tr>
<tr>
<td>$A = 136$ (0$g_{7/2}1d2s0h_{11/2}$ shell)</td>
<td>0.94</td>
</tr>
</tbody>
</table>

Quenching of $\sigma T$ matrix elements: theory

Two main sources:

1) LIMITED MODEL SPACE

SP states can be grouped into 3 sub-spaces, well separated in energy:
- inert core (completely filled levels with neutrons and protons paired to $J=0$)
- valence space (partially filled levels)
- external space (empty levels)

The only active degrees of freedom are given by nucleons inside the valence space (valence nucleons) while excitations of core nucleons and valence nucleons in the external space are “frozen” or, more properly, “not taken into account explicitly”
Quenching of $\sigma\tau$ matrix elements: theory

Two main sources:

2) NON-NUCLEONIC DEGREES OF FREEDOM

Processes in which the weak probe prompts a meson to be exchanged between two nucleons

→ meson-exchange two-body currents (2BC)

Two-body e.w. currents effects

The contribution of 2BC improves systematically the agreement between theory and experiment.

In-Medium SRG

GT fp-shell nuclei


From ChEFT 2N +3N interaction

L. Coraggio et al. Physical Review C 109, 014301 (2024)
$2\nu\beta\beta$ decay

\[ M_{GT}^{2\nu} = \sum_k \frac{\langle 0^+_f | |\vec{\sigma} \cdot \tau^-|k\rangle \langle k ||\vec{\sigma} \cdot \tau^-||0^+_i \rangle}{E_k + E_0} \]

Blue: bare $J_A$ at LO in ChPT (namely the GT operator $g_A$)

L. Coraggio et al. Physical Review C 109, 014301 (2024)
2νββ decay

Blue: bare $J_A$ at LO in ChPT (namely the GT operator $g_A$)

Green: effective $J_A$ at LO in ChPT

$$M_{2\nu GT}^{2\nu} = \sum_k \frac{\langle 0^+_f | |\vec{s} \cdot \tau^-||k\rangle \langle k||\vec{s} \cdot \tau^-||0^+_i\rangle}{E_k + E_0}$$
2νββ decay

\[ M_{GT}^{2\nu} = \sum_k \frac{\langle 0^+_i | |\vec{\sigma} \cdot \tau^-| |k\rangle \langle k | |\vec{\sigma} \cdot \tau^-| |0^+_i \rangle}{E_k + E_0} \]

**Blue:** bare \( J_A \) at LO in ChPT (namely the GT operator \( g_A \))

**Green:** effective \( J_A \) at LO in ChPT

**Black:** bare \( J_A \) at N3LO in ChPT

---

L. Coraggio et al. Physical Review C 109, 014301 (2024)
$2\nu\beta\beta$ decay

Theoretical and experimental data for $\beta\beta$ decay.

- **Blue**: bare $J_A$ at LO in ChPT (namely the GT operator $g_A$)
- **Green**: effective $J_A$ at LO in ChPT
- **Black**: bare $J_A$ at N3LO in ChPT
- **Red**: effective $J_A$ at N3LO in ChPT

Mathematical expression:

$$M^{2\nu}_{GT} = \sum_k \frac{\langle 0^+ | |\hat{\sigma} \cdot \tau^- | k \rangle \langle k | |\hat{\sigma} \cdot \tau^- | 0^+_i \rangle}{E_k + E_0}$$

L. Coraggio et al. Physical Review C 109, 014301 (2024)
Outline

• The problem of the renormalization of $\sigma\tau$ operator

• The impact on $0\nu\beta\beta$ decay matrix element
Quenching of $\sigma T$ matrix elements & $0\nu\beta\beta$ decay

The inverse of the $0\nu\beta\beta$-decay half-life is proportional to the squared nuclear matrix element (NME). This evidences the relevance to calculate the NME ($M^{0\nu}$)

\[
[T_{1/2}^{0\nu}]^{-1} = G^{0\nu} |M^{0\nu}|^2 \langle m_\nu \rangle^2 \propto g_A^4
\]

