Towards a non-perturbative description of inflation

Diego Cruces

ICC-Universitat de Barcelona
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2212.05112., 2203.13852, 2107.12735, 1807.09057
with C. Germani, T. Prokopec and A. Palomares

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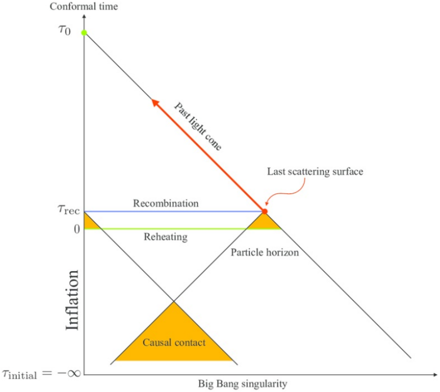
Outline

Introduction

Inflation (basics)

Alan H. Guth, Phys. Rev. D 23, 347 (1981)

- Solves many previous problems of the standard Big Bang evolution of the universe.
- $(aH)^{-1}$ exponentially decreases $\rightarrow$ negative pressure fluid.



$$S = \int d^4x \sqrt{-g}[\tfrac{1}{2}R + \tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)]$$

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$$

$$H^2 = \rho \qquad \ddot{\phi} + 3H\dot{\phi} + V_\phi = 0$$

$$H = \frac{\dot{a}}{a}$$

$$\epsilon_1 = -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2}{2H^2} \ll 1$$

- There are inhomogeneities both in the scalar field and in the metric which are important due to the exponential expansion of the universe.
- How do we study them?
  Cosmological perturbation theory.

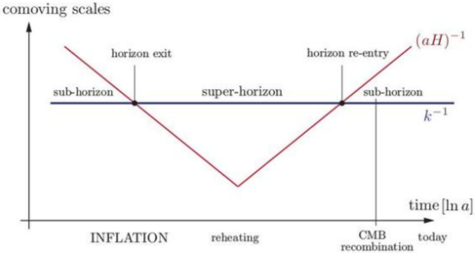
$$\phi \simeq \bar{\phi} + \delta\phi\,, \qquad g_{\mu\nu} \simeq \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$$

$$ds^2 = -(1+2A)dt^2 + 2a\partial_i B dx^i dt + a^2\left[(1+2D)\delta_{ij} - 2\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)E\right]dx^i dx^j,$$

We then define the comoving curvature perturbation

$$R = D + \frac{1}{3}\nabla^2 E - \frac{H}{\dot{\phi}}\delta\phi$$

$$\delta G_{\mu\nu} = \frac{1}{M_{PL}^2}\delta T_{\mu\nu} \quad \to \quad \frac{1}{a^3\epsilon_1}\frac{d}{dt}\left[a^3\epsilon_1\dot{R}\right] - \nabla^2 R = 0$$



- The long wavelength limit is equivalent to neglecting spatial gradients (terms proportionals to k in Fourier space).

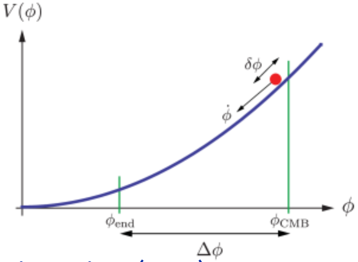
$$\frac{1}{a^3\epsilon_1}\frac{d}{dt}\left[a^3\epsilon_1\dot{R}_k\right] - \cancel{k^2 R_k} = 0\,,$$

- Solution

$$R_k(k \to 0) = C_1 + C_2\int\frac{dt}{a^3\epsilon_1}\,,$$

where C_1 and C_2 are constants coming from initial conditions.

- Slow roll inflation (SR)



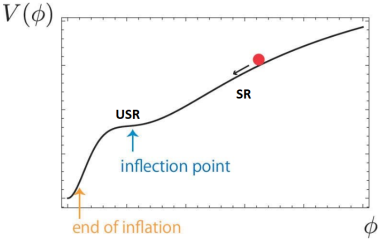
$$\cancel{\ddot{\phi}} + 3H\dot{\phi} + V_\phi = 0$$

$$\dot{\phi} \sim \text{constant}$$

$$\epsilon_i \simeq \text{constant} \ll 1$$

$$\epsilon_{i+1} = \frac{1}{H\epsilon_i}\frac{d}{dt}\epsilon_i$$

- Ultra slow-roll inflation (USR)



$$\ddot{\phi} + 3H\dot{\phi} + \cancel{V_\phi} = 0$$

$$\epsilon_1 \sim \dot{\phi}^2 \sim e^{-6Ht} \ll 1$$

$$\epsilon_2 \simeq -6$$

R_k can grow!

