Entanglement entropy: logarithmic terms

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Outline of the talk

- Introduction
- Introduction: history, definition and methods of computation
- Entanglement entropy of black holes
- Puzzle of non-minimal coupling
- Holography: entropy as minimal surface
- Log terms in EE of a Conformal Field Theory (CFT)
- Why log terms might be interesting?
- Conclusions

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Second big wave, 2006 - present: Ryu and Takayanagi, and many people after . . .

A pure (vacuum) state $|\psi >= \sum_{i,a} \psi_{ia} |A>_i \, |B>_a$ and density matrix $\rho_0(A,B) = |\psi >< \psi|$

$|A>$ states are inside surface Σ and $|B>$ are outside of Σ

Density matrix $\rho_B = \mathrm{Tr}\,_A \rho_0(A,B)$ and entropy $S_B = -\mathrm{Tr}\,\rho_B \ln \rho_B$

Since $\mathrm{Tr}\,\rho_A^k = \mathrm{Tr}\,\rho_B^k$ entropy $S_A = S_B$ depends on geometry of separation surface Σ and space-time geometry near Σ

That is why in earlier years it was called *geometric entropy*

Bombelli, Koul, Lee and Sorkin '86; Srednicki '93; Frolov and Novikov '93

In Quantum Field Theory entanglement entropy is UV divergent (function of UV cut-off ϵ)

to leading order EE is proportional to area of Σ

$$S \sim \frac{A(\Sigma)}{\epsilon^{d-2}} \quad \textit{if} \quad d > 2 \quad \text{and} \quad S \sim \frac{c}{6}\ln(1/\epsilon) \quad \textit{if} \quad d = 2$$

due to short-distance correlations across Σ

Bombelli, Koul, Lee and Sorkin '86; Srednicki '93; Holzhey, Larsen and Wilzcek '94

In 2d CFT c is central charge $< T >= \frac{c}{48\pi} R$

More generally, in d-dimensional curved space-time (with no boundary) EE is a Laurent series

$$S = \frac{s_{d-2}}{\epsilon^{d-2}} + \frac{s_{d-4}}{\epsilon^{d-4}} + \cdots + \frac{s_{d-2n}}{\epsilon^{d-2n}} + \cdots + s_0 \ln \epsilon + s(g)$$

$$s_{d-2-2n} = \sum_{(l+p)=n} \int_\Sigma \mathcal{R}^l k^{2p}$$

$\mathcal{R}$ is Riemann curvature and k is extrinsic curvature of Σ

Since there are 2 normal vectors to Σ only even powers of k may appear

Logarithmic term s_0 is non-zero if d is even

If space-time has boundary ∂M and if Σ intersects ∂M then the story is different: Log term may appear in any dimension d (odd or even)

Introduction: replica method

In Quantum Field Theory and in presence of rotational symmetry (in sub-space orthogonal to Σ)

$$\mathrm{Tr}\,\rho^n = Z[C_n]$$

is partition function on conical space with angle deficit $2\pi(1-n)$ at surface Σ so that EE is computed by differentiating w.r.t. n of effective action $W(n) = -\ln Z(n)$ on conical space

$$S = (n\partial_n - 1)W(n)|_{n=1}$$

Heat kernel method (field operator $\mathcal{D} = -\nabla^2 + \xi R$)

$$W = -\frac{1}{2}\int_{\epsilon^2}^{\infty}\frac{ds}{s}\mathrm{Tr}\,K(s)\,, \quad \mathrm{Tr}\,K_{\mathcal{M}_n} = \frac{1}{(4\pi s)^{d/2}}\sum_{k=0}(a_k^{reg} + a_k^{\Sigma})s^k$$

$$a_1^{\Sigma} = \frac{\pi}{3}\frac{1-\alpha^2}{\alpha}\int_{\Sigma} 1$$

$$a_2^{\Sigma} = \frac{\pi}{3}\frac{1-\alpha^2}{\alpha}\int_{\Sigma}(\frac{1}{6}-\xi)R - \frac{\pi}{180}\frac{1-\alpha^4}{\alpha^3}\int_{\Sigma}(R_{aa} - 2R_{abab})$$