- $G^{0\nu}$ → phase space factor
- $\langle m_\nu \rangle = |\sum_k m_k U_{ek}|$, effective mass of the Majorana neutrino

$U_{ek}$ being the lepton mixing matrix
Outline

• The problem of the renormalization of $\sigma\tau$ operator

• The impact on $0\nu\beta\beta$ decay matrix element

• The $\beta$ spectrum, the shape function
The total half-life of the $\beta$ decay is expressed in terms of the $k$-th partial decay half-life as

$$\frac{1}{T_{1/2}} = \sum_k \frac{1}{t_{1/2}^k}$$

$$t_{1/2}^k = \frac{\kappa}{C}$$

where $\kappa = 6144$ s
The total half-life of the $\beta$ decay is expressed in terms of the k-th partial decay half-life as

$$\frac{1}{T_{1/2}} = \sum_k \frac{1}{t_{1/2}^k} \quad t_{1/2}^k = \frac{\kappa}{\tilde{C}}$$

where $\kappa = 6144$ s

$\tilde{C}$ is the integrated shape function, whose integrand defines the $\beta$-decay energy spectrum

$$\tilde{C} = \int_{1}^{w_0} p_e w_e (w_0 - w_e)^2 F(Z, w_e) C_n(w_e) dw_e$$

- $F(Z, w_e) = F_0(Z, w_e)L_0(Z, w_e)$, $F$ is the Fermi function and takes into account the distortion of the electron wave function by the nuclear charge and $L_0$ accounts for the finite size effect.
- $w_e$ is the electron energy
- $C_n(w_e)$ is the shape factor of the n-th forbidden transition which depends on the nuclear matrix elements (NMEs) of the decay operators.

H. Behrens and W. Büring, Nuclear Physics A62 (1971) 11 1 – 144

G. De Gregorio et al. to be published on Phys. Rev. C
The $\beta$ decay spectrum

$$\tilde{C} = \int_1^{w_0} p_e w_e (w_0 - w_e)^2 F(Z, w_e) C_n(w_e) dw_e$$

In general $C_n$ is function of $W_e$.

For allowed transition: $C_n(w_e) = \text{Const} = B(\text{GT}) = g_A^2 \frac{|\langle f || \sum_k a_k \tau_k || i \rangle|^2}{2J_i + 1}$

Beta spectrum is insensitive to $B(\text{GT})$

The $\beta$ decay spectrum

\[ \tilde{C} = \int_1^{w_0} p_e w_e (w_0 - w_e)^2 F(Z, w_e) C_n(w_e) dw_e \]

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For allowed transition:

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Beta spectrum is insensitive to $B(\text{GT})$

Beta spectrum is insensitive to $L^{\text{th}}$-unique decays
The β decay spectrum

\[ \tilde{C} = \int_1^{w_0} p_e w_e (w_0 - w_e)^2 F(Z, w_e) C_n(w_e) \, dw_e \]

Beta spectrum is sensitive to L\textsuperscript{th}-nonunique decays


M. Paulsen et al. arXiv:2309.14014v1
The $\beta$ decay spectrum

\[ \tilde{C} = \int_{1}^{w_0} p_e w_e (w_0 - w_e)^2 F(Z, w_e) C_n(w_e) dw_e \]

Beta spectrum is sensitive to $L^{th}$-nonunique decays

1. One can extract the empirical value of $q$ from the $\beta$ spectrum on forbidden non-unique decays.

2. Alternatively, having a microscopic theory without free parameters, $\beta$ spectra offer a further benchmark of the theoretical framework.
Outline

• The problem of the renormalization of $\sigma \tau$ operator
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• The Realistic Shell Model
Realistic Shell-Model

Shell model: A well-established approach to obtain a microscopic description of both collective and single-particle properties of nuclei.
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The degrees of freedom of the core nucleons and the excitations of the valence ones above the model space are not considered explicitly.
Realistic Shell-Model

Shell model: A well-established approach to obtain a microscopic description of both collective and single-particle properties of nuclei.