Gradient expansion

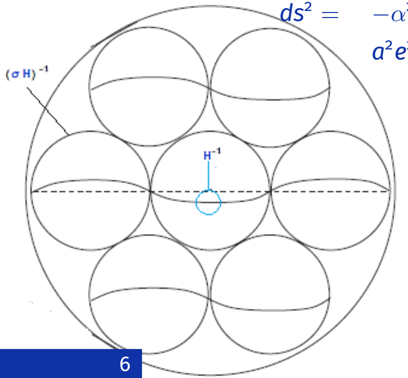
Gradient expansion

D. S. Salopek and J. R. Bond, Phys. Rev. D 42 (1990), 3936

- The characteristic scale of inhomogeneities is larger that the Hubble horizon scale $L \gg H^{-1}$

$$L \sim \mathcal{O}\left(\frac{1}{\sigma}\right) \to L \sim \frac{1}{\sigma H}$$

with $\sigma \ll 1$



$$\begin{aligned} ds^2 = \quad & -\alpha^2 d\bar{t}^2 \\ & a^2 e^{2D^{NL}} \tilde{\gamma}_{ij} \left(d\bar{x}^i + \beta^i d\bar{t}\right) \left(d\bar{x}^j + \beta^j d\bar{t}\right). \end{aligned}$$

$$ds^2 = -dt_p^2 + a(t_p)^2 \delta_{ij} dx_p^i dx_p^j.$$

$$H_p^2 = \rho_p$$

$$\frac{d^2}{dt_p^2}\phi_p + 3H_p \frac{d}{dt_p}\phi_p + V_{\phi_p} = 0$$

Gradient expansion vs linear perturbation theory

1. Perturbation theory is an expansion at leading order in the amplitude of the fluctuation but at all orders in the wavenumber k.
2. $\mathcal{O}(k^0)$ gradient expansion is an expansion at leading order in the wavenumber but at all orders in the amplitude of the fluctuation.

Gradient expansion

Problems

1. Initial conditions are not well defined in gradient expansion.
    - Stochastic formalism or δN formalism give perturbative initial conditions to the gradient expansion.
2. One cannot naively neglect all the spatial derivatives.
    - Why?

Non-local terms

- For adiabatic modes as in SR and USR

$$\dot{R} \propto \frac{1}{\epsilon_1}\nabla^2\Psi \qquad \Psi \quad \text{Bardeen potential}$$

- $\nabla^2\Psi$ does not vanish in the $k \to 0$ limit!
- $$\nabla^2\Psi \sim \mathcal{O}(k^0) \qquad \to \qquad \Psi \sim \mathcal{O}(k^{-2})$$
- Terms like $\nabla^2\Psi$ always decay, but this does not mean that they are k-suppressed.
- They play a crucial role for the correct description of the evolution of R at super-horizon scales.

Applications of gradient expansion

Applications of gradient expansion

δN formalism

δN FORMALISM

D. S. Salopek and J. R. Bond, Phys. Rev. D 42 (1990), 3936

At leading order in gradient expansion and neglecting decaying terms, the number of e-folds is:

$$N \simeq \int_{\bar{t}^{o}}^{\bar{t}^{e}} \left(\bar{H} + \dot{D}^{NL}\right) d\bar{t}\,,$$

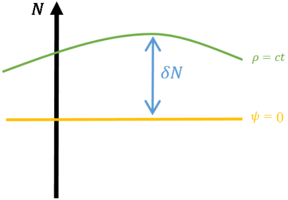
If we integrate from a flat hypersurface at time $\bar{t}^{o}$ to some other hypersurface specified by $\mathcal{A}$ we get

$$N_f^{\mathcal{A}} \simeq \int_{\bar{t}^{o}}^{\bar{t}^{e}} \bar{H} d\bar{t} + D_{\mathcal{A}}^{NL} = \bar{N} + D_{\mathcal{A}}^{NL}(\bar{t}^{e})\,,$$

