Integrability in Binary Black Holes Dynamics

4th BIG Meeting: Barcelona Initiative for Gravitation

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November 29, 2024

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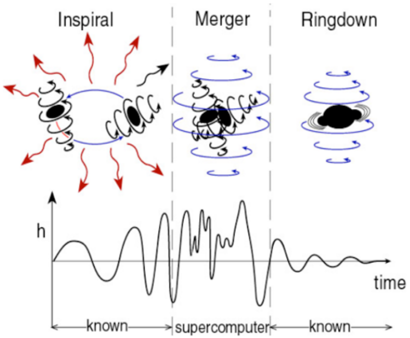
In collaboration with C.F. Sopuerta, J.L. Jaramillo, C. Vitel, B. Krishnan

- J. L. Jaramillo, M. Lenzi, and C. F. Sopuerta, Phys.Rev.D 110, 104049 (2024), J. L. Jaramillo, B. Krishnan, and C. F. Sopuerta, Int. J. Mod. Phys. D 32, 2342005 (2023)
- M. Lenzi and C. F. Sopuerta, Phys. Rev. D 104, 084053 (2021), Phys. Rev. D 104, 124068 (2021), Phys. Rev. D 107, 044010 (2023), Phys. Rev. D 107, 084039 (2023), Phys. Rev. D 109, 084030 (2024)
- E. Gasperin and J. L. Jaramillo, Class. Quant. Grav. 39, 115010 (2022), J. L. Jaramillo, R. Panosso Macedo, and L. Al Sheikh, Phys. Rev. X 11, 031003 (2021)

A Universality Conjecture



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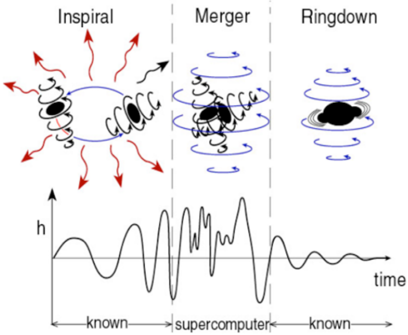

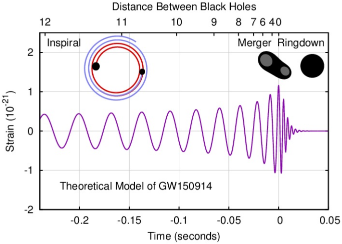

- *Asymptotic reasoning* : filter some DoFs to unveil the underlying (universal) patterns

- *Wave mean flow* : effective separation into “slow” and “fast” DoFs

J. L. Jaramillo, B. Krishnan, and C. F. Sopuerta, Int. J. Mod. Phys. D **32**, 2342005 (2023), J. L. Jaramillo and B. Krishnan, (2022), J. L. Jaramillo, M. Lenzi, and C. F. Sopuerta, Phys. Rev. D **110**, 104049 (2024)

Table of contents

BH perturbation theory

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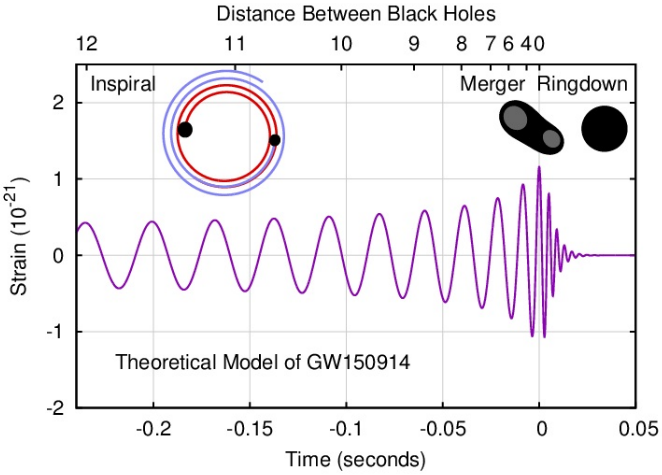

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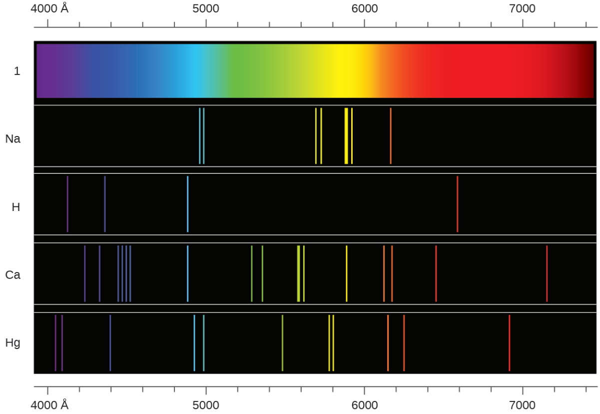
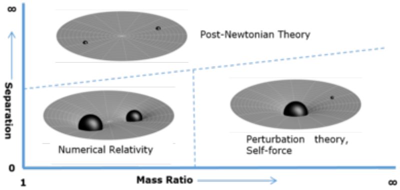
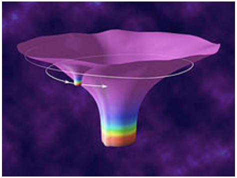