The degrees of freedom of the core nucleons and the excitations of the valence ones above the model space are not considered explicitly.

\[ V_{NN} (+V_{NNN}) \]
\[ \downarrow \]
\[ MANY \ BODY \ PERTURBATION \ THEORY \]
\[ \downarrow \]
\[ H_{eff} \]

Realistic shell-model calculations starting from a nuclear Hamiltonian and decay operators derived consistently.
$H \rightarrow H_{\text{eff}}$

$H |\psi_v\rangle = E_v |\psi_v\rangle \rightarrow H_{\text{eff}} |\varphi_\alpha\rangle = E_v |\varphi_\alpha\rangle$

$|\varphi_\alpha\rangle = \text{eigenvectors obtained diagonalizing } H_{\text{eff}} \text{ in the reduced model space } |\varphi_\alpha\rangle = P |\psi_v\rangle$
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$\langle \varphi_v | \Theta | \varphi_\lambda \rangle \neq \langle \Psi_v | \Theta | \Psi_\lambda \rangle$
Realistic Shell-Model

\[ H \rightarrow H_{eff} \]

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\[ \langle \varphi_v|\Theta|\varphi_\lambda\rangle \neq \langle \Psi_v|\Theta|\Psi_\lambda\rangle \]

We then require an effective operator \( \Theta_{eff} \) defined as follows

\[ \Theta_{eff} = \sum_{\nu\lambda} |\varphi_v\rangle \langle \Psi_v|\Theta|\Psi_\lambda\rangle \langle \varphi_\lambda| \]

\[ \langle \varphi_v|\Theta_{eff}|\varphi_\lambda\rangle = \langle \Psi_v|\Theta|\Psi_\lambda\rangle \]
• The problem of the renormalization of $\sigma$ operator
• The impact on $0\nu\beta\beta$ decay matrix element
• The $\beta$ spectrum, the shape function
• The realistic shell model
• Details of the Calculation
Details of the Calculations

Calculations for nuclei in the neighborhood of 0νββ candidates above 78Ni core:

- Second forbidden non-unique gs to gs β-decay of $^{94}$Nb and $^{99}$Tc
- Fourth forbidden non-unique gs to gs β-decay of $^{113}$Cd and $^{115}$In

One major proton and neutron shell above $^{78}$Ni

Interaction

$V_{\text{low-k}}$ of CD-Bonn with $\Lambda=2.6$ fm$^{-1}$ + Coulomb force for protons

TBME +SP energies of $H_{\text{eff}}$ & $\Theta_{\text{eff}}$ @ third order +3-body correlations
\[ ft = \frac{\kappa}{\bar{c}} \int_1^{w_0} p_e w_e (w_0 - w_e)^2 F(Z, w_e) \, dw_e \]

<table>
<thead>
<tr>
<th></th>
<th>Bare</th>
<th>Effective</th>
<th>Exp</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^{94})Nb</td>
<td>11.30</td>
<td>11.58</td>
<td>11.95 (7)</td>
</tr>
<tr>
<td>(^{99})Tc</td>
<td>11.580</td>
<td>11.876</td>
<td>12.325 (12)</td>
</tr>
<tr>
<td>(^{113})Cd</td>
<td>21.902</td>
<td>22.493</td>
<td>23.127 (14)</td>
</tr>
<tr>
<td>(^{115})In</td>
<td>21.22</td>
<td>21.64</td>
<td>22.53 (3)</td>
</tr>
</tbody>
</table>
RSM $\beta$ decay spectra

$^{94}\text{Nb}$

$^{99}\text{Tc}$

Intensity (arb. units)

$T_e$ (keV)

