Companion Exercises to Granada Lectures on Effective Field Theories (J.Santiago's version)

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1 Exercises

Exercise 1.1. Argue by dimensional analysis that all scaleless integrals except, possibly the following one,

$$\int_k \frac{1}{k^4} = \frac{\mathrm{i}}{16\pi^2}\left(\frac{1}{\varepsilon_{\mathrm{UV}}} - \frac{1}{\varepsilon_{\mathrm{IR}}}\right) = 0, \tag{1}$$

are identically zero. Using the identity

$$\frac{1}{k^4} = \frac{1}{k^2(k^2-M^2)} - \frac{M^2}{k^4(k^2-M^2)} \tag{2}$$

split the integral in Eq. (1) in two separate integrals, with the integrands in the previous identity, and discuss their UV and IR divergences. Compute them using dimreg and prove the identity in (1). You can use the general result

$$I_{n,m} \equiv \int_k \frac{1}{(k^2)^n}\frac{1}{(k^2-M^2)^m} = \frac{(-1)^{n+m}\mathrm{i}}{(4\pi)^{2-\varepsilon}(M^2)^{n+m-2+\varepsilon}}\frac{\Gamma(n+m-2+\varepsilon)\Gamma(2-n-\varepsilon)}{\Gamma(m)\Gamma(2-\varepsilon)}. \tag{3}$$

Solution:-

A generic scaleless integral of the form

$$I_n \equiv \mu^{-2\varepsilon}\int_k 1/k^n, \tag{4}$$

has mass dimension $4-n$. If $n \neq 4$ then it has to be proportional to a dimensionful scale, but there is no dimensionful scale in the integral so it has to be identically 0 by dimensional analysis. The case $n=4$ is special but we can write

$$I_4 = \mu^{-2\varepsilon}\int_k \frac{1}{k^4} = \mu^{-2\varepsilon}\int_k \frac{1}{k^2(k^2-M^2)} - \mu^{-2\varepsilon}\int_k \frac{M^2}{k^4(k^2-M^2)} \equiv I_4^{\mathrm{UV}} - I_4^{\mathrm{IR}}. \tag{5}$$

where the first integral is UV divergent but IR convergent and the opposite for the second one. Using Eq. (3) we obtain

$$I_4^{\mathrm{UV}} = I_{1,1} = \frac{\mathrm{i}}{16\pi^2}\left(\frac{1}{\bar{\varepsilon}_{\mathrm{UV}}} + 1 + \ln\frac{\mu^2}{M^2}\right), \tag{6}$$

$$I_4^{\mathrm{IR}} = M^2 I_{2,1} = \frac{\mathrm{i}}{16\pi^2}\left(\frac{1}{\bar{\varepsilon}_{\mathrm{IR}}} + 1 + \ln\frac{\mu^2}{M^2}\right). \tag{7}$$

Technically we have $\varepsilon_{\mathrm{UV}} > 0$ and $\varepsilon_{\mathrm{IR}} < 0$ but we can analytically continue one into the other and have a unique ε. Taking the difference we obtain the requested result.

Exercise 1.2. Use the identity

$$0 = \int_k \frac{\partial}{\partial k^\mu} \frac{k^\mu}{(k^2 - M^2)^n}, \tag{8}$$

to prove the following integration by parts identity

$$\int_k \frac{1}{(k^2 - M^2)^{n+1}} = \frac{d-2n}{2n} \frac{1}{M^2} \int_k \frac{1}{(k^2 - M^2)^n}, \qquad n \geq 1. \tag{9}$$

Solution:-

$$\begin{aligned}
\frac{\partial}{\partial k^\mu} \frac{k^\mu}{(k^2 - M^2)^n} &= \frac{d}{(k^2 - M^2)^n} - \frac{2nk^2}{(k^2 - M^2)^{n+1}} \\
&= \frac{d}{(k^2 - M^2)^n} - \frac{2n(k^2 - M^2 + M^2)}{(k^2 - M^2)^{n+1}} \\
&= \frac{d-2n}{(k^2 - M^2)^n} - \frac{2nM^2}{(k^2 - M^2)^{n+1}},
\end{aligned} \tag{10}$$

which, upon integration produces Eq. (9).

Exercise 1.3. Consider the following divergent integral

$$I(p) = \int_0^\infty \mathrm{d}k \frac{k}{k+p}. \tag{11}$$

Take as many derivatives with respect to p as needed to make the integral finite and compute it. Write the original integral, by integrating the result with respect to p, as a non-local function of p plus a polynomial in p (with possibly divergent coefficients).