- Scattering of particles and waves through Black Holes (Hawking radiation)
- Binary ringdown signal and Quasi-Normal modes (Black Hole spectroscopy)
- GW with perturbative sources (e.g. EMRI's)





Decoupling Einstein equations

- Perturbed Einstein equations at linear order

$$g_{\mu\nu} = \widehat{g}_{\mu\nu} + h_{\mu\nu} \quad \longrightarrow \quad \widehat{G}_{\mu\nu} = 0\,, \quad \delta G_{\mu\nu} = 0$$

- Background metric splitting reflecting spherical symmetry

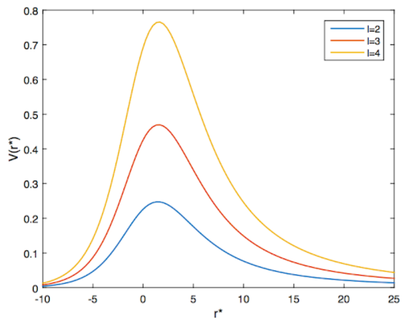
$$\widehat{g}_{\mu\nu} = \begin{pmatrix} g_{ab} & 0 \\ 0 & r^2\Omega_{AB} \end{pmatrix} \quad \longrightarrow \quad \begin{array}{l} g_{ab}\mathrm{d}x^a\mathrm{d}x^b = -f(r)\,\mathrm{d}t^2 + \mathrm{d}r^2/f(r) \\ \Omega_{AB}\mathrm{d}\Theta^A\mathrm{d}\Theta^B = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2 \end{array}$$

- Harmonic and parity expansion of h $\quad h_{\mu\nu} = \sum_{\ell,m} h_{\mu\nu}^{\ell m,\mathrm{odd}} + h_{\mu\nu}^{\ell m,\mathrm{even}}$

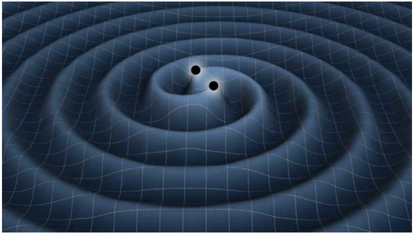
Master equations

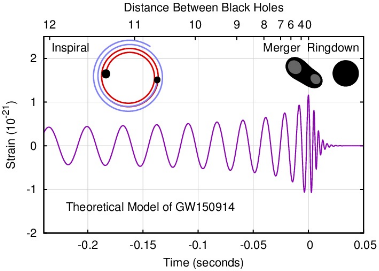
$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}-V_\ell^{\mathrm{even/odd}}\right)\Psi_{\mathrm{even/odd}}^{\ell m}=0$$

- Master functions $\Psi = \Psi(h, \partial h)$
- Effective potential $V_\ell^{\mathrm{even/odd}}$
- Tortoise coordinate $dx/dr = 1/f(r)$
- "fast" (Ψ) and "slow" (V) DoF



BH binary systems





1. GW signal can be written in terms of master functions

$$h_{+/\times} \propto \Psi \Rightarrow h = h_+ - ih_\times \propto \Psi$$

2. Energy and angular momentum emission (luminosity) at infinity

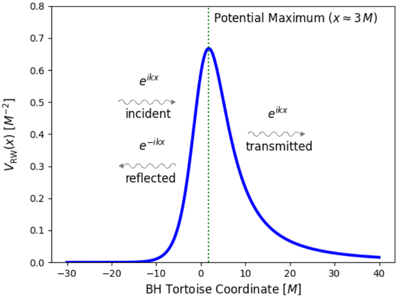
$$\frac{dE}{dt} \propto |\dot{\Psi}|^2, \quad \frac{dJ}{dt} \propto \Psi\dot{\Psi}$$

3. Extreme Mass Ratio Inspirals (EMRIs) detectable by LISA

K. Martel and E. Poisson, Phys. Rev. D **71**, 104003 (2005)

Frequency domain master equation

$$\Psi = e^{ikt}\psi \quad \longrightarrow \quad \psi_{,xx} - V\psi = -k^2\psi$$



- Master equation describes scattering of waves and particles
- Quasi-normal modes correspond to vanishing incident wave, $\omega_n \in \mathbb{C}$

J. A. H. Futterman, F. A. Handler, and R. A. Matzner, **Scattering from Black Holes,** (Cambridge University Press, May 2012)

