## The ultraviolet limit of the power spectrum and Lagrangian perturbation theory

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Based on: [JB, Z. Vlah, O. Antipin, to appear]
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## Perturbation theory of LSS

The study of the Large Scale Structure (LSS) has emerged as a core technique to understand the properties of the Universe.

LSS is mainly studied via numerical simulations.
However, it is important to develop analytical approaches to investigate the nonlinear growth of cosmic structures.

Standard Perturbation Theory (SPT) Lagrangian Perturbation Theory (LPT)


## Lagrangian perturbation theory

We consider cold dark matter interacting via Newtonian gravity.
For a fluid element at position $\mathbf{q}$ at some initial time, its position $\mathbf{x}$ at later times is written in terms of the displacement field $\psi(\mathbf{q}, \mathrm{t})$

$$
\mathbf{x}(\mathbf{q}, t)=\mathbf{q}+\psi(\mathbf{q}, t)
$$

The matter power spectrum is

$$
P(\mathbf{k})+(2 \pi)^{3} \delta_{D}(\mathbf{k})=\int d^{3} q e^{i \mathbf{k} \cdot \mathbf{q}}\left\langle e^{i \mathbf{k} \cdot \Delta \psi(\mathbf{q})}\right\rangle
$$

where $\Delta \psi(\mathbf{q}) \equiv \psi\left(\frac{\mathbf{q}}{2}\right)-\psi\left(-\frac{\mathbf{q}}{2}\right)$
Lagrangian perturbation theory (LPT) expands $\psi(q)$ as

$$
\psi(\mathbf{q}, t)=\psi^{(1)}(\mathbf{q}, t)+\psi^{(2)}(\mathbf{q}, t)+\psi^{(3)}(\mathbf{q}, t)+\ldots
$$

and solves the EOM perturbatively.

## The large $k$ expansion

We consider the matter power spectrum

$$
P(\mathbf{k})+(2 \pi)^{3} \delta_{D}(\mathbf{k})=\int d^{3} q e^{i \mathbf{k} \cdot \mathbf{q}}\left\langle e^{i \mathbf{k} \cdot \Delta \psi(\mathbf{q})}\right\rangle
$$

and expand the relative displacement field $\Delta \psi(\mathrm{q})$ as

$$
\Delta \psi_{i}(\mathbf{q}) \sim \sum_{a=0} \frac{1}{2^{2 a}(2 a+1)!} M_{i \mu_{1} \mu_{2} \ldots \mu_{2 a+1}}^{(2 a+1)} q^{\mu_{1}} q^{\mu_{2}} \ldots q^{\mu_{2 a+1}}
$$

where $\left.M_{i \mu_{1} \mu_{2} \ldots \mu_{2 a+1}}^{(2 a+1)} \equiv \frac{\partial^{2 a+1} \psi^{i}(\mathbf{q})}{\partial q_{\mu_{1}} \partial q_{\mu_{2}} \ldots \partial q_{\mu_{2 a+1}}}\right|_{\mathbf{q}=0}$
We have

$$
P(k)=\frac{1}{k^{3}}\left\langle\int d^{3} z e^{i \hat{k}^{i} z^{j}\left(\delta_{i j}+M_{i j}^{(1)}\right)+\frac{i}{24 k^{2}} \hat{k}^{i} z^{j} z^{k} z^{l} M_{i j k l}^{(3)}+\ldots}\right\rangle \sim \frac{1}{k^{3}} \sum_{i=0} \mathcal{C}_{i}\left(\frac{k_{\mathrm{UV}}}{k}\right)^{2 i}
$$

## The large k expansion

$$
P(k) \sim \frac{1}{k^{3}} \sum_{i=0} \mathcal{C}_{i}\left(\frac{k_{\mathrm{UV}}}{k}\right)^{2 i}
$$

The coefficients can be expressed in terms of probability distribution functions (PDFs) for the derivatives of $\psi(\mathbf{q}, \mathrm{t})$. E.g.

$$
\mathcal{C}_{0}=\left\langle\int d^{3} z e^{i \hat{k^{i} z^{j}}\left(\delta_{i j}+M_{i j}^{(1)}\right)}\right\rangle=(2 \pi)^{3} \mathcal{P}_{M_{13}^{(1)}, M_{23}^{(1)}, M_{33}^{(1)}}[0,0,-1]
$$

[S. Chen, M. Pietroni, JCAP 06 (2020) 033]
The PDFs are dominated by the non-linear dynamics and can be computed in simulations.