$\text{Exp}$

$\text{Bare}$
RSM $\beta$ decay spectra

$^{94}\text{Nb}$ and $^{99}\text{Tc}$ decay spectra showing intensity vs. kinetic energy ($T_e$) for bare and effective cases.
RSM $\beta$ decay spectra

$^{94}$Nb

$^{99}$Tc

Intensity (arb. units)

$T_e$ (keV)

Exp

Bare

Effective

q=0.27

q=0.50
RSM $\beta$ decay spectra

$^{113}$Cd

$^{115}$In
RSM $\beta$ decay spectra

113Cd

115In

Intensity (arb. units) vs. $T_e$ (keV)

Exp, Bare, Effective
RSM $\beta$ decay spectra

$^{113}$Cd

$^{115}$In
Conclusions and Perspectives

Conclusions

- The theoretical $log f t$ moves towards experiment by employing the theoretical effective operators.
Conclusions

• The theoretical log ft moves towards experiment by employing the theoretical effective operators.

• The shape of the calculated energy spectra is rather insensitive to the choice of the β-decay operator, bare or effective, and in both cases the reproduction of the observed normalized energy spectra is more than satisfactory.
Conclusions and Perspectives

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• The shape of the calculated energy spectra is rather insensitive to the choice of the $\beta$-decay operator, bare or effective, and in both cases the reproduction of the observed normalized energy spectra is more than satisfactory.

• Using the bare operator, but introducing a quenching factor of the axial constant to improve the reproduction of the experimental logfts, there is a distortion of the shape of the energy spectra that affects the agreement with the observed ones.
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Perspectives

• Study of forbidden $\beta$ decays in other mass regions, close to nuclei that are candidates for detecting the $0\nu\beta\beta$ decay
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Perspectives

• Study of forbidden $\beta$ decays in other mass regions, close to nuclei that are candidates for detecting the $0\nu\beta\beta$ decay

• Tackle the forbidden $\beta$-decay problem starting from the derivation of electroweak currents by way of the chiral Perturbation theory
Details of the Calculations

- **Nuclear Hamiltonian**: Entem-Machleidt N3LO two-body potential plus N2LO three-body potential (Λ = 500 MeV)

- **Axial current** $J_A$ calculated at N3LO in ChPT

- **Heff** obtained calculating the **Q-box** up to the 3rd order in $V_{NN}$ (up to 2p-2h core excitations) and up to the 1st order in $V_{NNN}$

- **Effective operators** are consistently derived by way of the MBPT

- **fp-shell nuclei**: four proton and neutron orbitals outside $^{40}\text{Ca}$: 0f7/2, 0f5/2, 1p3/2, 1p1/2

- **fpg-shell nuclei**: four proton and neutron orbitals outside $^{56}\text{Ni}$: 0f5/2, 1p3/2, 1p1/2, 0g9/2

RSM calculations, starting from ChPT two- and three-body potentials and two-body meson-exchange currents for spectroscopic and spin-isospin dependent observables of $^{48}\text{Ca}$, $^{76}\text{Ge}$, $^{82}\text{Se}$
Quenching of $\sigma T$ matrix elements: meson exchange currents

Nowadays, EFT provides a powerful approach where both nuclear potentials and two electroweak currents may be consistently constructed, the latter appearing as subleading corrections to the one-body GT operator $\sigma T^\pm$.
The axial current $J_A$