Known master equations

Odd parity

$$\Psi_{\mathsf{RW}} = \frac{r^a}{r}\tilde{h}_a$$

$$\Psi_{\mathsf{CPM}} = \frac{2r}{(\ell-1)(\ell+2)}\varepsilon^{ab}\left(\tilde{h}_{b:a} - \frac{2}{r}r_a\tilde{h}_b\right)$$

T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063–1069 (1957), C. T. Cunningham, R. H. Price, and V. Moncrief, Astrophys. J. **224**, 643–667 (1978)

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Even parity

$$\Psi_{\mathrm{ZM}} = \frac{2r}{\ell(\ell+1)}\left\{\tilde{K} + \frac{2}{\lambda}\left(r^a r^b \tilde{h}_{ab} - r r^a \tilde{K}_{:a}\right)\right\}$$

$$\lambda(r) = (\ell+2)(\ell-1) - \Lambda r^2 - 3\,(f-1)$$

F. J. Zerilli, Phys. Rev. D **2**, 2141–2160 (1970), V. Moncrief, Ann. Phys. (N.Y.) **88**, 323 (1974)

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Perturbative gauge invariants

$$\tilde{h}_a = h_a - \frac{1}{2}h_{2:a} + \frac{r_a}{r}h_2\,, \qquad \tilde{h}_{ab} = h_{ab} - \kappa_{a:b} - \kappa_{b:a}\,,$$

$$\tilde{K} = K + \frac{\ell(\ell+1)}{2}G - 2\frac{r^a}{r}\kappa_a$$

What are all the possible master equations that one can obtain for the vacuum perturbations of a Schwarzschild BH?

M. L. and C. F. Sopuerta, Phys. Rev. D **104**, 084053 (2021)

Assumptions

What are all the possible master equations that one can obtain for the vacuum perturbations of a Schwarzschild BH?

1. Linear in the metric perturbations and first-order derivatives
2. Time independent coefficients
3. Arbitrary perturbative gauge

M. L. and C. F. Sopuerta, Phys. Rev. D **104**, 084053 (2021)

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Infinite pairs of master functions and potentials (V, Ψ)

- Standard branch
- Darboux branch

M. L. and C. F. Sopuerta, Phys. Rev. D **104**, 084053 (2021)

The standard branch

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}-{}_{\mathrm{S}}V_\ell^{\mathrm{odd/even}}\right){}_{\mathrm{S}}\Psi^{\mathrm{odd/even}}=0$$

- Standard branch potentials

$$ {}_{\mathrm{S}}V_\ell^{\mathrm{odd/even}}=\begin{cases} V_\ell^{\mathrm{RW}} & \text{odd parity}\\ V_\ell^{\mathrm{Z}} & \text{even parity}\end{cases}$$

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- Most general master function

$$ {}_{\mathrm{S}}\Psi^{\mathrm{odd/even}}=\begin{cases} \mathcal{C}_1\Psi^{\mathrm{CPM}}+\mathcal{C}_2\Psi^{\mathrm{RW}} & \text{odd parity} \\ \mathcal{C}_1\Psi^{\mathrm{ZM}}+\mathcal{C}_2\boxed{\Psi^{\mathrm{NE}}} & \text{even parity}\end{cases}$$

$$\Psi^{\mathrm{NE}}(t,r)=\frac{1}{\lambda(r)}t^a\left(r\tilde{K}_{:a}-\tilde{h}_{ab}r^b\right)$$

The Darboux branch

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}-{}_{\rm D}V_\ell^{\rm odd/even}\right){}_{\rm D}\Psi^{\rm odd/even}=0$$

- Family of potentials ${}_{\rm D}V_\ell^{\rm odd/even}$ satisfying

$$\left(\frac{\delta V_{,x}}{\delta V}\right)_{,x}+2\left(\frac{V_{\ell,x}^{\rm RW/Z}}{\delta V}\right)_{,x}-\delta V=0\,,$$

with $\delta V={}_{\rm D}V_\ell^{\rm odd/even}-V_\ell^{\rm RW/Z}$.

M. L. and C. F. Sopuerta, Phys. Rev. D **104**, 084053 (2021), Phys. Rev. D **109**, 084030 (2024)

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  with $\delta V={}_{\rm D}V_\ell^{\rm odd/even}-V_\ell^{\rm RW/Z}$.