McKean and Singer '67; Cheeger '83; Dowker '77; Fursaev '94

If no rotational symmetry (extrinsic curvature k of Σ is non-zero) one considers *squashed cones*

$$\int_{\mathcal{M}_n} R = n\int_{\mathcal{M}} R + 4\pi(1-n)\int_\Sigma 1$$

D.D. Sokolov and A. Starobinsky '77

$$\int_{\mathcal{M}_n} R^2 = n\int_{\mathcal{M}} R^2 + 8\pi(1-n)\int_\Sigma R$$

$$\int_{\mathcal{M}_n} R^2_{\mu\nu} = n\int_{\mathcal{M}} R^2_{\mu\nu} + 4\pi(1-n)\int_\Sigma (R_{aa} - \frac{1}{2}k^2)$$

$$\int_{\mathcal{M}_n} R^2_{\alpha\beta\mu\nu} = n\int_{\mathcal{M}} R^2_{\alpha\beta\mu\nu} + 8\pi(1-n)\int_\Sigma (R_{abab} - \mathrm{Tr}\, k^2)$$

$$R_{ab} = R_{\mu\nu} n^\mu_a n^\nu_a \quad \textit{and} \quad R_{abab} = R_{\alpha\beta\mu\nu} n^\alpha_a n^\beta_b n^\mu_a n^\nu_b$$

Fursaev and SS '94; Fursaev, Patrushev and SS '13

Topological Euler number

$$\chi_4[\mathcal{M}_n] = n\chi_4[\mathcal{M}] + (1-n)\chi_2[\Sigma]$$

Conformal invariant

$$\int_{\mathcal{M}_n} W^2 = n\int_{\mathcal{M}} W^2 + 8\pi(1-n)\int_\Sigma [W_{abab} - \mathrm{Tr}\,\hat{k}^2]$$

$\hat{k}^a_{\mu\nu} = k^a_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}\mathrm{Tr}\,k^a$, $a = 1, 2$ is conformal invariant constructed from extrinsic curvature.

Fursaev and SS '94; Fursaev, Patrushev and SS '13

Historically the study of EE was motivated by attempts to find a stat. mechanical explanation of Bekenstein-Hawking entropy

If Σ is black hole horizon then its extrinsic curvature vanishes $k^a = 0\,,\ a = 1, 2$

rotational symmetry is generated by Killing vector

$$S_{d=4} = \frac{A(\Sigma)}{48\pi\epsilon^2} - \frac{1}{144\pi}\int_\Sigma [R(1-6\xi) - \frac{1}{5}(R_{aa} - 2R_{abab})]\ln\epsilon$$

Myers, Demers, Lafrance '94; SS '94

EE of the Schwarzschild black hole

$$S_{Sch} = \frac{A(\Sigma)}{48\pi\epsilon^2} + \frac{1}{45}\ln\frac{r_+}{\epsilon}$$

for any value of ξ

SS '94

This is entire entanglement entropy including UV finite part!

If Riemann curvature appears in field operator (as in $\mathcal{D} = -\nabla^2 + \xi R$) should we take into account its distributional part when consider on conical space $\mathcal{M}_n$?

If we do then (for scalar field) one finds for heat kernel

$$a_k^\Sigma \to a_k^\Sigma - 4\pi\xi(1-n)\int_\Sigma \mathbf{a}_{k-1}^{reg}$$

and for entropy (SS '95)

$$S_{con} = \frac{A(\Sigma)}{48\pi\epsilon^2}(1-6\xi) - \frac{1}{144\pi}\int_\Sigma [R(1-6\xi)^2 - \frac{1}{5}(R_{aa} - 2R_{abab})] \ln\epsilon$$

In Log term no changes if $\xi = 1/6$ (conformal case) since $\mathbf{a}_1^{reg} = 0$ in this case

s_0 is invariant under conformal rescaling preserving horizon

No modification in Log term for the Schwarzschild black hole

Area term is not positive definite in general. That means this entropy does not correspond to a well-defined density matrix.