The matrix elements of the axial current $J_A$ are derived through a chiral expansion up to N3LO, and employing the same LECs as in 2NF and 3NF

$$J_A^{LO} = -g_A \sum_i \sigma_i \tau_{i,\pm}$$

$$J_A^{N^2LO} = \frac{g_A}{2m_N^2} \sum_i K_i \times (\sigma_i \times K_i) \tau_{i,\pm}$$

$$J_A^{N^3LO} (1PE; k) = \sum_{i<j} \frac{g_A}{2f_\pi^2} \{4c_3 \tau_j k + (\tau_i \times \tau_j)_\pm \times \left[ (c_4 + \frac{1}{4m} \sigma_i \times k - \frac{i}{2m} K_i) \right] \} \sigma_j \cdot k \frac{1}{\omega_k^2}$$

$$J_A^{N^3LO} (CT; k) = \sum_{i<j} z_0 (\tau_i \times \tau_j)_\pm (\sigma_i \times \sigma_j)$$

$$z_0 = \frac{g_A}{2f_\pi^2 m_N^2} \left[ \frac{m_N}{4g_A \Lambda_X} \right] \left[ c_D + \frac{m_N}{3} (c_3 + 2c_4) + \frac{1}{6} \right]$$

Effective Shell-Model Hamiltonian

\[ H |\Psi_\nu \rangle = E_\nu |\Psi_\nu \rangle \]

\[ H = H_0 + H_1 = \sum_{i=1}^{A} T_i + U_i + \sum_{i<j} (V_{ij} - U_i) \]
Effective Shell-Model Hamiltonian

\[ H |\Psi_v\rangle = E_v |\Psi_v\rangle \]

\[ H = H_0 + H_1 = \sum_{i=1}^{A} T_i + U_i + \sum_{i<j} (V_{ij} - U_i) \]

In the shell model, the nucleus is represented as an inert core, plus n valence nucleons moving in a limited number of SP orbits above the closed core and interacting through a model-space effective interaction. The valence or model space is defined in terms of the eigenvectors of \( H_0 \).
Effective Shell-Model Hamiltonian

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In the shell model, the nucleus is represented as an inert core, plus n valence nucleons moving in a limited number of SP orbits above the closed core and interacting through a model-space effective interaction. The valence or model space is defined in terms of the eigenvectors of \( H_0 \)

\[ |\phi_i\rangle = [a_1^+ a_2^+ a_3^+ ... a_n^+]_i |c\rangle \quad i=1,2,...,d \]

**GOAL:**

\[ H |\Psi_v\rangle = E_v |\Psi_v\rangle \quad \Rightarrow \quad H_{eff} P |\Psi_v\rangle = E_v P |\Psi_v\rangle \]
Effective Shell-Model Hamiltonian

We are looking for a new hamiltonian $H$ that has the same eigenvalues of the A-nucleon system hamiltonian $H$, and satisfies the decoupling equation between the model space $P$ and its complement $Q$:

$$QHP = 0 \text{ so that } H_{\text{eff}} = P\mathcal{H}P$$
We are looking for a new hamiltonian $\mathcal{H}$ that has the same eigenvalues of the A-nucleon system hamiltonian $H$, and satisfies the decoupling equation between the model space $P$ and its complement $Q$:

$$QHP = 0 \text{ so that } H_{\text{eff}} = P\mathcal{H}P$$

$$\mathcal{H} = \Omega^{-1}H\Omega$$

$$\begin{pmatrix} PHP & PHQ \\ QHP & QHQ \end{pmatrix} \rightarrow \begin{pmatrix} PHP & PHQ \\ 0 & QHQ \end{pmatrix}$$
Effective Shell-Model Hamiltonian

We are looking for a new hamiltonian $\mathcal{H}$ that has the same eigenvalues of the A-nucleon system hamiltonian $H$, and satisfies the decoupling equation between the model space $P$ and its complement $Q$:

$$QHP = 0 \text{ so that } H_{\text{eff}} = P\mathcal{H}P$$

$$\mathcal{H} = \Omega^{-1}H\Omega$$

$$(PHP \ PHQ) \rightarrow (PHP \ PHQ)$$

Suzuki-Lee

$\Omega = e^{\omega}$

$$\omega = \begin{pmatrix} 0 & 0 \\ Q\omega P & 0 \end{pmatrix}$$

$$H_{\text{eff}}(\omega) = PH_1P + PH_1Q \frac{1}{\epsilon - QHQ} QH_1P - PH_1Q \frac{1}{\epsilon - QHQ} QH_1^{\text{eff}}(\omega)$$
Effective Shell-Model Hamiltonian

$$\hat{Q}(\epsilon) = PH_1 P + PH_1 Q \frac{1}{\epsilon - QHQ} \Omega H_1 P$$
Effective Shell-Model Hamiltonian

\[ \hat{Q}(\epsilon) = PH_1 P + PH_1 Q \frac{1}{\epsilon - Q H Q} Q H_1 P \]