- Most general (potential dependent) master function

$${}_{\rm D}\Psi^{\rm odd/even}=\begin{cases}\mathcal{C}\left(\Sigma^{\rm odd}\,\Psi^{\rm CPM}+\Phi^{\rm odd}\right) & \text{odd parity}\\ \mathcal{C}\left(\Sigma^{\rm even}\,\Psi^{\rm ZM}+\Phi^{\rm even}\right) & \text{even parity}\end{cases}$$

M. L. and C. F. Sopuerta, Phys. Rev. D **104**, 084053 (2021), Phys. Rev. D **109**, 084030 (2024)

Darboux covariance

Darboux transformation

- Darboux transformation between (v, Φ) and (V, Ψ)

$$\left(-\partial_t^2 + \partial_x^2 - v\right)\Phi = 0 \longrightarrow \begin{cases} \Psi = \Phi_{,x} + W\,\Phi \\ V = v + 2\,W_{,x} \\ W_{,x} - W^2 + v = \mathcal{C} \end{cases} \longrightarrow \left(-\partial_t^2 + \partial_x^2 - V\right)\Psi = 0$$

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$$\left(\frac{\delta V_{,x}}{\delta V}\right)_{,x} + 2\left(\frac{v_{,x}}{\delta V}\right)_{,x} - \delta V = 0$$

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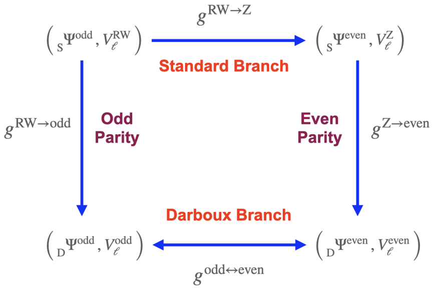
- Darboux covariance of perturbations of spherically-symmetric BHs



M. Lenzi and C. F. Sopuerta,
Phys. Rev. D **104**, 124068
(2021)

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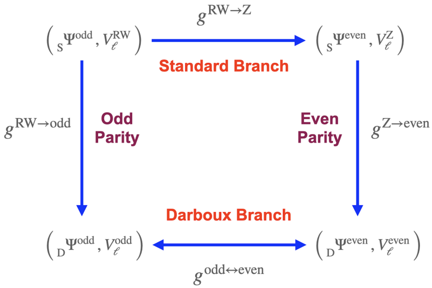

Isospectral symmetry

$$\Psi = e^{ikt}\psi$$

$$\phi_{,xx} - v\phi = -k^2\phi$$

$$\psi_{,xx} - V\psi = -k^2\psi$$

Darboux transformation

- Darboux transformation between (v, Φ) and (V, Ψ)

$$\left(-\partial_t^2 + \partial_x^2 - v\right)\Phi = \boxed{\sigma} \longrightarrow \begin{cases} \Psi = \Phi_{,x} + W\,\Phi \\ V = v + 2\,W_{,x} \\ W_{,x} - W^2 + v = \mathcal{C} \end{cases} \longrightarrow \left(-\partial_t^2 + \partial_x^2 - V\right)\Psi = \boxed{S}$$

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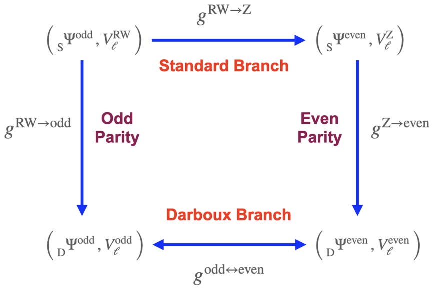

Darboux covariance with perturbative sources

$$\boxed{S = \sigma_{,x} + W\sigma}$$

M. Lenzi and C. F. Sopuerta, Phys. Rev. D **109**, 084030 (2024)

Standard view on DTs

DT in frequency domain $\Psi(t,r) = e^{ikt}\psi(x;k)$

$$L_V\psi(x;k) \equiv \left(\partial_x^2 - V\right)\psi(x;k) = -k^2\psi(x;k)$$

$$L_V\psi_0 = -k_0^2\psi_0 \;\longrightarrow\; W(x) = -(\ln\psi_0)_{,x} \;\longrightarrow\; \begin{cases} L_v\phi = -k^2\phi \\ \phi = \mathcal{W}[\psi,\psi_0]/\psi_0 \end{cases}$$

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- Darboux transformation between RW and ZM

$$\begin{aligned} \psi_0 &= \tfrac{\lambda(r)}{2}\mathrm{e}^{-ik_0x} \\ k_0 &= \tfrac{i(\ell+2)!}{6M\,(\ell-1)!} \end{aligned} \quad\longrightarrow\quad \begin{aligned} W_0(x) &= \tfrac{6M\,f(r)}{\lambda(r)r^2} + ik_0 \\ V_{\rm RW}^{\rm Z} &= \pm W_{0,x} + W_0^2 + k_0^2 \end{aligned}$$

S. Chandrasekhar, Proc. Roy. Soc. Lond. A **369**, 425–433 (1980)

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- Darboux generating function as a superpotential

$$\begin{aligned} V &= W_{,x} + W^2 + \mathcal{C} \\ v &= -W_{,x} + W^2 + \mathcal{C} \end{aligned} \quad\longrightarrow\quad \begin{aligned} (\partial_x + W)(\partial_x - W)\,\psi &= -\hat{k}^2\psi \\ (\partial_x - W)(\partial_x + W)\,\phi &= -\hat{k}^2\phi \end{aligned}$$