In this talk we relate the large k expansion to LPT.

## The Zel'dovich approximation

## ZEL'DOVICH POWER SPECTRUM

$$
(2 \pi)^{3} \delta_{D}(k)+P_{Z}(k)=\int d^{3} q e^{i \vec{q} \cdot \vec{k}} e^{-\frac{1}{2} k_{i} k_{j} A_{i j}(\vec{q})}=\int d^{3} q e^{i \vec{q} \cdot \vec{k}} e^{-\frac{1}{2} k^{2}\left(X(q)+\mu^{2} Y(q)\right)}
$$

$\mathrm{A}_{\mathrm{ij}}$ is the two-point function

$$
A_{i j}(\vec{q})=\left\langle\Delta \Psi_{i}(\mathbf{q}) \Delta \Psi_{j}(\mathbf{q})\right\rangle=X(q) \delta_{i j}+Y(q) \hat{q}_{i} \hat{q}_{j}
$$

$$
\mu \equiv \frac{\mathbf{k} \cdot \mathbf{q}}{k q}
$$

where e.g.

$$
Y(q)=2 \xi_{2}(q) \quad \xi_{2}(q)=\int_{0}^{\infty} \frac{d k}{2 \pi^{2}} P_{L}(k) j_{2}(q k)
$$

This is the leading order in LPT.

## Large $k$ and LPT: leading order

We expand $\mathrm{X}(\mathrm{q})$ and $\mathrm{Y}(\mathrm{q})$ around $\mathrm{q}=0$ as

$$
\begin{aligned}
& \quad X(q)=q^{2} \sum_{n=0}^{\infty} x_{n}\left(k_{\mathrm{UV}} q\right)^{2 n} \quad Y(q)=q^{2} \sum_{n=0}^{\infty} y_{n}\left(k_{\mathrm{UV}} q\right)^{2 n} \\
& x_{n}=2 \frac{(-1)^{n}}{(2 n+5) \Gamma(2 n+4)} \sigma_{n} \\
& y_{n}=2(n+1) x_{n}
\end{aligned} \quad \sigma_{n} \equiv \frac{k_{\mathrm{UV}}^{2}}{2 \pi^{2}} \int_{0}^{\infty} P_{L}(k)\left(\frac{k}{k_{\mathrm{UV}}}\right)^{2 n+2} d k
$$

The Zel'dovich power spectrum becomes

$$
P_{Z}(k)=\frac{2 \pi}{k^{3}} \int_{0}^{\infty} \int_{-1}^{1} z^{2} e^{i \mu z} \exp \left(-\frac{1}{2} \sum_{n=0}^{\infty}\left(\frac{k_{\mathrm{UV}}}{k}\right)^{2 n} z^{2 n+2}\left(x_{n}+\mu^{2} y_{n}\right)\right) d \mu d z
$$

By expanding the above for large k we have

## General coefficients

$P(k) \sim \frac{1}{k^{3}} \sum_{i=0} \mathcal{C}_{i}\left(\frac{k_{\mathrm{UV}}}{k}\right)^{2 i}$

The coefficients are non-perturbative in $\sigma_{0}$.