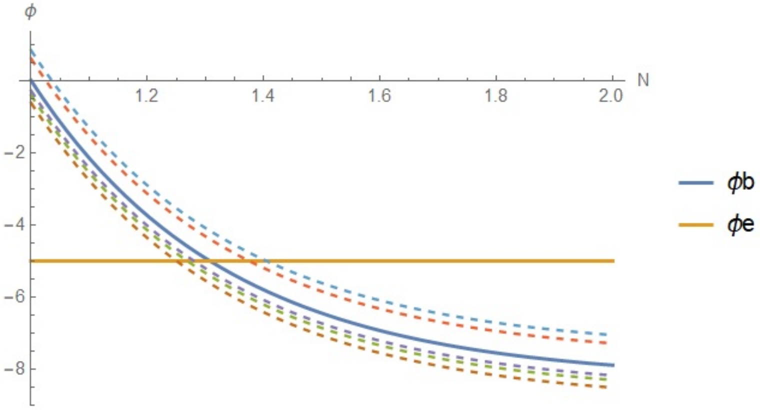
We define the δN formalism as

$$\delta N \equiv N_f^{\mathcal{A}} - \bar{N} = D_{\mathcal{A}}^{NL}(\bar{t}^{e})\,.$$

Different final hypersurfaces $\rightarrow$ different curvatures!



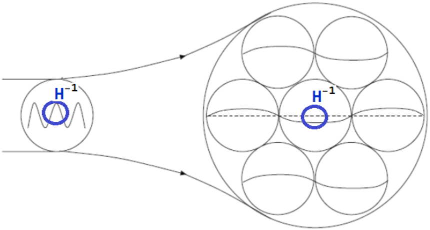
δN FORMALISM IN TERMS OF INITIAL CONDITIONS.



Non-linear but still perturbative!

Problems with δN formalism.

- δN formalism only gives initial conditions for one mode!



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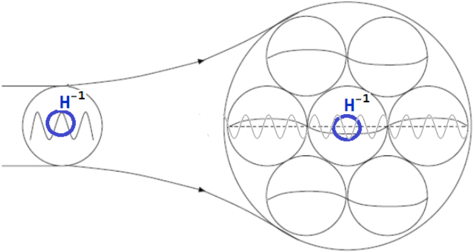

Problems with δN formalism.

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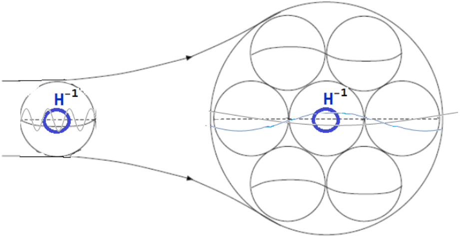


Applications of gradient expansion

Stochastic approach to inflation

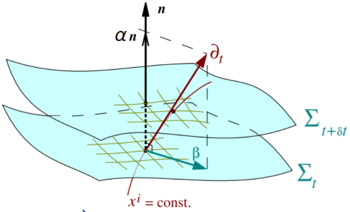
Stochastic approach to inflation

A. A. Starobinsky, Lect. Notes Phys. 246 (1986).

- Evolution of the full quantum field
  - IR part ($k < \sigma a H$)
    - Gradient expansion
      - Equation of motion
  - UV part ($k > \sigma a H$)
    - Linear perturbation theory
      - Noises

ADM formalism

$$R^{(3)} - K_{ij}K^{ij} + K^2 = \frac{2}{M_{PL}^2} T_{\mu\nu} n^\mu n^\nu$$



$$\left(\frac{H}{\alpha^{IR}}\right)^2 - \frac{1}{3}\left(\frac{1}{2}\left(\frac{\dot{\phi}^{IR}}{\alpha^{IR}}\right)^2 + V\left(\phi^{IR}\right)\right) = \frac{\dot{\phi}^{IR}}{3\left(\alpha^{IR}\right)^2}\xi_1$$

$$\xi_1 = -\sigma a H\left(1-\epsilon_1\right)\int \frac{d\mathbf{k}}{(2\pi)^{3/2}}\delta\left(k-\sigma a H\right)\varphi_{\mathbf{k}}$$

A very important aspect:

- $\varphi_{\mathbf{k}}$ must be computed over an stochastic background.

$$\frac{1}{3\left(\alpha^{IR}\right)^3}\left\{6H^2\mathcal{A}_{\mathbf{k}} + \alpha^{IR}\dot{\phi}^{IR}\dot{\varphi}_{\mathbf{k}} - \left(\dot{\phi}^{IR}\right)^2\mathcal{A}_{\mathbf{k}} - V_\phi\left(\phi^{IR}\right)\varphi_{\mathbf{k}} + \frac{k^4}{3a^2}\mathcal{E}_{\mathbf{k}}\right\} = 0$$

...