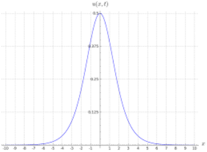
S. Chandrasekhar, Proc. Roy. Soc. Lond. A **369**, 425–433 (1980)

Preliminary Summary

- Infinite hierarchy of master equations
- Infinite allowed BH potentials, related by Darboux transformations
- Physical equivalence of the possible descriptions
- Separation into "slow" and "fast" degrees of freedom

Integrable structures in Cauchy slices

$$V_{,\sigma} - 6VV_{,x} + V_{,xxx} = 0$$



- Korteweg-de Vries deformations: isospectral symmetries of the master equation

- A triangle of integrable structures: KdV-Virasoro-Schwarzian derivative

- Conformal transformations of the master equation: Schwarzian derivative modification in the potential

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J. L. Jaramillo, M. Lenzi, and C. F. Sopuerta, Phys. Rev. D **110**, 104049 (2024)
M. Lenzi and C. F. Sopuerta, Phys. Rev. D **104**, 124068 (2021), Phys. Rev. D **107**, 044010 (2023)

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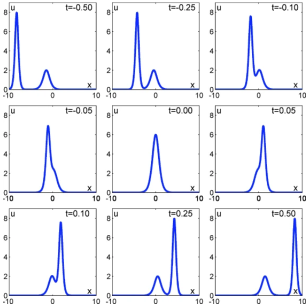
- Soliton solutions to the KdV equation



N. J. Zabusky and M. D. Kruskal, Phys. Rev. Lett. **15**, 240–243 (1965)

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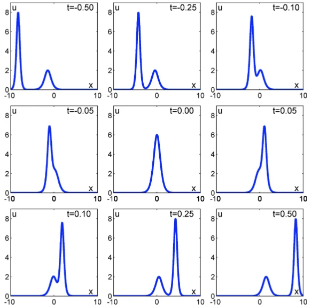

Soliton resolution conjecture

Generic global-in-time nonlinear wave dynamics decouple universally at late times into soliton solutions plus radiation.

T. Tao, Bulletin of the American Mathematical Society **46**, 1–33 (2009)

Korteweg-de Vries isospectral deformations

$$V_{,\sigma} - 6VV_{,x} + V_{,xxx} = 0$$

- Darboux transformation + inverse scattering solves the KdV equation

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Phys. Rev. Lett. **19**, 1095–1097 (1967)

Korteweg-de Vries isospectral deformations

$$V_{,\sigma} - 6VV_{,x} + V_{,xxx} = 0$$

- Darboux transformation + inverse scattering solves the KdV equation
- KdV deformations of the frequency domain master equation

$$\begin{cases} V(x) \to V(\sigma, x) \\ \psi(x) \to \psi(\sigma, x) \\ k \to k(\sigma) \end{cases}$$

M. Lenzi and C. F. Sopuerta, Phys. Rev. D **104**, 124068 (2021)

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M. Lenzi and C. F. Sopuerta, Phys. Rev. D **104**, 124068 (2021)

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- KdV equation as an integrable Hamiltonian system with infinite conserved quantities

$$\partial_\sigma V = \{V, \mathcal{H}\} \quad \longrightarrow \quad \mathcal{H}_n[V] = \int_{-\infty}^{\infty} dx \, P_n(V, V_{,x}, V_{,xx}, \dots)$$

L. D. Faddeev and V. E. Zakharov, Funct. Anal. Appl. **5**, 280–287 (1971)

Korteweg-de Vries isospectral deformations

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- Darboux transformation + inverse scattering solves the KdV equation
- KdV deformations of the frequency domain master equation

$$\begin{cases} V(x) \to V(\sigma, x) \\ \psi(x) \to \psi(\sigma, x) \\ k \to k(\sigma) \end{cases} \implies \boxed{(k^2)_{,\sigma} = 0}$$

- KdV equation as an integrable Hamiltonian system with infinite conserved quantities

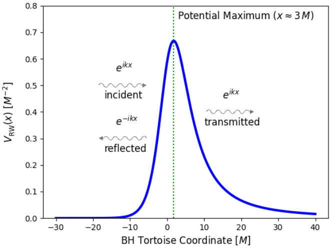
$$\partial_\sigma V = \{V, \mathcal{H}\} \quad \longrightarrow \quad \mathcal{H}_n[V] = \int_{-\infty}^{\infty} dx\, P_n(V, V_{,x}, V_{,xx}, \ldots)$$

$$\boxed{\mathcal{H}_n[V] = \mathcal{H}_n[V_{\mathrm{RW}}]}$$

M. Lenzi and C. F. Sopuerta, Phys. Rev. D **104**, 124068 (2021)

BH scattering

$$\psi(x,k,\sigma) = \begin{cases} a(k,\sigma)e^{ikx} + b(k,\sigma)e^{-ikx} \\ e^{ikx} \end{cases}$$