SS '95; Larsen, Wilzcek '95

Puzzle of non-minimal coupling: Kabat's contact terms

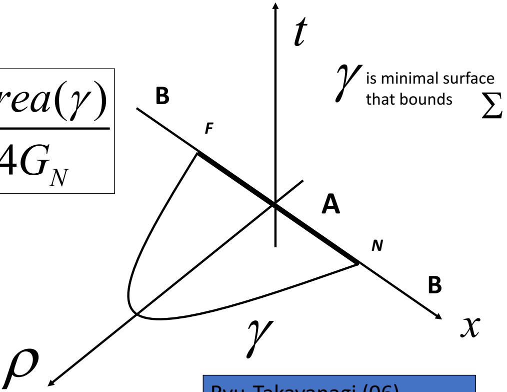
Similar story for gauge fields (contact terms of D. Kabat '95)

$$S_{con} = \frac{A[\Sigma]}{8\pi\epsilon^2}(\frac{d-2}{6} - 1)$$

is negative in dimensions $d < 8$

Holographic Entanglement Entropy

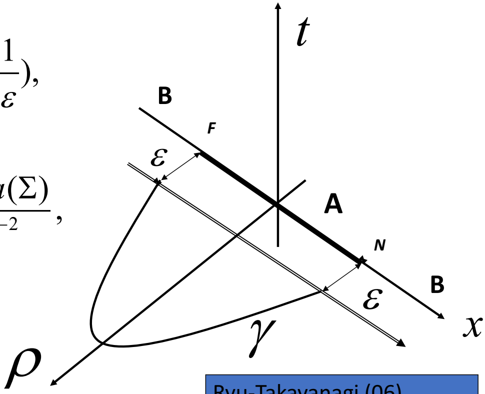
$$S = \frac{Area(\gamma)}{4G_N}$$



Ryu-Takayanagi (06)

UV/IR duality

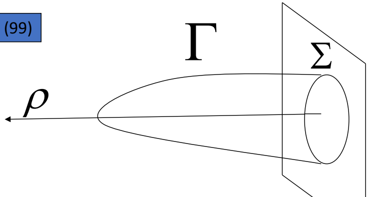
$$S = \frac{c}{6}\ln(\frac{1}{\varepsilon}),$$
$$d = 2$$
$$S \sim \frac{Area(\Sigma)}{\varepsilon^{d-2}},$$
$$d > 2$$



Ryu-Takayanagi (06)

Graham-Witten surface anomaly

Graham-Witten (99)



$$Area(\Gamma) = \frac{Area(\Sigma)}{\varepsilon^2} + c(\Sigma)\ln\varepsilon + ..$$

Graham-Witten anomaly

4d black hole on boundary

5d AdS bulk metric:

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(x,\rho) dx^i dx^j$$

$$g(x,\rho) = g_{(0)}(x) + g_{(2)}(x)\rho + g_{(4)}(x)\rho^2 + h_{(4)}(x)\rho^2 \ln\rho + ..$$

$$g_{(0)ij}(x) \quad : \qquad ds^2 = f(r)d\tau^2 + f^{-1}(r) + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

(4d metric of static black hole)

$$g_{(2)ij}(x) = -\frac{1}{2}(R_{ij} - \frac{1}{6} R g_{(0)ij}) \qquad g_{(4)ij}(x) = \langle T_{ij}^{CFT}(x) \rangle$$

Fefferman-Graham (85), Henningson-Skenderis (98),
Balasubramanian-Kraus (99), de Haro-Skenderis-S.S. (2000)

Entanglement Entropy of Black holes
due to N=4 super Yang-Mills

$$S_{div} = \frac{A(\Sigma)}{4\pi\varepsilon^2} N^2 - \frac{N^2}{2\pi} \int_\Sigma (\frac{1}{4} R_{ii} - \frac{1}{6} R) \ln \varepsilon$$

$$\frac{1}{G_N} = \frac{2N^2}{\pi}$$, N is number of colors in the CFT

Log terms in EE of a Conformal Field Theory (CFT)

Trace anomaly in $d = 4$

$$< T^\mu_\mu >= -\frac{A}{5760\pi^2} E_4 + \frac{B}{1920\pi^2} W^2$$

$E_4 = R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2$ is Euler density

$A_0 = 1\,,\ A_{1/2} = 11\,,\ A_1 = 62 \qquad B_0 = 1\,,\ B_{1/2} = 6\,,\ B_1 = 12$