Solution:-

$I(p)$ is linearly divergent in the UV ($k \to \infty$). Its first integral is logarithmically divergent and the second one is finite,

$$I'(p) = -\int_0^\infty \mathrm{d}k \, \frac{k}{(k+p)^2}, \quad I''(p) = 2\int_0^\infty \mathrm{d}k \, \frac{k}{(k+p)^3} = \frac{1}{p}. \tag{12}$$

Integrating twice, with respect to p, we obtain

$$\int\int \mathrm{d}p I''(p) = \int \mathrm{d}p \, [c_1 + \ln p] = p \ln p - p + c_1 p + c_2. \tag{13}$$

where c_1 and c_2 are integration constants that contain the UV divergences of the original integral. The important point is that they arise from the integration and therefore are always propostional to polynomial in momenta, a sign of local operators generating them.

Exercise 1.4. Use the following topological identities for connected diagrams

$$L = P - V + 1, \quad 2P + E = \sum_v n_v, \tag{14}$$

where the sum in the second expression runs over all vertices in the diagram, L stands for the number of loops, P for the number of internal propagators, V for the number of vertices, n_v is the number of particles on vertex v and E is the number of external particles, to show that

$$\left[(n_i - 2) - (n_j - 2)C_j \frac{\partial}{\partial C_j}\right] C'^{(1)}_{i,D} = -2LC'^{(1)}_{i,D}. \tag{15}$$

Where $C'^{(1)}_{i,D}$ is a polynomial of WCs arising from diagram D.

Solution:-

$C'^{(1)}_i$ is computed as a sum of contributions from Feynman diagrams

$$C'^{(1)}_i = \sum_D C'^{(1)}_{i,D}, \tag{16}$$

where D runs over all diagrams that contribute and $C_{,D}i'^{(1)}$ is a polynomial in all the WCs of the model. Using the topological identities we have

$$\begin{aligned}\left[(n_i - 2) - (n_j - 2)C_j \frac{\partial}{\partial C_j}\right] C'^{(1)}_{i,D} &= \left[E - 2 - \sum_v (n_v - 2)\right] C'^{(1)}_{i,D} \\ &= [E - 2 - 2P - E + 2V]\, C'^{(1)}_{i,D} = -2LC'^{(1)}_{i,D}.\end{aligned} \tag{17}$$

In the first identity we have used that $n_i = E$ and that

$$C_j \frac{\partial}{\partial C_j}, \tag{18}$$

acting on $C'^{(1)}_{i,D}$ just runs over all vertices returning $C'^{(1)}_{i,D}$.

Exercise 1.5. Consider the following Lagrangian for a real scalar

$$\mathcal{L} = \mathcal{L}_4 + \mathcal{L}_6, \tag{19}$$

$$\mathcal{L}_4 = -\frac{1}{2}\phi(\partial^2 + m^2)\phi - \lambda\phi^4, \tag{20}$$

$$\mathcal{L}_6 = \frac{\alpha_{61}}{\Lambda^2}\phi^6 + \frac{\beta_{62}}{\Lambda^2}\phi^3\partial^2\phi. \tag{21}$$

Perform the field redefinition $\phi \to \phi + \beta_{62}\phi^3/\Lambda^2$ and obtain the Lagrangian in terms of the new interpolating field. Compute now the EoM for ϕ from $\mathcal{L}_4$ and introduce the solution on the second operator of $\mathcal{L}_6$. Compare the resulting Lagrangians. Compute now the effect of the field redefinition up to dimension 8. Compute the EoM for ϕ using the full $\mathcal{L}$ and insert the solution back in the second operator of $\mathcal{L}_6$. Compare the resulting Lagrangians.

Solution:-

After the field redefinition the Lagrangian goes to

$$\begin{aligned}\mathcal{L} \to & \frac{1}{2} - \frac{1}{2}\phi(\partial^2 + m^2)\phi - \left(\lambda + \beta_{62}\frac{m^2}{\Lambda^2}\right)\phi^4 \\ & + \frac{1}{\Lambda^2}\left(\alpha_{61} - 4\lambda\beta_{62} - \frac{\beta_{62}^2}{2}\frac{m^2}{\Lambda^2}\right)\phi^6 + \frac{1}{\Lambda^4}\left[(6\alpha_{61}\beta_{62} - 6\lambda\beta_{62}^2)\phi^8 + \frac{39}{10}\beta_{62}^2\phi'^5\partial^2\phi'\right],\end{aligned} \tag{22}$$

where we have used integration by parts to prove the following two identities

$$\phi^4(\partial_\mu\phi)(\partial^\mu\phi) = -\frac{1}{5}\phi^5\partial^2\phi, \tag{23}$$

$$\phi^3\partial^2\phi^3 = \frac{9}{5}\phi^5\partial^2\phi. \tag{24}$$

The EoM from $\mathcal{L}$ are given by

$$\partial^2\phi = -m^2\phi - 4\lambda\phi^3 + 6\frac{\alpha_{61}}{\Lambda^2}\phi^5 + \frac{\beta_{62}}{\Lambda^2}(3\phi^2\partial^2\phi + \partial^2\phi^3). \tag{25}$$