- Following the same procedure with every ADM equation and being careful about the $k \to 0$ limit:
  DC, C. Germani, Phys.Rev.D 105 (2022) 2, 023533
  DC, Universe 8 (2022) 6, 334

$$\pi^{IR} = \frac{\partial \phi^{IR}}{\partial N} + \xi_1 \,,$$

$$\frac{\partial \pi^{IR}}{\partial N} + \left(3 - \frac{\left(\pi^{IR}\right)^2}{2M_{PL}^2}\right)\pi^{IR} + \left(3M_{PL}^2 - \frac{\left(\pi^{IR}\right)^2}{2}\right)\frac{V_\phi\left(\phi^{IR}\right)}{V\left(\phi^{IR}\right)} = -\xi_2 \,,$$

$$\partial_i\left(\frac{\partial}{\partial N}\left(\frac{1}{3}\nabla^2 E^{IR}\right)\right) - \frac{\partial_i \alpha^{IR}}{\alpha^{IR}} + \frac{\partial \phi^{IR}}{\partial N}\frac{\partial_i \phi}{2M_{PL}^2} = -\partial_i \xi_4 \,,$$

Stochastic approach to inflation

- Typically in the literature people study the stochastic formalism at leading order in ϵ_1 (without considering ∇^2 terms):

$$\pi^{IR} = \frac{\partial \phi^{IR}}{\partial N} + \xi_1 \, ,$$

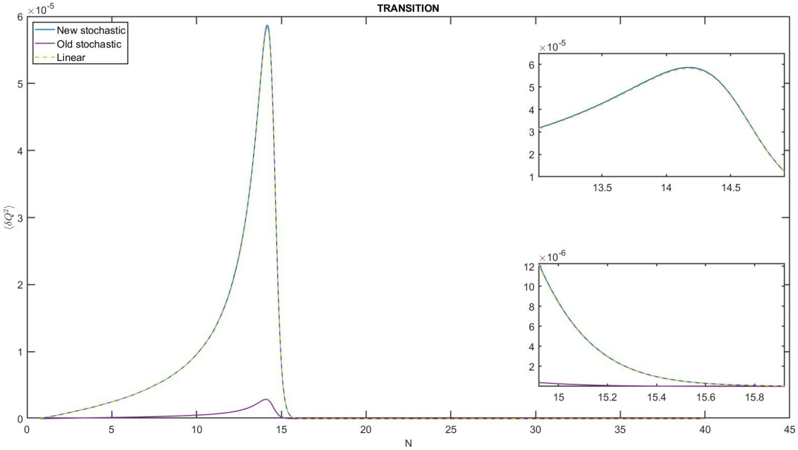
$$\frac{\partial \pi^{IR}}{\partial N} + \left(3 - \cancel{\frac{\left(\pi^{IR}\right)^2}{2M_{PL}^2}} \right) \pi^{IR} + \left(3M_{PL}^2 - \cancel{\frac{\left(\pi^{IR}\right)^2}{2}} \right) \frac{V_\phi\left(\phi^{IR}\right)}{V\left(\phi^{IR}\right)} = -\xi_2 \, ,$$

$$\cancel{\partial_i \left(\frac{\partial}{\partial N} \left(\frac{1}{3} \nabla^2 E^{IR} \right) \right) - \frac{\partial_i \alpha^{IR}}{\alpha^{IR}} + \frac{\partial \phi^{IR}}{\partial N} \frac{\partial_i \phi}{2M_{PL}^2} = -\partial_i \xi_4} \, ,$$

- Noises still in a stochastic background!
  - Very difficult to solve the system both analytically and numerically.
  - An alternative is to compute the noises in a deterministic background $\rightarrow$ small noise approximation $\rightarrow$ linear perturbation theory.
  - At least we can use it as a consistency check of our formalism.

SR-USR-SR Transition $V(\phi) = V_0 \left(1 + \beta (\phi - \phi_0)^3\right)$

Why? Inflection point $\rightarrow$ PBHs.



CONCLUSIONS

- Gradient expansion is a very powerful tool to study inflationary inhomogeneities in a non-perturbative way. However, it present 2 main problems:
  1. Initial conditions not well defined
     - Solved perturbatively with the δN formalism.
     - Solved non-perturbatively in the stochastic formalism $\rightarrow$ very difficult.
  2. One must be very careful when dropping spatial derivatives in this approximation.

Thank you!