The QCD static potential

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static potential
→ dominant interaction between heavy $q$-$\bar{q}$ at low energy $T = 0$
- attractive
- coulomb-like at small $r$ (linearly rising at large $r$)
$T \neq 0$
- the short-distance potential screened
- yukawa-like with screening mass $\propto T$

proposed signal for QGP formation:
suppression of heavy $q$-$\bar{q}$ bound state production at high $T$

$T.\ Matsui\ and\ H.\ Satz,\ Phys.\ Lett.\ B\ 178,\ 416-422\ (1986)$
at LO and in thermal equilibrium:

$V(r)$ from f-transform of temporal component of thermal gluon propagator in zero frequency limit

real part: (using $\alpha = g^2 C_F/(4\pi)$)

$$G_{lo}(0, p) = -\frac{1}{m_D^2 + p^2}$$

$$V_{1lo}(\vec{r}) = g^2 C_F \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{r}} G_{lo}(0, p) = -\frac{\alpha}{r} e^{-m_D r}$$

will consider:

1. effects of an anisotropic momentum distribution on the real part

MEC, G. Kustatter and A. Mukherjee, arXiv:2405.05622

2. corrections in thermal equilibrium beyond LO

anisotropy

anisotropic distribution function:

\[ n(\vec{k}) = C_\xi \, n(kH_\xi(\vec{\nu})) \quad \text{with} \quad n(k) = \frac{1}{e^{\beta(k)} + 1} \quad \text{and} \quad \vec{\nu} = \hat{k} \]

spheroidal anisotropy \((\hat{n}_3 = \hat{z})\)

\[ H^2_\xi(\vec{\nu}) = 1 + \xi(\vec{n}_3 \cdot \vec{\nu})^2 \]

\[ P. \text{Romatschke and M. Strickland, Phys. Rev. D 68, 036004 (2003).} \]

generalization \((\hat{n}_1 \cdot \hat{n}_3 = 0)\)

\[ H^2_\xi(\vec{\nu}) = (1 + \xi_0) + \xi_2(\vec{n}_1 \cdot \vec{\nu})^2 + \xi_9(\vec{n}_3 \cdot \vec{\nu})^2 + \xi_6(\vec{n}_1 \cdot \vec{\nu})(\vec{n}_3 \cdot \vec{\nu}) + \xi_4(\vec{n}_1 \cdot \vec{\nu})^4 \]
\[ + \xi_8(\vec{n}_1 \cdot \vec{\nu})^3(\vec{n}_3 \cdot \vec{\nu}) + \xi_11(\vec{n}_1 \cdot \vec{\nu})^2(\vec{n}_3 \cdot \vec{\nu})^2 + \xi_{13}(\vec{n}_1 \cdot \vec{\nu})(\vec{n}_3 \cdot \vec{\nu})^3 + \xi_{14}(\vec{n}_3 \cdot \vec{\nu})^4. \]

set \(\xi_i\): iso distro expanded/contracted in dirn \(\hat{\nu}\) if \(H_\xi(\vec{\nu}) < \) \(\Rightarrow 1\)

\[ MEC, B.M. \text{Forster and S. Makar, Phys. Rev. C 104, 064908 (2021).} \]
normalization:

\[ [m_D^2]_\xi = 8g^2 \frac{C_\xi}{(2\pi)^3} \int d\Omega \int dk \ k \ n_f(kH) \equiv m_D^2 \]

motivation: makes threshold of potential independent of \( \xi_i \)  
- will calculate binding energy from schrödinger equation  
⇒ want to change \( \xi_i \) without moving the whole potential up/down

the calculation:

1. \( n(\vec{k}) \to \) calculate \( \Pi_{\mu\nu} \) in the hard-loop approximation
2. find the propagator by inverting \( D_{\mu\nu}^{-1} = D_{0\mu\nu}^{-1} - \Pi_{\mu\nu} \)
3. temporal component \( D_{00}(\vec{p}) \to V(\vec{r}) \)
spheroidal distribution:

\[ \hat{V} = \frac{V}{(g^2 C_F m_D)} \]

if \( q-\bar{q} \) pair aligned/anti-aligned with dirn of anisotropy (\( x = \cos(\theta) \))

\[ \rightarrow \text{potential is deeper/shallower than isotropic} \]

** for more general distributions the structure is richer**
results: weak anisotropy

\[ \bar{m}_D = m_D (1 - \sum_i \xi_i h_i) \text{ where } h_i \text{ are numerical coefficients} \]

\[ V(r, \theta, \phi) = -\alpha \bar{m}_D - \frac{\alpha e^{-r m_D}}{r} (1 - \sum_i \xi_i f_i (rm_D, \theta, \phi)) - \alpha (1 - e^{-r m_D}) (m_D - \bar{m}_D) \]

satisfies:

\[ \lim_{\xi_i \to 0} V(r, \theta, \phi) = -\alpha m_D - \frac{\alpha}{r} e^{-r m_D} \]

\[ \lim_{r \to 0} V(r, \theta, \phi) = -\frac{\alpha}{r} \]

\[ \lim_{r \to \infty} V(r, \theta, \phi) = -\alpha m_D \]
effect of anisotropy on binding energy

method for weak anisotropy:

• ansatz: $V(\vec{r}) \rightarrow$ effective screening mass $\tilde{m}_D(rm_D, \theta, \phi)$

• reformulate result with an angle averaged screening masses
  - constructed from integrals weighted with spherical harmonics
  ⇒ physical effects of the anisotropy are packaged into screening masses that depend on the anisotropy parameters


⇒ calculate binding energies by solving 1d Schrödinger equation
results: ground state $E_b$ minus binding energy of isotropic state

<table>
<thead>
<tr>
<th></th>
<th>$\xi_2$</th>
<th>$\xi_4$</th>
<th>$\xi_{14}$</th>
<th>$\xi_{11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-0.617$</td>
<td>-4.30</td>
<td>-1.52</td>
<td>-1.52</td>
<td>-0.866</td>
</tr>
</tbody>
</table>

$\xi_9 = 0.95$ and $g = 1.85$

parameter in top row is 0.8 and all others set to zero

$|E_b|$ increases - anisotropy promotes binding in the ground state
- *not always true for excited states*

$$H_\xi^2(\vec{v}) = 1 + \xi_2(\vec{n}_1 \cdot \vec{v})^2 + \xi_9(\vec{n}_3 \cdot \vec{v})^2 + \xi_4(\vec{n}_1 \cdot \vec{v})^4 + \xi_{11}(\vec{n}_1 \cdot \vec{v})^2(\vec{n}_3 \cdot \vec{v})^2 + \xi_{14}(\vec{n}_3 \cdot \vec{v})^4 + \ldots$$
motivation:
static potential has an imaginary part
\[ \text{Im}[V] > \text{Re}[V] \] when screening effects become important
- bound states disappear because decay (become wide resonances)
- not because \( V \) is screened *too shallow to support them*


- work beyond leading order
- in the temperature range where bound states start melting
  ● a check of the idea of quarkonium dissociation
  ● provides a wider set of physically motivated forms of the potential
- *to use as input for methods to extract \( V \) from lattice correlators*
potential obtained from real time QCD (rectangular) wilson loop

\[
W(t, r) = \frac{1}{N_c} \left\langle \mathcal{P} \exp \left( ig \int A_\mu(z) z^\mu \right) \right\rangle
\]

\[
V(r) = \lim_{t \to \infty} \frac{i}{t} \ln[W(t, r)]
\]

- thermalized plasma
- \( M_q \gg \) all other physical scales
  static quark and antiquark are (unthermalised) probe particles
- couple to \( A_0 \) on \( t \)-ordered branch of CTP contour
- we use Coulomb gauge and dimensional regularization
\[ \tilde{V}_{lo}(p): g \ll 1 \& \text{typical } \tilde{p} \text{ exchange btwn } q-\bar{q} \text{ is } p \ll T \]

\[ \tilde{V}_{lo}(p) = g^2 C_F G_{lo}(0, p) \]
\[ G_{lo}(0, p) = -\frac{1}{m_D^2 + p^2} + \frac{i\pi T m_D^2}{p (m_D^2 + p^2)^2} \]

we calculate beyond LO corrections to \( \tilde{V}_{lo}(p) \)
we consider \( p \sim g^a T \text{ with } 1/3 < a < 2/3 \)
\( \Rightarrow \) which gives \( m_D \ll p \ll T \)

- upper bound on \( p \): from condition \( \text{Re } \tilde{V}_{lo}(p) \sim \text{Im } \tilde{V}_{lo}(p) \)
  - width is so big the bound state decays
- lower bound on \( p \): require \( p \) semi-hard
  - calculation of beyond leading order potential is simplified

consequences: \( V(r) \) valid for \( r m_D \ll 1 \ll r T \)
how to calculate static potential beyond leading order

- expand $W$ to higher order in $g$
- dress the propagator in the LO contribution

- iterate the LO potential (not shown)

determine how to dress lines/vertices for $p \sim g^a T$ with $\frac{1}{3} < a < \frac{2}{3}$
comment about power counting

calculation of nlo htl $n$-point functions

→ follow prescription . . .

for the static potential there are two important differences
1. fermion lines have the form $\frac{1}{p_0 \pm i\eta} (M_q \gg \text{all other scales})$
2. external frequencies are taken to zero

⇒ external momenta don’t flow through the diagram as expected

** power counting is different from standard thermal field theory

• we take into account corrections to LO:
  real part larger than $g^2$ & imag part larger than $(g^{3a}, g^{2-a})$

• denominators $\sim p^2 + m_D^2$ kept unexpanded (*damped approximation*)