The leading coefficient is

$$
\mathcal{C}_{0}=15 \sqrt{5}\left(\frac{2 \pi}{\sigma_{0}}\right)^{3 / 2} e^{-\frac{5}{2 \sigma_{0}}}
$$

[S. Chen, M. Pietroni, JCAP 06 (2020) 033]
[S. Konrad, M. Bartelmann Mon.Not.Roy.Astron.Soc. 515 (2022) 2]
The coefficients can be written as

$$
\mathcal{C}_{n}=\frac{1}{n!} \sum_{j=1}^{n}\left(-\frac{1}{2}\right)^{j} \int_{-1}^{1} Y_{n, j}\left(g_{1}, \ldots, g_{-j+n+1}\right) H_{j, n}\left(\mu, \sigma_{0}\right) d \mu
$$

$$
\begin{aligned}
& g_{n} \equiv n!x_{n}\left(1+2 \mu^{2}(n+1)\right) \\
& H_{j, n}\left(\mu, \sigma_{1}\right)=2^{j+n} 15^{j+n+1}\left(\left(2 \mu^{2}+1\right) \sigma_{0}\right)^{-j-n-2}\left[\sqrt{30\left(2 \mu^{2}+1\right) \sigma_{0}} \Gamma\left(j+n+\frac{3}{2}\right)\right. \\
& \\
& \left.\quad \times{ }_{1} F_{1}\left(j+n+\frac{3}{2} ; \frac{1}{2} ;-\frac{15 \mu^{2}}{4 \sigma_{0} \mu^{2}+2 \sigma_{0}}\right)-30 i \mu \Gamma(j+n+2){ }_{1} F_{1}\left(j+n+2 ; \frac{3}{2} ;-\frac{15 \mu^{2}}{4 \sigma_{0} \mu^{2}+2 \sigma_{0}}\right)\right]
\end{aligned}
$$

## Large $k$ and LPT: 1-loop EFT

## 1-LOOP POWER SPECTRUM

$$
(2 \pi)^{3} \delta^{D}(k)+P_{1 \text {-loop }}(k)=2 \pi \int_{0}^{\infty} \int_{-1}^{1} q^{2} e^{i k \mu q} e^{-\frac{i}{6} k^{3} \mu^{3} T(q)} e^{-\frac{1}{2} i k^{3} \mu V(q)} e^{-\frac{1}{2} k^{2}\left(X(q)+\mu^{2} Y(q)\right)} d q d \mu
$$

$\mathrm{V}(\mathrm{q})$ and $\mathrm{T}(\mathrm{q})$ come from the three-point function $\mathrm{W}_{\mathrm{ijk}}$

$$
W_{i j k}(\vec{q})=\left\langle\Delta \Psi_{i}(\mathbf{q}) \Delta \Psi_{j}(\mathbf{q}) \Delta \Psi_{k}(\mathbf{q})\right\rangle=V(q) \hat{q}_{\{i} \delta_{j k\}}+T(q) \hat{q}_{i} \hat{q}_{j} \hat{q}_{k}
$$

We have e.g.

$$
Y(q)=\frac{1}{\pi^{2}} \int_{0}^{\infty} \underline{j_{2}(q k)}\left(P_{L}(k)+\frac{9 Q_{1}(k)}{98}+\frac{10 R_{1}(k)}{21}+\alpha_{1} k^{2} P_{L}(k)\right) d k
$$

Bessel function
Counterterms
and the same for $\mathrm{X}(\mathrm{q}), \mathrm{V}(\mathrm{q}), \mathrm{T}(\mathrm{q})$.

## Large k and LPT: 1-loop EFT

We again expand $\mathrm{X}(\mathrm{q}), \mathrm{Y}(\mathrm{q}), \mathrm{V}(\mathrm{q}), \mathrm{T}(\mathrm{q})$ around $\mathrm{q}=0$ as e.g.

$$
\begin{aligned}
& Y(q)=q^{2} \sum_{n=0}^{\infty} y_{n}\left(k_{\mathrm{UV}} q\right)^{2 n} \quad y_{n}=2(n+1) \cdot \frac{(-1)^{n}}{(2 n+5) \Gamma(2 n+4)} \tau_{1, n} \\
& \tau_{1, n} \equiv \frac{k_{\mathrm{UV}}^{2}}{2 \pi^{2}} \int_{0}^{\infty}\left(P_{L}(k)+\frac{9}{98} Q_{1}(k)+\frac{10}{21} R_{1}(k)+\alpha_{1} k^{2} P_{L}(k)\right)\left(\frac{k}{k_{\mathrm{UV}}}\right)^{2 n+2} d k
\end{aligned}
$$

