Modelling the distribution of cosmic rays: Applying the discontinuous Galerkin method to the Vlasov-Fokker-Planck equation

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Cosmic ray acceleration in a nutshell

Cosmic rays (energetic and charged particles) are accelerated at collisionless shocks in astrophysical plasmas. The particles are scattered by plasma waves on the up- and downstream site of the shock. Since the plasma waves on the downstream site move with a fraction of the shock's speed, the cosmic rays are scattered by moving scattering centres and thus gain energy. The distribution of cosmic rays is modelled with the Vlasov-Fokker-Planck (VFP) equation.

The Vlasov-Fokker-Planck equation

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\nabla f + q\left(\boldsymbol{E}+\frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right)\cdot\frac{\partial f}{\partial \boldsymbol{p}} = \left(\frac{\partial f}{\partial t}\right)_{\text{Sc}}$$

f is called the distribution function. It describes the cosmic ray's number density in the phase space, i.e. $\mathrm{d}N = f\mathrm{d}x^3\mathrm{d}p^3$. The $\boldsymbol{E}$- and $\boldsymbol{B}$-fields are supported by the thermal plasma, through which the particles propagate. The distribution function f evolves in time because

- the particles are moving at velocity $\boldsymbol{v}$,
- they are accelerated by the Lorentz force
- and they are scattered by plasma waves.

The scattering randomises the motion of the particles, i.e. anisotropies in the cosmic ray distribution will decay. The 'classical' transport equation takes advantage of this and, consequently, only contains the isotropic part of the distribution function. In some astrophysical scenarios the time scales are such, that it is necessary to consider higher order anisotropies. It is possible to include them if in momentum space spherical coordinates are used, the distribution function f is expanded in spherical harmonics and the scattering operator is defined to be

$$\left(\frac{\partial f}{\partial t}\right)_{\text{Sc}} = \frac{\nu}{2}\Delta_{\theta,\varphi}\,.$$

ν is the scattering frequency. This operator implies that the anisotropies decay, but the expansion can be truncated at an arbitrary order.

The spherical harmonic expansion of the distribution function

The number of variables of the distribution function $f = f(\boldsymbol{x},\boldsymbol{p},t)$can be reduced, if we use spherical if we expand f in (real) spherical harmonics, namely

$$f(\boldsymbol{x},\boldsymbol{p},t) = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{s=0}^{1} f_{lms}(\boldsymbol{x},p,t)Y_{lms}(\theta,\varphi)\,.$$

The spherical harmonics Y_{lms} contain the information on the cosmic ray's direction of travel. It is possible to derive a system of differential equations for the expansion coefficients f_{lms} by

- Substituting the expansion of f into the VFP equation,
- multiplying it with $Y_{l'm's'}$,
- integrating it over the solid angle $\mathrm{d}\Omega$

$$\partial_t\boldsymbol{f} + \boldsymbol{A}\nabla\boldsymbol{f} + \boldsymbol{B}\partial_p\boldsymbol{f} + \boldsymbol{C}\boldsymbol{f} = 0\,.$$

$(\boldsymbol{f})_i = f_{lms}$ is a vector containing all the expansion coefficients. $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are matrices.

Prototype

For testing purposes the system of differential equations determining the expansion coefficients f_{lms} is simplified:

1. It is assumed that there are no relevant $\boldsymbol{E}$- and $\boldsymbol{B}$-fields.
2. The dimension of the configuration space is set to one.

Thus, the prototype VFP equation is

$$\partial_t\boldsymbol{f} + \boldsymbol{A}\partial_x\boldsymbol{f} + \boldsymbol{C}\boldsymbol{f} = 0\,.$$

This equation describes the evolution of a distribution of cosmic rays, which may initially be anisotropic: The energy of the particles does not change. The distribution will be advected along the x-direction and the anisotropies (modelled by the expansion coefficients with $l \geq 1$) decay with time. This type of equation is called an *advection-reaction equation*.