When inserted in the redundant operator, we obtain

$$\begin{aligned}\frac{\beta_{62}}{\Lambda^2}\phi^3\partial^2\phi \to & \frac{\beta_{62}}{\Lambda^2}\phi^3\left[-m^2\phi - 4\lambda\phi^3 + 6\frac{\alpha_{61}}{\Lambda^2}\phi^5 + \frac{\beta_{62}}{\Lambda^2}(3\phi^2\partial^2\phi + \partial^2\phi^3)\right] \\ = & -\beta_{62}\frac{m^2}{\Lambda^2}\phi^4 - 4\lambda\beta_{62}\frac{1}{\Lambda^2}\phi^6 + 6\alpha_{61}\beta_{62}\frac{1}{\Lambda^4}\phi^8 + \frac{24}{5}\beta_{62}^2\frac{1}{\Lambda^4}\phi^5\partial^2\phi.\end{aligned} \tag{26}$$

Comparing Eq. (26) with Eq (22) we see that the use of EoM correctly recovers the effect of the field redefinitions at leading order, written in blue in Eq. (26), while the terms quadratic in β_{61} come with the incorrect coefficient or are even not present.

Exercise 1.6. Using partial fractioning, show the following result,

$$I_{\mathrm{F}} = \mu^{2\varepsilon}\int_k \frac{1}{k^2-M^2}\frac{1}{k^2-m^2} = \frac{\mathrm{i}}{16\pi^2}\left\{\frac{1}{\bar{\varepsilon}}+1+\ln\frac{\mu^2}{M^2}+\frac{m^2}{M^2-m^2}\ln\frac{m^2}{M^2}\right\}. \tag{27}$$

Solution:-

$$\begin{aligned}
I_F &= \frac{\mu^{2\varepsilon}}{M^2-m^2}\int_k\left(\frac{1}{k^2-M^2}-\frac{1}{k^2-m^2}\right)\\
&= \frac{1}{M^2-m^2}\frac{\mathrm{i}}{16\pi^2}\left(M^2\left[\frac{1}{\bar{\varepsilon}}+1-\ln\frac{M^2}{\mu^2}\right]-m^2\left[\frac{1}{\bar{\varepsilon}}+1-\ln\frac{m^2}{\mu^2}\right]\right)\\
&= \frac{\mathrm{i}}{16\pi^2}\left(\frac{1}{\bar{\varepsilon}}+1+\ln\mu^2-\frac{M^2}{M^2-m^2}\ln M^2+\frac{m^2}{M^2-m^2}\left[\ln m^2+\color{blue}{\ln M^2-\ln M^2}\right]\right)\\
&= \frac{\mathrm{i}}{16\pi^2}\left(\frac{1}{\bar{\varepsilon}}+1+\ln\frac{\mu^2}{M^2}+\frac{m^2}{M^2-m^2}\ln\frac{m^2}{M^2}\right). \qquad (28)
\end{aligned}$$

Where we have denoted in blue a term that has been added and subtracted to get the final result.

Exercise 1.7. Using `MatchmakerEFT` [1] to compute the beta functions of the WCs of the following EFT,

$$\mathcal{L}_{\mathrm{EFT}} = -\frac{1}{2}\phi(\partial^2+m_\phi^2)\phi+\bar{\psi}(\mathrm{i}\not{\partial}-m_\psi)\psi-\eta\bar{\psi}\psi\phi+\frac{C_s}{2}\bar{\psi}\psi\bar{\psi}\psi+\frac{C_v}{2}\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi+\frac{C_t}{2}\bar{\psi}\sigma^{\mu\nu}\psi\bar{\psi}\sigma_{\mu\nu}\psi, \tag{29}$$

where we have defined $\sigma^{\mu\nu}\equiv(\mathrm{i}/2)[\gamma^\mu,\gamma^\nu]$.