We are computing the coefficients of the large $k$ expansion in LEFT.

$$
\begin{gathered}
!+=-\cdot-{ }_{c} P(k) \sim \frac{1}{k^{3}} \sum_{i=0} \mathcal{C}_{i}\left(\frac{k_{\mathrm{UV}}}{k}\right)^{2 i} \\
\mathcal{C}_{0}=2 \pi \int_{0}^{\infty} \int_{-1}^{1} z^{2} \exp \left(i \mu z-\frac{z^{2}\left(245\left(2 \mu^{2}+1\right) \tau_{1,0}-24 i \mu z\left(\left(5 \mu^{2}+3\right) \tau_{2,0}-7 \tau_{3,0}\right)\right)}{7350}\right) d \mu d z
\end{gathered}
$$

## Symmetry constraints at one loop

At one-loop the combination $W_{0}=T+5 \mathrm{~V}$ appears.

$$
W_{0}(q)=\frac{1}{\pi^{2}} \int_{0}^{\infty}\left(\frac{-3}{7 k}\right) j_{1}(q k)\left(Q_{1}(k)-3 Q_{2}(k)+2 R_{1}(k)-6 R_{2}(k)+\alpha_{2} k^{2} P_{L}(k)\right) d k
$$

The small q expansion consistent with large k scaling is

$$
\begin{aligned}
& W_{0}(q)=q^{3} \sum_{n=0}^{\infty} w_{n}\left(k_{\mathrm{UV}} q\right)^{2 n} \quad w_{n}=\frac{3 \sqrt{\pi}}{7} \frac{(-1)^{n+1}}{\Gamma(n+2) \Gamma\left(n+\frac{7}{2}\right)} \tau_{3, n} \\
& \tau_{3, n} \equiv \frac{k_{\mathrm{UV}}^{3}}{2 \pi^{2}} \int_{0}^{\infty}\left(Q_{1}(k)-3 Q_{2}(k)+2 R_{1}(k)-6 R_{2}(k)+\alpha_{2} k^{2} P_{L}(k)\right)\left(\frac{k}{2 k_{\mathrm{UV}}}\right)^{2 n+2} d k
\end{aligned}
$$

However one would naively conclude that $W_{0}(q) \sim q$
Solution

$$
J_{1}(q k) \underset{q \rightarrow 0}{\sim} \frac{q k}{3}
$$

$\int_{0}^{\infty}\left(Q_{1}(k)-3 Q_{2}(k)+2 R_{1}(k)-6 R_{2}(k)+\alpha_{2} k^{2} P_{L}(k)\right) k d k=0$
These hold for every $\mathrm{P}_{\mathrm{L}}(\mathrm{k})$ and come from symmetry.

$$
\alpha_{2}=0
$$

## Large k and LPT: the power spectrum

We start from the matter power spectrum

$$
P(\mathbf{k})+(2 \pi)^{3} \delta_{D}(\mathbf{k})=\int d^{3} q e^{i \mathbf{k} \cdot \mathbf{q}}\left\langle e^{i \mathbf{k} \cdot \Delta \psi(\mathbf{q})}\right\rangle
$$

and use the cumulant theorem to expand it as

$$
\begin{gathered}
\left.\left\langle e^{i \mathbf{k} \cdot \Delta \psi(\mathbf{q})}\right\rangle\right\rangle=\exp \left[\sum_{N=1}^{\infty} \frac{(-i)^{N}}{N!} k^{N} \hat{k}_{\mu_{1}} \ldots \hat{k}_{\mu_{N}}\left\langle\Delta \Psi_{\mu_{1}}(\mathbf{q}) \ldots \Delta \Psi_{\mu_{N}}(\mathbf{q})\right\rangle_{c}\right] \equiv \exp \left[\sum_{N=1}^{\infty} \frac{(-i)^{N}}{N!} k^{N} \mathcal{B}_{N}\left(q, k_{\mathrm{UV}}, \mu\right)\right] \\
\text { where (1) } \mathcal{B}_{N}\left(q, k_{\mathrm{UV}}, \mu\right) \sim q^{N} \sum_{n=0}^{\infty} b_{N n}(\mu)\left(k_{\mathrm{UV}} q\right)^{2 n}
\end{gathered}
$$