Exact calculation of the \( \hat{Q} \) – box is computationally prohibitive (for manybody system) we perform a perturbative expansion

\[ \frac{1}{\epsilon - Q H Q} = \sum_{n=0}^{\infty} \frac{(Q H_1 Q)^n}{(\epsilon - Q H_0 Q)^{n+1}} \]

Second order diagrams
Effective Shell-Model Operators

Consistently with the derivation of the effective Hamiltonian we have to derive effective transition operators, by way of the many-body perturbation theory

$$|\varphi_v\rangle = P|\Psi_v\rangle$$

Obviously, for any decay-operator $\Theta$

$$\langle\varphi_v|\Theta|\varphi_\lambda\rangle \neq \langle\Psi_v|\Theta|\Psi_\lambda\rangle$$
Effective Shell-Model Operators

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Obviously, for any decay-operator \( \Theta \)

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We then require an effective operator \( \Theta_{\text{eff}} \) defined as follows

\[ \Theta_{\text{eff}} = \sum_{\nu\lambda} |\varphi_v\rangle \langle \Psi_v | \Theta | \Psi_\lambda \rangle \langle \varphi_\lambda | \] \[ \langle \varphi_v | \Theta_{\text{eff}} | \varphi_\lambda \rangle = \langle \Psi_v | \Theta | \Psi_\lambda \rangle \]

Second order one body diagrams for a one-body operator

Forbidden $\beta$-decay ME

$$\phi_{nljm} = \begin{pmatrix} G_{nljm}(r) \\ F_{nljm}(r) \end{pmatrix}$$

$$\frac{dg_k(r)}{dr} + \frac{\kappa + 1}{r} g_k(r) - (E + m - V(r)) f_k(r) = 0$$

$$\frac{df_k(r)}{dr} - \frac{\kappa - 1}{r} f_k(r) + (E - m - V(r)) g_k(r) = 0$$

$$V/A M^{(N)}_{KL}(k_e, m, n, \rho) = \frac{1}{J_i} \sum_{\pi, \nu} V/A m^{(N)}_{KL}(\pi, \nu)(k_e, m, n, \rho) \times \text{OBTD}(\Psi_f, \Psi_i, \pi, \nu, K)$$

$$V m^{(N)}_{KL}(k_e, m, n, \rho) = \langle \phi_{\kappa \pi \mu} \parallel \left( \frac{r}{R} \right)^{L+2N} \mathcal{I}(k_e, m, n, \rho, r) T_{KL} \parallel \phi_{\kappa \nu \mu} \rangle$$

$$A m^{(N)}_{KL}(k_e, m, n, \rho) = \langle \phi_{\kappa \pi \mu} \parallel \left( \frac{r}{R} \right)^{L+2N} \mathcal{I}(k_e, m, n, \rho, r) \gamma_5 T_{KL} \parallel \phi_{\kappa \nu \mu} \rangle$$

$$T_{KL}^{M} = \begin{cases} Y_{LM} \delta_{KL} & S = 0 \\ (-1)^{L-K+1} \gamma_5 [Y_L \otimes \sigma]_{KM} & S = 1 \end{cases}$$
Non-relativistic ME

\[ \phi_{nljm} = \begin{pmatrix} G_{nljm}(r) \\ F_{nljm}(r) \end{pmatrix} \]