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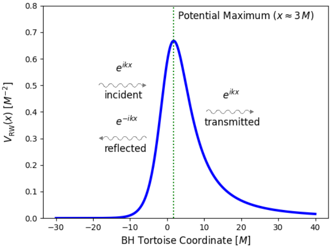

- Bogoliubov coefficients completely determine the physics (greybody factors and QNMs)
  - Greybody factors

$$T(k,\sigma) = \left|\frac{1}{a(k,\sigma)}\right|^2, \quad R(k,\sigma) = \left|\frac{b(k,\sigma)}{a(k,\sigma)}\right|^2$$

  - QNMs: k_i such that $a(k_i,\sigma) = 0$

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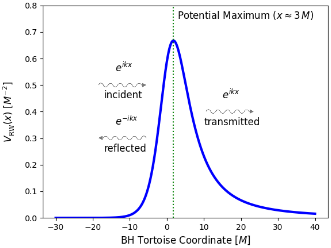

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  - QNMs: k_i such that $a(k_i,\sigma) = 0$
- Greybody factors and QNMs are conserved by DT and KdV deformations

BH greybody factors from KdV integrals

- Trace identities: a set of integral equations that relate the KdV integrals to the greybody factors

$$\ln a(k,\sigma) = \sum_{n=1}^{\infty} \frac{\mu_n}{k^n} \quad \longrightarrow \quad (-1)^{n+1}\frac{\mathcal{H}_{2n+1}}{2^{2n+1}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, k^{2n} \ln T(k)$$

L. D. Faddeev and V. E. Zakharov, Funct. Anal. Appl. **5**, 280–287 (1971)

BH greybody factors from KdV integrals

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BH moment problem

The greybody factors in BH scattering processes are uniquely determined by the KdV integrals of the BH potential via a (Hamburger) moment problem

$$\mu_{2n} = \int_{-\infty}^{\infty} dk \, k^{2n} p(k)$$

where

$$\mu_{2n} = (-1)^n \frac{\mathcal{H}_{2n+1}}{2^{2n+1}} \,, \quad p(k) = -\frac{\ln T(k)}{2\pi}$$

M. Lenzi and C. F. Sopuerta, Phys. Rev. D **107**, 044010 (2023)

Moment problem

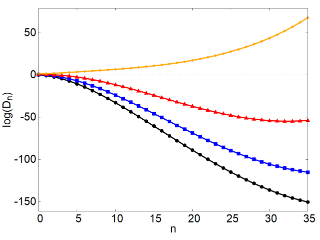
$$\mu_n = \int_{\mathcal{I}} dx\, x^n\, p(x) \quad n = 0, 1, 2, \ldots$$

- Existence: Is there a function $p(x)$ on $\mathcal{I}$ whose moments are given by $\{\mu_n\}$?
- Uniqueness: Do the moments $\{\mu_n\}$ determine uniquely a distribution $p(x)$ on $\mathcal{I}$?
- Solution: How can we construct all such probability distributions?

Moment problem: Existence and Uniqueness

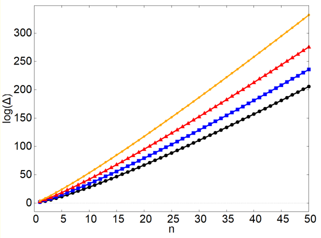
Existence

$$D_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \mu_2 & \mu_3 & \cdots & \mu_{n+2} \\ \vdots & \vdots & \cdots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} > 0$$



Uniqueness

$$\Delta(n) = C^n (2n)! - \hat{\mu}_{2n} > 0$$



Moment problem: Solution

Solution through Padé approximants

$$T(k) \quad \simeq \quad \exp\left(-2\,\pi\,\sigma\,k\,\sum_{i=1}^{L}\lambda_i e^{-t_i\,\sigma^2 k^2}\right) \quad \lambda_i = \lambda_i\left[\{\mathcal{H}_n\}\right] \quad t_i = t_i\left[\{\mathcal{H}_n\}\right]$$

1. Evaluate the first n KdV integrals
2. Obtain the moments from the KdV integrals and construct the MGF

$$M(t) = \int_0^\infty d\xi\, e^{-t\xi}\,\tilde{p}(\xi) = \sum_{n=0}^{\infty}\frac{\tilde{\mu}_n}{n!}(-t)^n$$