The structure follows from dimensional analysis, translation invariance, and parity. We have

$$
P(k)=\frac{2 \pi}{k^{3}} \int_{0}^{\infty} \int_{-1}^{1} z^{2} e^{i \mu z} \exp \left[\sum_{N=1}^{\infty} \frac{(-i)^{N}}{N!} \sum_{n=0}^{\infty} b_{N n}(\mu) z^{N+2 n}\left(\frac{k_{\mathrm{UV}}}{k}\right)^{2 n}\right] d \mu d z
$$

By expanding the exponential for small $k_{U V} / k$ at fixed $N$, one generates the large $k$ expansion with the corresponding coefficients $\mathrm{C}_{i}$ at the N -th order in LEFT

## Integral constraints and counterterms

Consider odd N. Hence we

$$
\text { (1) } \mathcal{B}_{N}\left(q, k_{\mathrm{UV}}, \mu\right) \sim q^{N} \sum_{n=0}^{\infty} b_{N n}(\mu)\left(k_{\mathrm{UV}} q\right)^{2 n}
$$ can write

(2) odd $N: \mathcal{B}_{N}=\sum_{i=0}^{\frac{N-1}{2}} \mu^{2 i+1} \sum_{j=0}^{\frac{N-1}{2}} e_{i, 2 j+1} \Xi_{N, 2 j+1}\left(q, k_{\mathrm{UV}}\right)$
where

$$
\mu \equiv \frac{\mathbf{k} \cdot \mathbf{q}}{k q}
$$

$$
\Xi_{N, l}\left(q, k_{\mathrm{UV}}\right)=\int_{0}^{\infty}(f_{N, l}\left(k, k_{\mathrm{UV}}\right)+\underbrace{[\mathrm{ct}]_{N, l}}_{\text {Counterterms }}\left(k, k_{\mathrm{UV}}\right)) j_{l}(k q) d k
$$

Consistency between 1 and (2) implies an infinite series of integral constraints for the functions and counterterms appearing in Lagrangian EFT.

An analogous result can be obtained for even N .

## Integral constraints and counterterms

The integral constraints are
$45\left(\sum_{2}\right.$
$2 \int_{0}^{\infty}\left(\sum_{j=0}^{\gamma} e_{i, 2 j} a_{\gamma-j, 2 j}[\mathrm{ct}]_{N, 2 j}\left(k, k_{\mathrm{UV}}\right)\right) k^{2 \gamma} d k=0$

$$
\gamma=1, \ldots, \frac{N-2}{2}, \quad i=0, \ldots, \frac{N}{2}
$$

$45\left(\sum_{2}^{2}\right.$
N IS ODD
$2 \int_{0}^{\infty}\left(\sum_{j=0}^{\gamma} e_{i, 2 j+1} a_{\gamma-j, 2 j+1}[\mathrm{ct}]_{N, 2 j+1}\left(k, k_{\mathrm{UV}}\right)\right) k^{2 \gamma+1} d k=0$
The $\mathrm{a}_{\mathrm{ij}}$ are the coefficients of the Maclaurin series of the spherical Bessel functions.

$$
\gamma=0, \ldots, \frac{N-3}{2}, \quad i=0, \ldots, \frac{N-1}{2}
$$

## Comparisons in $\mathrm{d}=1+1$ dimensions



The Zel'vovich approximation (blue) and the large $k$ expansion at various orders.


The large k expansion is a saddle point expansion (around $q=0$ ). Including subleading saddles improves the results.