Proposal for Log term in EE (based on conformal invariance and holography)

$$s_0^{CFT} = \frac{A}{180}\chi[\Sigma] - \frac{B}{240\pi}\int_\Sigma [W_{abab} - \mathrm{Tr}\,\hat{k}^2] \qquad \text{(SS '08)}$$

$\chi[\Sigma]$ is Euler number and $\hat{k}^a\,,\ a = 1,2$ is traceless part of extrinsic curvature

Applied for black holes this formula gives:

For extremal black holes (with near horizon geometry $H_2 \times S_2$)

$$s_0 = \frac{A}{90}$$

For the Schwarzschild black holes

$$s_0 = \frac{A - 3B}{90}$$

Note: for $\mathcal{N} = 4$ SYM in our normalization one has that $A = 3B$.

also related works of A. Sen and collaborators '11-'13 on supergravity vs microscopic entropy

Two test geometries in Minkowski spacetime ($W_{abab} = 0$):

$\Sigma = S_2 : \chi = 2, \ \hat{k}^a = 0 \, , a = 1, 2$ $\boxed{s_0 = \frac{A}{90}}$

(this case is conformally equivalent to extremal black hole)

$\Sigma = \mathrm{Cylinder}_2 \colon \ \chi = 0, \ \mathrm{Tr} \, \hat{k}^2 = \frac{1}{2R^2}$ $\boxed{s_0 = \frac{B}{240} \frac{L}{R}}$

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conformal scalar fields if Σ is sphere
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holographic CFT and its deformations (many papers)

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gauge fields if Σ *is sphere*

$$s_0 = \frac{62}{90}\ (predicted) \quad VS \quad s_0 = \frac{32}{90}\ (calculated)$$

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gauge fields if Σ is cylinder

$$s_0 = \frac{12}{240}\frac{L}{R}\ (predicted) \quad VS \quad s_0 = \frac{7}{240}\frac{L}{R}\ (calculated)$$

(Huerta and Pedraza '18)

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Any way: The proposal works for strongly coupled $\mathcal{N}=4$ SYM. Why it should not work for weakly coupled super-gauge multiplet (scalars, Dirac fermions and gauge fields)?

Why log terms might be interesting: case of black holes?

Log modification of BH entropy (Fursaev '94; SS '97)

$$S(M) = 4\pi \frac{M^2}{M_{PL}^2} + \sigma \ln M$$

where σ depends on multiplet of massless fields

$$\sigma = \frac{1}{45}(N_0 + \frac{7}{2}N_{1/2} - 13N_1 - \frac{233}{4}N_{3/2} + 212N_2 + 91N_A)$$

in Standard Model with graviton $\sigma = 164/45$ (without graviton $\sigma = -16/15$)
It produces modification in Hawking temperature

$$1/T_H = 8\pi \frac{M}{M_{PL}^2} + \frac{\sigma}{M}$$

so that $T_H \sim M$ for small black holes

Evaporation rate

$$\frac{dM}{dt} = -T_H^4 M^2$$

If $\sigma > 0$ then black hole evaporation time is infinite

(possible consequences for primordial black holes?)

Conclusion: why log terms are interesting after all?

- Log terms are universal, do not depend on regularization

- Log terms are geometrical: topology of entangling surface and conformal geometric invariants

- Log terms are related to conformal anomaly (still have to understand gauge fields)

- Many interesting future directions: boundaries, interactions, strings..

Thank you for your attention!

Increasing activity since 2015:

Herzog, Huang, Jensen ('15 and '17); Fursaev ('15), Jensen, O'Bannon ('15); SS ('15), Fursaev, SS ('16); Huang ('16); Berthiere, SS ('16); Astaneh, SS ('17), Astaneh, Fursaev, Berthiere, SS ('17); Herzog, Huang ('17); Chu, Miao, Guo ('17); Rodriguez-Gomez, Russo ('17 and '18); Seminara, Sisti, Tonni ('17 and '18); Berthiere ('18)