→ extends region that coordinate space potential is valid
in the self-energy diagram we need:
- power correction to htl gluon bubble
- one loop gluon bubble with loop momenta semi-hard

\[ \text{Re}[V] \text{ in A. K. Rebhan, Phys. Rev. D 48, R3967 (1993)} \]

for all other diagrams (ladder diagrams):
- htl propagators and bare vertices

\textit{also: static quark self-energies}

\rightarrow \text{ constant contributions that we have not calculated}
coordinate space potential beyond leading order

\[ \hat{r} = r m_D \]

\[ l_j(\hat{r}) = \int_0^\infty d\hat{p} \sin (\hat{p}\hat{r}) (\hat{p}^2 + 1)^{-j} \]

\[ V_{1lo} = - \frac{g^2 C_F}{4\pi \hat{r}} \left( m_D e^{-\hat{r}} - 2i T l_2(\hat{r}) \right) \]

\[ \text{Re}[V_2] = \frac{g^4 N_c C_F T}{64\pi^2 \hat{r}} \left\{ 8 (l_2(\hat{r}) - l_1(\hat{r})) + \frac{e^{-\hat{r}}}{16} \left( 3\pi^2 - 16 + \frac{\hat{r}}{6} \left( 16 - \pi^2 \right) \right) \right\} \]

\[ i\text{Im}[V_2] = -i \frac{g^3 C_F T}{16\pi^2 \hat{m}_D} \left\{ \frac{3\pi^2 - 16}{32 \hat{r}} l_2(\hat{r}) + \frac{7}{3} N_c e^{-\hat{r}} - \frac{2g \hat{m}_D}{\pi \hat{r}} \left( N_c - \frac{N_f}{2} \right) (l_1(\hat{r}) - l_2(\hat{r})) \right\} \]
results: compare to lattice calculations

- include contributions to $V(r)$ from $p \sim m_D$
  - have a universal form at any order in $g$
    since $m_D r \ll 1$ can expand exponential $e^{i\vec{p} \cdot \vec{r}}$ in this region
    $\to$ a polynomial in $r$ (up to possible logarithms)
- add contributions:
  \[
  \text{Re}[V] = C + g^3 r_0 T
  \]
  \[
  \text{Im}[V] = g^3 i_0 T + g^5 i_2 r^2 T^3
  \]
coefficients obtained by fitting to lattice results
$C$ is a global constant that adjusts the origins of the energies

note also:
- heavy $q$ self-energy contributions don’t need to be calculated
- they are absorbed into the fitted constants

use $g = 1.8$ from fit to $T = 0$ lattice data with $r \in (0.0, 0.3)$ fm

find $(C, r_0, i_0, i_2)$ with fit to all available $T$ and $r \in (0.02, 0.3)$ fm

- real part of potential varies little with $T$ (like data)
- imaginary part gets big control from soft region

– solid bands are uncertainties in fitted coefficients inherited from lattice data

![Graphs showing real and imaginary parts of potential vs. distance](image-url)
solve the schrödinger equation using our result for $\text{Re}[V]$
→ binding energies and $\Gamma = -\langle \text{Im}[V] \rangle$
find coefficients by fitting to all available temperatures
- error bars from fitting to upper/lower values

⇒ reasonable description of data for both $E_{\text{bind}}$ and $\Gamma$
fitted soft contribution
contribution to $V(r)$ from $p \sim m_D$

\[
\text{Re}[V] = C + g^3 r_0 T
\]
\[
\text{Im}[V] = g^3 i_0 T + g^5 i_2 r^2 T^3
\]

find values of coefficients by fitting to 2 sets of lattice data

in our calculation all scales are explicit

→ expect same size for numerical coefficients from the two fits

\[
(r_0, i_0, i_2) = (0.049, -0.021 \pm 0.002, 0.205 \pm 0.001)
\]
\[
(r_0, i_0, i_2) = (0.078 \pm -0.004, -0.026 \pm 0.009, 0.053 \pm 0.002)
\]

$i_2$ from the first fit is significantly larger

$C=219$ MeV from first calculation

in second the coulomb binding energy is subtracted ($C$ plays no role)
dissociation:
- bound states disappear because decay (become wide resonances)
- not because $V$ is screened too shallow to support them

$T_{	ext{diss}} \approx$ temperature where ground state $E_{\text{bind}} = \Gamma = -2\langle \text{Im} V \rangle$
- define $E_{\text{bind}}$ as eigenvalue of $V$ with threshold set to 0

lo result: $T_{\text{diss}} = 193.2$ MeV

beyond-lo result: $T_{\text{diss}} = 151.8 \pm 1.2$ MeV ← using first fit

** outlying result for $i_2$

beyond-lo result: $T_{\text{diss}} = 225 \pm 10$ MeV ← using second fit

consistent with other lattice studies predict $> \text{crossover } T$

conclusions

- anisotropy changes real part of the LO potential
- general effect is to promote binding

- calculated beyond-lo corrections to momentum space potential
- when the typical momentum transfer $p$ fulfils $m_D \ll p \ll T$
- relevant region to obtain dissociation $T$ for heavy quarkonium

$\Rightarrow V(r)$ determined up to a polynomial in $r^2$ (and possible $\log(r)$)
- encodes the contribution for $p \lesssim m_D$
- describes reasonably well 2 different sets of lattice data (LO fails)