The discontinuous Galerkin method

The discontinuous Galerkin (dG) method is a method to solve advection-reaction equations numerically. The dG method is a finite element (FE) method. FE methods

- split the domain into cells,
- define a set of polynomials on each cell,
- and a set of conditions to determine the coefficients of these polynomials.

The discrete approximation of the solution to the differential equation is a linear combination of these polynomials, namely

$$\boldsymbol{f}_h = \sum_{j=1}^{N}\alpha_j\boldsymbol{\Phi}_j\,.$$

N is the total number of polynomials, the number of degrees of freedom. In 'classical' finite element methods, it is required that the polynomials are continuous at the cell boundaries. This reduces the number of degrees of freedom. This is not the case for the *discontinuous* Galerkin method.

The computation of the discrete approximation $\boldsymbol{f}_h$ requires the determination of the coefficients α_j. It is possible to get a linear system of equations for the αs by multiplying the differential equation with one of the finite element's polynomials and integrating over the domain, i.e.

$$\sum_K\sum_{j=1}^{n}\int_K\left[\frac{\partial\alpha_j}{\partial t}\boldsymbol{\Phi}_j\cdot\boldsymbol{\Phi}_i - \alpha_j\left(\boldsymbol{A}\boldsymbol{\Phi}_j\right)\cdot\frac{\partial\boldsymbol{\Phi}_i}{\partial x} + \alpha_j\left(\boldsymbol{C}\boldsymbol{\Phi}_j\right)\cdot\boldsymbol{\Phi}_i\right]\mathrm{d}x + \alpha_j\left(\boldsymbol{A}\boldsymbol{\Phi}_j\right)\cdot\boldsymbol{\Phi}_i\big|_{x_{K,l}}^{x_{K,r}} = 0\,.$$

- The sum over K is a sum over all cells.
- n denotes the number of polynomials belonging to *one* cell.
- The integration by parts yields a boundary term, which is ill-defined. The polynomials are not continuous at the cell boundaries. It is replaced by a numerical flux term.
- The equation holds for all indices $i \in \{1,\dots,N\}$ and, hence, yields a system of ordinary differential equations for the αs.
- Applying a time discretisation method (e.g. forward Euler) gives a linear system of equations for the αs in each time step

Numerical results

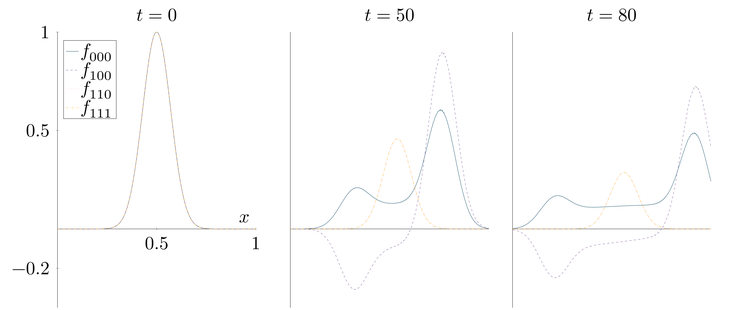

Figure 1. The evolution of a distribution function with a dipole at $t = 0$.

- The f_{000} term is the isotropic part of the distribution function f. It is initialised as a Gaussian distribution.
- The expansion coefficients f_{100}, f_{110} and f_{111} introduce a dipole in the distribution of the cosmic rays. All three of them are initialised as a Gaussian distribution as well.
- Two components of this dipole, namely f_{110} and f_{111}, do not interact with the isotropic part of the distribution function.
- The third component f_{100} is the x-component of the dipole and changes the isotropic part as depicted.

Next steps

The next steps in the development of the dG solver of the VFP equation are

- The inclusion of $\boldsymbol{E}$- and B-fields.
- To work with a two dimensional configuration space.
- To use the solution of the VFP solver to compute the feedback of the cosmic rays onto the plasma.

References

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