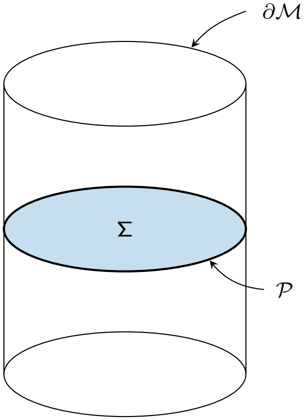
A richer structure (yet to be fully uncovered) of Weyl anomaly:

$$\int_{\mathcal{M}_d} \sqrt{g}\,\langle T_{\mu\nu}\rangle\, g^{\mu\nu} = a\chi(\mathcal{M}_d) + b_k \int_{\mathcal{M}_d} \sqrt{\gamma} I_k(W)$$

$$+a'\chi(\partial\mathcal{M}_d) + b'_k \int_{\partial\mathcal{M}_d} \sqrt{\gamma} J_k(W,\hat{K}) + c_n \int_{\partial\mathcal{M}_d} \sqrt{\gamma}\mathcal{K}_n(\hat{K})\,,$$

$\chi[\mathcal{M}_d]$ is Euler number of manifold $\mathcal{M}_d$, $I_k(W)$ are conformal invariants constructed from the Weyl tensor, $\mathcal{K}_n(\hat{K})$ are polynomial of degree $(d-1)$ of the trace-free extrinsic curvature, $K_{\mu\nu} = K_{\mu\nu} - \frac{1}{d-2}\gamma K$ is trace free extrinsic curvature of boundary; $\hat{K}_{\mu\nu} \to e^{\sigma}\hat{K}_{\mu\nu}$ if $g_{\mu\nu} \to e^{\sigma} g_{\mu\nu}$.

CFT on manifolds with boundaries: entanglement entropy



d = 3 :

$$\int_{\mathcal{M}_3} \langle T \rangle = \frac{c_1}{96}\chi[\partial\mathcal{M}_3] + \frac{c_2}{256\pi}\int_{\partial\mathcal{M}_3} \mathrm{Tr}\,\hat{K}^2$$

Charges (c_1, c_2):

$(-1, 1)$ for scalar filed (Dirichlet b.c.)
$(1, 1)$ for scalar field (conformal Robin b.c)
$(0, 2)$ for Dirac field (mixed b.c.)

A curious observation: for free fields c_2 equals to C_T (that appears in TT 2-point correlation function); is there a general proof that $c_2 = C_T$? or a counter-example?

Log term in entanglement entropy:

$$s_{log} = \frac{c_1}{24}\mathcal{N}$$

$\mathcal{N}$ is number of intersections of Σ and $\partial\mathcal{M}_3$

d = 4 :

$$\int \langle T \rangle = -\frac{a}{180}\chi[\mathcal{M}_4] + \frac{b}{1920\pi^2}\left(\int_{\mathcal{M}_4} \mathrm{Tr}\, W^2 \; - 8\int_{\partial\mathcal{M}_4} W^{\mu\nu\alpha\beta} N_\mu N_\beta \hat{k}_{\nu\alpha}\right) + \frac{c}{280\pi^2}\int_{\partial\mathcal{M}_4} \mathrm{Tr}\, \hat{k}^3$$

$$s_{log} = \frac{a}{720\pi}\left[\int_\Sigma R_\Sigma + 2\int_{\mathcal{P}} k_p\right] - \frac{b}{240\pi}\int_\Sigma [W_{ijij} - \mathrm{Tr}\, \hat{k}_i^2] + d\, F_d + e\, F_e$$

$$\text{where} \quad F_d = -\frac{1}{40\pi}\int_{\mathcal{P}} \hat{k}_{\mu\nu} v^\mu v^\nu \quad F_e = -\frac{1}{\pi}\int_{\mathcal{P}} (N\cdot p_i)(\hat{k}_i)_{\mu\nu} v^\mu v^\nu$$

Theory	a	b	c	d	boundary condition
real scalar	1	1	1	1	Dirichlet
real scalar	1	1	$\frac{7}{9}$	$-\frac{2}{3}$	conformal Robin
Dirac spinor	11	6	5	1	mixed
gauge boson	62	12	8	7	absolute/relative

Complete agreement with holographic computation for $\mathcal{N} = 4$ SYM provided boundary conditions preserve 1/2 SUSY
Astaneh, SS ('17); Astaneh, Berthiere, Fursaev, SS ('17)