Helicity states for two- and three-gluon glueballs

Author: C.Chevalier Co-author: V.Mathieu

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Nuclear and subnuclear physics unit
University of Mons

Glueballs

QCD confinement rules allows the existence of pure gluonic states, so-called **glueballs**.

[1] Vadacchino (2023) *arXiv:2305.04869*
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Concerning the dynamic of the system, it is (often) ruled by a phenomenological QCD-inspired Hamiltonian.

Constituent Approaches

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Constituent gluon

QCD Boson

- Intrinsic parity: -1
- Mass: 0+

Gluons are considered as massless or endowed with constituent mass depending on the model.

- Spin degrees-of-freedom: 2

Helicity degrees-of-freedom have to be considered ($\lambda = \pm 1$).

Resistance : neutral **Weakness : coulorfull**

(thanks to Dall.e and Lexica.art for the artworks)

The Helicity Formalism

From one-body helicity states... [4]

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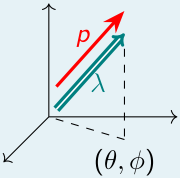

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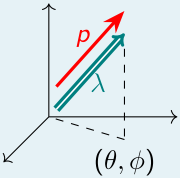

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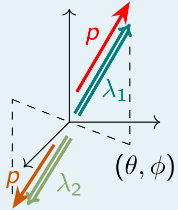

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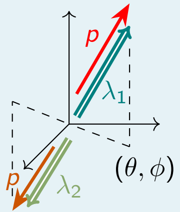

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Application to glueballs

Two-gluon glueball state

Decomposition of a two-gluon glueball in the helicity basis :

$$|\Psi; JM; \lambda_1\lambda_2\rangle = \int \frac{p^2 \mathrm{d}p}{4w_1(p)w_2(p)} \Psi(p) |p; JM; \lambda_1\lambda_2\rangle.$$

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Four sets of symmetric parity eigenstates

$$|S_+; J^P = (2k)^+\rangle\,, \qquad |S_-; J^P = (2k)^-\rangle\,,$$

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$\hookrightarrow$ No $J = 1$ state !

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Glueball spectrum

Let us have a look at results

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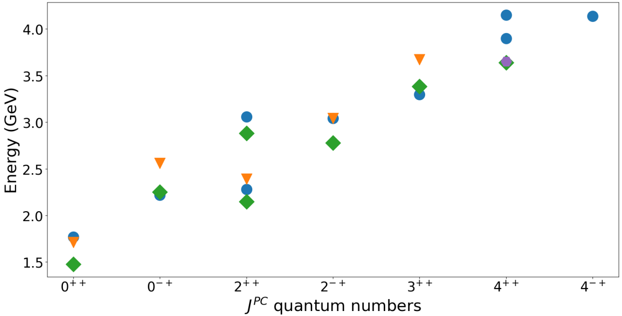

Figure: Comparison of two-gluon glueball spectra. Upper bounds obtained with a single Gaussian trial state (blue circles) are compared to lattice QCD results from [1] (orange triangles), [2] (green diamonds) and [3] (purple hexagon).

[1] Chen et al. (2006) Phys.Rev.D, **73**, 014516 [2] Meyer (2005) Phys.Lett.B, **605**, 344
[3] Liu (2002) Mod.Phys.Lett.A, **17**, 1419

Three-gluon glueballs

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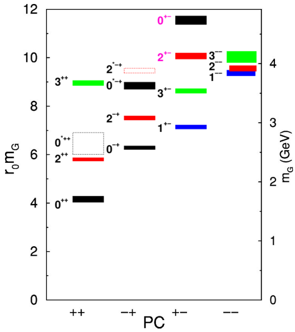

Figure: Glueball spectrum from lattice QCD calculations (from Morningstar, Peardon (1999) Phys.Rev.D **60** 034509)

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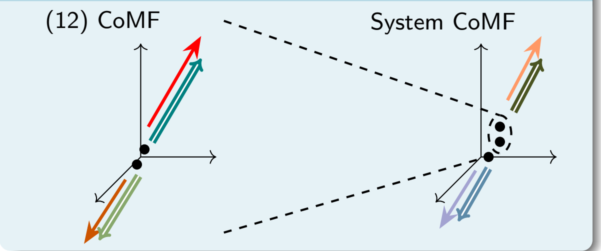
Wick suggests to construct three-body helicity states thanks to two successive two-body couplings.

Diagram: Wick's J-helicity states



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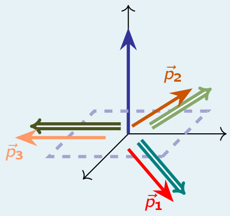

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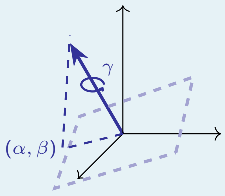

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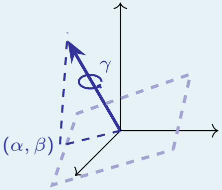

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$\hookrightarrow$ Same as for two-body !