Solution:-

The result is

$$\begin{aligned}
16\pi^2\beta_{m_\phi^2} &= 4\eta^2(m_\phi^2-6m_\psi^2), & (30)\\
16\pi^2\beta_{m_\psi} &= m_\psi[3\eta^2+m_\psi^2(6C_s-8C_v-24C_t)], & (31)\\
16\pi^2\beta_\eta &= \eta[5\eta^2+m_\psi^2(18C_s-24C_v-72C_t)], & (32)\\
16\pi^2\beta_{C_s} &= 6\eta^2C_s, & (33)\\
16\pi^2\beta_{C_v} &= 12\eta^2C_t, & (34)\\
16\pi^2\beta_{C_t} &= 2\eta^2(C_v+C_t). & (35)
\end{aligned}$$

Exercise 1.8. Match the following two amplitudes,

$$\mathrm{i}\mathcal{M}_F = \bar{u}_3 u_1 \bar{u}_4 u_2 \frac{\mathrm{i}\lambda^2}{M^2}\left(1 + \frac{p_1^2 + p_3^2 - 2p_1\cdot p_3}{M^2}\right), \tag{36}$$

and

$$\begin{aligned}\mathrm{i}\mathcal{M}_E = \mathrm{i}\bar{u}_4 u_1 \bar{u}_3 u_2 \Big\{ & C_{\psi^4} - C^{(1)}_{d^2\psi^4}[p_1^2 + p_2^2] - (C^{(1)}_{d^2\psi^4})^*[p_3^2 + p_4^2] \\ & + C^{(2)}_{d^2\psi^4}[p_1\cdot p_3 + p_2\cdot p_4] + C^{(3)}_{d^2\psi^4}[p_1\cdot p_4 + p_2\cdot p_3]\Big\},\end{aligned} \tag{37}$$

and show that the system has a unique solution

$$C_{\psi^4} = \frac{\lambda^2}{M^2}, \quad C^{(1)}_{d^2\psi^4} = -\frac{\lambda^2}{2M^4}, \quad C^{(2)}_{d^2\psi^4} = -\frac{\lambda^2}{M^4}, \quad C^{(3)}_{d^2\psi^4} = 0. \tag{38}$$

Solution:-

$$\begin{aligned}\mathrm{i}\mathcal{M}_E = {} & \mathrm{i}\bar{u}_4 u_1 \bar{u}_3 u_2 \Big\{ C_{\psi^4} - C^{(1)}_{d^2\psi^4}[p_1^2 + p_2^2] - (C^{(1)}_{d^2\psi^4})^*[p_3^2 + p_4^2] \\ & + C^{(2)}_{d^2\psi^4}[p_1\cdot p_3 + p_2\cdot p_4] + C^{(3)}_{d^2\psi^4}[p_1\cdot p_4 + p_2\cdot p_3]\Big\} \\ = {} & \bar{u}_4 u_1 \bar{u}_3 u_2 \mathrm{i}\Big\{ C_{\psi^4} + \big[C^{(3)}_{d^2\psi^4} - C^{(1)}_{d^2\psi^4} - \big(C^{(1)}_{d^2\psi^4}\big)^*\big]p_1^2 + \big[C^{(2)}_{d^2\psi^4} - C^{(1)}_{d^2\psi^4} - (C^{(1)}_{d^2\psi^4})^*\big]p_2^2 \\ & - 2\big(C^{(1)}_{d^2\psi^4}\big)^* p_3^2 + \big[C^{(2)}_{d^2\psi^4} + C^{(3)}_{d^2\psi^4} - 2\big(C^{(1)}_{d^2\psi^4}\big)^*\big]p_1\cdot p_2 \\ & + \big[C^{(2)}_{d^2\psi^4} - C^{(3)}_{d^2\psi^4} + 2\big(C^{(1)}_{d^2\psi^4}\big)^*\big]p_1\cdot p_3 \\ & + \big[-C^{(2)}_{d^2\psi^4} + C^{(3)}_{d^2\psi^4} + 2\big(C^{(1)}_{d^2\psi^4}\big)^*\big]p_2\cdot p_3\Big\} - (3 \leftrightarrow 4),\end{aligned} \tag{39}$$

where in the second equality we have used momentum conservation to eliminate $p_4 = p_1 + p_2 - p_3$. The term that survives in the limit of vanishing momenta gives

$$C_{\psi^4} = \frac{\lambda^2}{M^2}. \tag{40}$$

Equating the terms proportional to p_3^2 we get

$$C^{(1)}_{d^2\psi^2} = -\frac{\lambda^2}{2M^4}. \tag{41}$$

Equating now the one proportional to p_1^2 we get

$$C^{(3)}_{d^2\psi^2} = \frac{\lambda^2}{M^4} + C^{(1)}_{d^2\psi^2} + (C^{(1)}_{d^2\psi^2})^* = 0. \tag{42}$$

Finally, using the term proportional to p_2^2 we get

$$C^{(2)}_{d^2\psi^2} = C^{(1)}_{d^2\psi^2} + (C^{(1)}_{d^2\psi^2})^* = -\frac{\lambda^2}{M^4}. \tag{43}$$

It is easy to see that all other terms are correctly reproduced with these values.

References

[1] Adrian Carmona, Achilleas Lazopoulos, Pablo Olgoso, and Jose Santiago. Matchmakereft: automated tree-level and one-loop matching. *SciPost Phys.*, 12(6):198, 2022.