\[ \frac{d g_k(r)}{dr} + \frac{\kappa + 1}{r} g_k(r) - (E + m - V(r)) f_k(r) = 0 \]

\[ \frac{d f_k(r)}{dr} - \frac{\kappa - 1}{r} f_k(r) + (E - m - V(r)) g_k(r) = 0 \]

\[ V m_{KK0}^{(N)}(\pi, \nu)(k_e, m, n, \rho) = \sqrt{2} g_V \left[ G_{KK0}(k_\pi, k_\nu) \int_0^\infty g_\pi(r, k_\pi) \left( \frac{r}{R} \right)^{K+2N} \mathcal{I}(k_e, m, n, \rho, r) g_\nu(r, k_\nu) r^2 dr \right. \]

\[ \left. + G_{KK0}(-k_\pi, -k_\nu) \int_0^\infty f_\pi(r, k_\pi) \left( \frac{r}{R} \right)^{K+2N} \mathcal{I}(k_e, m, n, \rho, r) f_\nu(r, k_\nu) r^2 dr \right] \]

(27)

\[ A m_{KL1}^{(N)}(\pi, \nu)(k_e, m, n, \rho) = \text{sign}(K - L + \frac{1}{2}) \sqrt{2} g_A \left[ G_{KK0}(k_\pi, k_\nu) \int_0^\infty g_\pi(r, k_\pi) \left( \frac{r}{R} \right)^{L+2N} \mathcal{I}(k_e, m, n, \rho, r) g_\nu(r, k_\nu) r^2 dr \right. \]

\[ \left. + G_{KK0}(-k_\pi, -k_\nu) \int_0^\infty f_\pi(r, k_\pi) \left( \frac{r}{R} \right)^{L+2N} \mathcal{I}(k_e, m, n, \rho, r) f_\nu(r, k_\nu) r^2 dr \right]. \]

(28)
$\phi_{nljm} = \begin{pmatrix} G_{nljm}(r) \\ F_{nljm}(r) \end{pmatrix}$

\[
\begin{align*}
\frac{dg_\kappa(r)}{dr} + \frac{\kappa + 1}{r} g_\kappa(r) - (E + m - V(r)) f_\kappa(r) &= 0 \\
\frac{df_\kappa(r)}{dr} - \frac{\kappa - 1}{r} f_\kappa(r) + (E - m - V(r)) g_\kappa(r) &= 0
\end{align*}
\]

\[
V_{m_{KL1}^{(N)}(\pi, \nu)}(k_e, m, n, \rho) = \text{sign}(K - L + \frac{1}{2}) \sqrt{2} g_V \left[ G_{KL1}(\kappa_\pi, -\kappa_\nu) \int_0^\infty g_\pi(r, \kappa_\pi) \left( \frac{r}{R} \right)^{L+2N} \mathcal{I}(k_e, m, n, \rho, r) f_\nu(r, \kappa_\nu) r^2 dr \right] \\
- G_{KK0}(\kappa_\pi, \kappa_\nu) \int_0^\infty f_\pi(r, \kappa_\pi) \left( \frac{r}{R} \right)^{L+2N} \mathcal{I}(k_e, m, n, \rho, r) g_\nu(r, \kappa_\nu) r^2 dr 
\]

(29)

\[
A_{m_{KK0}^{(N)}}(\pi, \nu)(k_e, m, n, \rho) = \sqrt{2} g_A \left[ G_{KK0}(\kappa_\pi, -\kappa_\nu) \int_0^\infty g_\pi(r, \kappa_\pi) \left( \frac{r}{R} \right)^{K+2N} \mathcal{I}(k_e, m, n, \rho, r) f_\nu(r, \kappa_\nu) r^2 dr \right] \\
- G_{KK0}(\kappa_\pi, \kappa_\nu) \int_0^\infty f_\pi(r, \kappa_\pi) \left( \frac{r}{R} \right)^{K+2N} \mathcal{I}(k_e, m, n, \rho, r) g_\nu(r, \kappa_\nu) r^2 dr 
\]