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Berman's definition :

Wick's definition :

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- Three particles treated on equal footing.

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Includes a summation on all the allowed (j_{12}, λ_{12}) (→ Passage in Wick's one)

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Coefficients depend on a continuous degree of freedom

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  - ↪ Easier to compute two-body matrix elements.

Methodology (Work in progress)

Symmetry implemented with Berman's definition ⇒ Passage in Wick's one ⇒ Matrix element computed on Wick's states

Includes a summation on all the allowed (j_{12}, λ_{12})

Coefficients depend on a continuous degree of freedom

Prospects & Conclusion

Two-gluon glueball spectrum is easily reproduced in constituent approaches, as long as only helicity degrees-of-freedom are considered.

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- two different helicity bases are to be used,
- to convey from one basis to the other one brings a bit of complexity.

Figure: Rare picture of two constituent gluons meeting each other to make a glueball.

A few references about glueballs

A non-exhaustive list

Experimentally-oriented review :

- Crede and Meyer (2009) Prog.Part.Nucl.Phys., **63**, 74

Intermediary review :

- Llanes-Estrada (2021) Eur.Phys.J.Spec.Top, **230**, 1575

Theoretically-oriented reviews :

- Vadacchino (2023) *arXiv:2305.04869*
- Mathieu, Kochelev and Vento (2009) Int.J.Mod.Phys.E, **18**, 1

Constituent approaches : (among many other studies)

- Mathieu, Buisseret and Semay (2008) Phys.Rev.D, **77**, 114022
- Szczepaniak and Swanson (2003) Phys.Lett.B, **577**, 61

Decomposition of a two-gluon glueball in the helicity basis :

$$|\Psi; JM; \lambda_1\lambda_2\rangle = \int \frac{p^2 \mathrm{d}p}{4w_1(p)w_2(p)} \Psi(p) \, |p; JM; \lambda_1\lambda_2\rangle \, .$$

Technique : Spectrum calculation

Hamiltonian matrix elements are evaluated on trial states by switching from the helicity basis to the *LS* one.

Model : as simple as possible

Hamiltonian formulation with utra-relativistic kinematics, linear confinement and Coulombic short-range interaction,

$$H_{\mathrm{GB}} = 2\sqrt{p^2} + \frac{9\sigma}{4} r - 3\frac{\alpha_s}{r}$$

where $\alpha_s = 0.450$ and $\sigma = 0.185\,\mathrm{GeV}^2$.

Slide-ppendix : the two-gluon glueballs spectrum

State	$E_{\text{or.SGA}}$	LQCD [1]	LQCD [2]
$\|\Psi; S_+, 0^+\rangle$	1.769	1.710	1.475
$\|\Psi; S_-, 0^-\rangle$	2.216	2.560	2.250
$\|\Psi; D_+, 2^+\rangle$	2.279	2.390	2.150
$\|\Psi; S_+, 2^+\rangle$	3.060	N.A.	2.880
$\|\Psi; S_-, 2^-\rangle$	3.043	3.040	2.780
$\|\Psi; D_-, 3^+\rangle$	3.297	3.670	3.385
$\|\Psi; D_+, 4^+\rangle$	3.897	N.A.	3.640
$\|\Psi; S_+, 4^+\rangle$	4.150	N.A.	N.A.
$\|\Psi; S_-, 4^-\rangle$	4.139	N.A.	N.A.

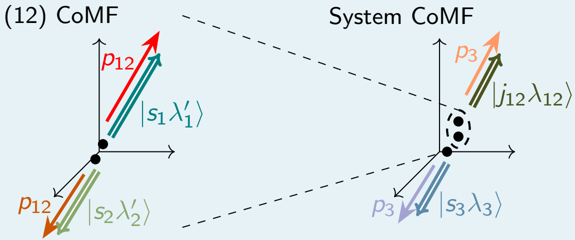
Table: Comparison of two-gluon glueball spectra. Upper bounds obtained with a single Gaussian trial state, $E_{\text{or.SGA}}$, are compared to lattice QCD results [1,2]. A supplementary LQCD calculations [3] which predicts a $J^P = 4^+$ state of 3.650 GeV can be mentioned. Energies are provided in GeV.