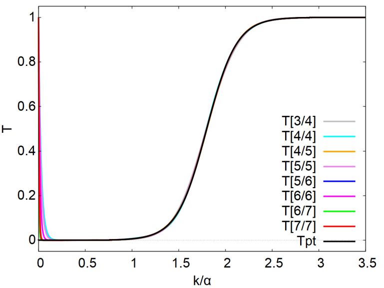
3. Construct Padé approximants of order $[K/L]$, with $K + L < n$
4. Evaluate the poles t_i and residues λ_i of the Padé approximants
5. Apply the Laplace inversion formula

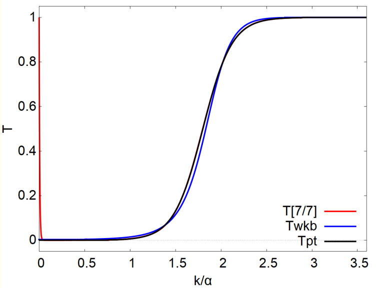
M. Lenzi and C. F. Sopuerta, Phys. Rev. D **107**, 084039 (2023)

Moment problem: Solution

Solution through Padé approximants: Pöschl-Teller

$$T(k) \quad \simeq \quad \exp\left(-2\,\pi\,\sigma\,k\,\sum_{i=1}^{L}\lambda_i e^{-t_i\,\sigma^2 k^2}\right) \qquad \lambda_i = \lambda_i\left[\{\mathcal{H}_n\}\right] \qquad t_i = t_i\left[\{\mathcal{H}_n\}\right]$$

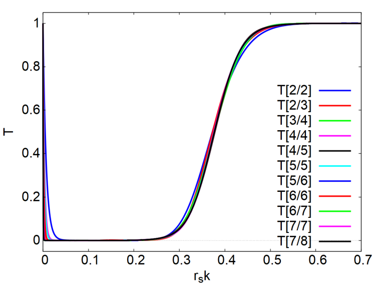


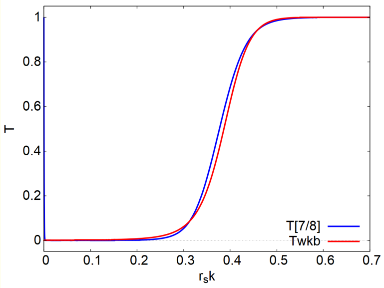


Moment problem: Solution

Solution through Padé approximants: Regge-Wheeler

$$T(k) \simeq \exp\left(-2\,\pi\,\sigma\,k \sum_{i=1}^{L} \lambda_i e^{-t_i\,\sigma^2 k^2}\right) \quad \lambda_i = \lambda_i\,[\{\mathcal{H}_n\}] \quad t_i = t_i\,[\{\mathcal{H}_n\}]$$





- Bi-Hamiltonian structure of KdV: Gardner-Zakharov-Faddeev brackets and Magri brackets

$$\partial_\sigma V = \{V, \mathcal{H}_2\}_{\mathrm{GFZ}} = \{V, \mathcal{H}_1\}_{\mathrm{M}} \ ,$$

F. Magri, J. Math. Phys. **19**, 1156–1162 (1978)

The KdV-Virasoro-Schwarzian derivative triangle

- Bi-Hamiltonian structure of KdV: Gardner-Zakharov-Faddeev brackets and Magri brackets

$$\partial_\sigma V = \{V, \mathcal{H}_2\}_{\mathrm{GFZ}} = \{V, \mathcal{H}_1\}_{\mathrm{M}} \ ,$$

- Magri brackets are the classical realization of the Virasoro algebra

$$V(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+1}} \longrightarrow \pi i \{L_n, L_m\}_{\mathrm{M}} = (n-m)L_{n+m} - \frac{n(n^2-1)}{2}\delta_{n+m,0}$$

J.-L. Gervais, Phys. Lett. B **160**, 277–278 (1985), J.-L. Gervais, Physics Letters B **160**, 279–282 (1985)

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- (BH) potentials as a CFT energy-momentum tensor:
  - Infinitesimal conformal transformation of V: $w(z) = z + \epsilon(z)$

$$\delta_\epsilon V(w) = \{V(w), F_\epsilon\}_{\mathrm{M}} \ , \quad F_\epsilon = -\frac{1}{2}\int dz\, \epsilon(z) V(z)$$

  - Finite conformal transformation of V: Schwarzian derivative

$$V(w) = \left(\frac{dw}{dz}\right)^{-2}\left[V(z) + \frac{1}{2}\mathcal{S}(w(z))\right] \ , \quad \mathcal{S}(w(z)) \equiv \frac{w_{zzz}}{w_z} - \frac{3}{2}\left(\frac{w_{zz}}{w_z}\right)^2$$