(30)
Treatment of relativistic ME within a non-relativistic framework

1) In the non-relativistic limit the small and large component are connected

\[
\frac{dg_\kappa(r)}{dr} + \frac{\kappa + 1}{r}g_\kappa(r) - (E + M - V(r))f_\kappa(r) = 0,
\]

\[
\frac{df_\kappa(r)}{dr} - \frac{\kappa - 1}{r}f_\kappa(r) + (E - M - V(r))g_\kappa(r) = 0
\]

T (E-M) \ll 2M and V(r) \ll 2M

\[
f_\kappa(r) = \frac{1}{2M_N} \left( \frac{d}{dr} + \frac{\kappa + 1}{r} \right)g_\kappa(r)
\]

\[
\left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{\kappa(\kappa + 1)}{r^2} + 2M_N[T - V(r)] \right)g_\kappa(r) = 0.
\]

Treatment of relativistic ME within a non-relativistic framework

1) In the non-relativistic limit, the small and large components are connected

\[
\frac{d g_\kappa(r)}{dr} + \frac{\kappa + 1}{r} g_\kappa(r) - (E + M - V(r)) f_\kappa(r) = 0,
\]

\[
\frac{d f_\kappa(r)}{dr} - \frac{\kappa - 1}{r} f_\kappa(r) + (E - M - V(r)) g_\kappa(r) = 0
\]

\[T \ (E-M) \ll 2M \text{ and } V(r) \ll 2M\]

\[f_\kappa(r) = \frac{1}{2M_N} \left( \frac{d}{dr} + \frac{\kappa + 1}{r} \right) g_\kappa(r) \quad \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{\kappa(\kappa + 1)}{r^2} + 2M_N [T - V(r)] \right) g_\kappa(r) = 0.\]

2) Resort to CVC relations

\[- \sqrt{L(2L+1)} V F_{LL-11} + \frac{(qR)^2}{2L+3} V F_{LL+11} - W_0 R V F_{LL0} = R C_L.\]

\[v F_{211} = -\frac{1}{\sqrt{10}} \gamma v F_{220}, \quad v F_{431} = -\frac{1}{\sqrt{36}} \gamma v F_{440}\]

TABLE III. $^{99}$Tc, $^{113}$Cd, and $^{115}$In $\beta$ decay relativistic form factors determined with and without recurring to the CVC relations, and the non-relativistic form factors connected with the relativistic ones by CVC. The values are in adimensional units.

<table>
<thead>
<tr>
<th></th>
<th>Bare</th>
<th>Effective</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{99}$Tc</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V_{F_{211}}$</td>
<td>0.000</td>
<td>0.008</td>
</tr>
<tr>
<td>$V_{F_{211}^{\text{CVC}}}$</td>
<td>-0.030</td>
<td>-0.017</td>
</tr>
<tr>
<td>$V_{F_{220}}$</td>
<td>0.286</td>
<td>0.161</td>
</tr>
<tr>
<td>$^{113}$Cd</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V_{F_{431}}$</td>
<td>0.0003</td>
<td>-0.008</td>
</tr>
<tr>
<td>$V_{F_{431}^{\text{CVC}}}$</td>
<td>0.032</td>
<td>0.015</td>
</tr>
<tr>
<td>$V_{F_{440}}$</td>
<td>-0.521</td>
<td>-0.237</td>
</tr>
<tr>
<td>$^{115}$In</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V_{F_{431}}$</td>
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<td>-0.009</td>
</tr>
<tr>
<td>$V_{F_{431}^{\text{CVC}}}$</td>
<td>0.031</td>
<td>0.017</td>
</tr>
<tr>
<td>$V_{F_{440}}$</td>
<td>-0.473</td>
<td>-0.267</td>
</tr>
</tbody>
</table>