Property: Relation to two-body J-helicity states

$$|p\theta\phi; j_{12}\lambda_{12}s_3\lambda_3; p_{12}s_1\lambda'_1s_2\lambda'_2\rangle \propto |m_{12}; p\theta\phi; j_{12}\lambda_{12}; s_1\lambda'_1s_2\lambda'_2\rangle \otimes |m_3; p(\pi+\phi)(\pi-\theta); s_3\lambda_3\rangle\,,$$

and $|p_3; JM; j_{12}\lambda_{12}s_3\lambda_3; p_{12}s_1\lambda'_1s_2\lambda'_2\rangle$
$$\propto \int \mathrm{d}\cos\theta\,\mathrm{d}\phi\; D^{J*}_{M\,\lambda_{12}-\lambda_3}(\phi,\theta,0)\,|p\theta\phi; j_{12}\lambda_{12}s_3\lambda_3; p_{12}s_1\lambda'_1s_2\lambda'_2\rangle\,.$$

Diagram: Wick's p-helicity states



Slide-ppendix : the Helicity Formalism

Explicit definition of Wick's three-body helicity states

Property: Relation to two-body J-helicity states

$$|p\theta\phi; j_{12}\lambda_{12}s_3\lambda_3; p_{12}s_1\lambda'_1 s_2\lambda'_2\rangle \propto |\boldsymbol{m_{12}; p\theta\phi; j_{12}\lambda_{12}; s_1\lambda'_1 s_2\lambda'_2}\rangle \otimes |m_3; p(\pi+\phi)(\pi-\theta); s_3\lambda_3\rangle\,,$$

and $|p_3; JM; j_{12}\lambda_{12}s_3\lambda_3; p_{12}s_1\lambda'_1 s_2\lambda'_2\rangle$

$$\propto \int \mathrm{d}\cos\theta\,\mathrm{d}\phi\; D^{J*}_{M\,\boldsymbol{\lambda_{12}}-\boldsymbol{\lambda_3}}(\phi,\theta,0)\,|p\theta\phi; j_{12}\lambda_{12}s_3\lambda_3; p_{12}s_1\lambda'_1 s_2\lambda'_2\rangle\,.$$

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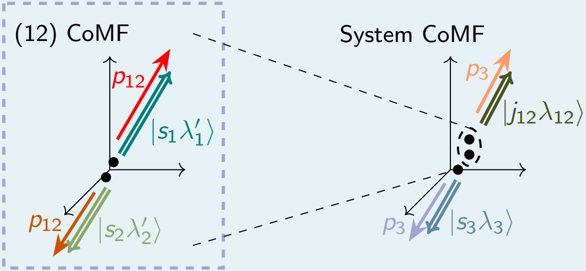

Parity and symmetry of Berman's states

Concerning Berman's state parity and symmetry, one can show that

$$\begin{aligned}
\Pi \left| JM\mu; w_1 w_2 w_3; \lambda_1\lambda_2\lambda_3 \right\rangle \\
&= \eta_1\eta_2\eta_3(-1)^{-s_1-s_2-s_3-\mu} \left| JM\mu; w_1 w_2 w_3; -\lambda_1 - \lambda_2 - \lambda_3 \right\rangle , \\
\mathbb{P}_{12} \left| JM\mu; w_1 w_2 w_3; \lambda_1\lambda_2\lambda_3 \right\rangle \\
&= (-1)^{J+\mu+\lambda_1+\lambda_2-\lambda_3} \left| JM-\mu; w_2 w_1 w_3; \lambda_2\lambda_1\lambda_3 \right\rangle , \\
\mathbb{P}_{13} \left| JM\mu; w_1 w_2 w_3; \lambda_1\lambda_2\lambda_3 \right\rangle \\
&= (-1)^{J-\mu-\lambda_1-\lambda_2-\lambda_3} e^{-i\varphi_{13}\mu} \left| JM-\mu; w_3 w_2 w_1; \lambda_3\lambda_2\lambda_1 \right\rangle , \\
\mathbb{P}_{23} \left| JM\mu; w_1 w_2 w_3; \lambda_1\lambda_2\lambda_3 \right\rangle \\
&= (-1)^{\lambda_1-\lambda_2-\lambda_3+J+\mu} e^{i\varphi_{23}\mu} \left| JM-\mu; w_1 w_3 w_2; \lambda_1\lambda_3\lambda_2 \right\rangle .
\end{aligned}$$

where

$$\varphi_{ij} = \frac{p_k^2 - p_i^2 - p_j^2}{2 p_i p_j}$$

As a result, parity and symmetry mixes different helicities as well as different μ values.

From Berman's definition to Wick's one

For three massless particles, one can show that

$$|JM\mu; w_1 w_2 w_3; \lambda_1\lambda_2\lambda_3\rangle \propto \sum_{j_{12}=|\lambda_1-\lambda_2|}^{\infty} \sum_{\lambda_{12}=-j_{12}}^{j_{12}} e^{i\pi\lambda_{12}/2} \sqrt{\frac{2j_{12}+1}{2}}\, d^{j_{12}}_{\lambda_{12}\,\lambda_1-\lambda_2}(u)$$
$$d^J_{\mu\,\lambda_{12}-\lambda_3}(\pi/2)\,|p_3;\; JM;\; j_{12}\lambda_{12}s_3\lambda_3;\; p_{12}s_1\lambda_1s_2\lambda_2\rangle$$

where

$$\begin{cases} u = \arccos\left(\left(w_1 - w_2\right)/w_3\right), \\ p_{12} = \sqrt{(w_1+w_2)^2 - w_3^2}/2, \\ p_3 = w_3. \end{cases}$$

.