Conformal transformation of the master equation

$$\psi_{,xx} - V\psi = -k^2\psi$$

- Perform the following general transformation

$$\begin{cases} x & \mapsto \quad x = x(\tilde{x})\,, \\ \psi & \mapsto \quad \psi(x) = \omega(\tilde{x})\tilde{\psi}(\tilde{x}) \end{cases} \quad \longrightarrow a(\tilde{x})\tilde{\psi}_{,\tilde{x}\tilde{x}} + b(\tilde{x})\tilde{\psi}_{,\tilde{x}} + c(\tilde{x})\tilde{\psi} = -k^2\omega\tilde{\psi}$$

- Cancel first order derivative terms to preserve the operator structure, i.e. $b(\tilde{x}) = 0$ to obtain

$$\tilde{\psi}_{\tilde{x}\tilde{x}} + \left(k^2 x_{\tilde{x}}^2 - \tilde{V}\right)\tilde{\psi} = 0\,, \quad \tilde{V}(\tilde{x}) = \left(\frac{d\tilde{x}}{dx}\right)^{-2}\left[V(x) + \frac{1}{2}\mathcal{S}(\tilde{x}(x))\right]$$

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The Schwarzian derivative tracks the KdV (hidden) integrable structure

Hyperboloidal foliations

KdV structures in hyperboloidal slices

- Review of hyperboloidal foliations

- Covariance of the hyperboloidal slicing under general scale tranformations

- Conformal transformations of the master equation

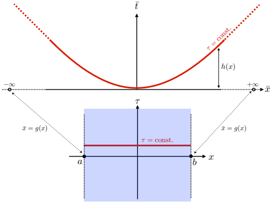
J. L. Jaramillo, M. Lenzi, and C. F. Sopuerta, Phys. Rev. D **110**, 104049 (2024)
E. Gasperin and J. L. Jaramillo, Class. Quant. Grav. **39**, 115010 (2022), J. L. Jaramillo, R. Panosso Macedo, and L. Al Sheikh, Phys. Rev. X **11**, 031003 (2021)

Hyperboloidal slicing

$$\left(-\partial_t^2 + \partial_x^2 - V_\ell\right)\phi = 0$$

- Perform the following transformation

$$(t,x) \to (\sigma,\xi) \;:\; \begin{cases} t &= \tau - h(\xi) \\ x &= g(\xi) \end{cases}$$



- With $\psi = \partial_\tau \phi$ the master equation becomes

$$\partial_\tau \begin{pmatrix} \phi \\ \psi \end{pmatrix} = i\mathbb{L} \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \quad \mathbb{L} = \frac{1}{i} \begin{pmatrix} 0 & 1 \\ \mathcal{L}_1 & \mathcal{L}_2 \end{pmatrix}$$

$$\begin{cases} \mathcal{L}_1 = \frac{1}{w(\xi)}\left[\partial_\xi\left(p(\xi)\partial_\xi\right) - q_\ell(\xi)\right] \\ \mathcal{L}_2 = \frac{1}{w(\xi)}\left[2\gamma(\xi)\partial_\xi + \partial_\xi\gamma(\xi)\right] \end{cases}$$

$$w(\xi) = \frac{1}{g'}\left(g'^2 - h'^2\right)$$

$$p(\xi) = \frac{1}{g'} \qquad \gamma(\xi) = \frac{h'}{g'}$$

$$q_\ell(\xi) = g' V_\ell$$

$$\begin{cases} \mathcal{L}_1 = \frac{1}{w(\xi)} \left[\partial_\xi \left(p(\xi)\partial_\xi\right) - q_\ell(\xi)\right] & \mathcal{L}_1 \quad \text{bulk} \\ \mathcal{L}_2 = \frac{1}{w(\xi)} \left[2\gamma(\xi)\partial_\xi + \partial_\xi\gamma(\xi)\right] & \mathcal{L}_2 \quad \text{boundary} \end{cases}$$

- Define an energy scalar product (crucial to assess QNM instability)

$$\langle\varphi_1, \varphi_2\rangle = \frac{1}{2}\int_a^b \left(w\bar{\psi}_1\psi_2 + p\partial_\xi\bar{\phi}_1\partial_\xi\phi_2 + q_\ell\bar{\phi}_1\phi_2\right) d\xi\,, \qquad \varphi = \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

- Non-selfadjointness is due to dissipation at the boundaries

$$\mathbb{L}^\dagger = \mathbb{L} + \mathbb{L}^\partial\,, \quad \mathbb{L}^\partial = \frac{1}{i}\begin{pmatrix} 0 & 1 \\ 0 & \mathcal{L}_2^\partial \end{pmatrix}, \quad \mathcal{L}_2^\partial = 2\frac{\gamma}{w}\left[\delta(\xi - a) - \delta(\xi - b)\right]$$

E. Gasperin and J. L. Jaramillo, Class. Quant. Grav. **39**, 115010 (2022), J. L. Jaramillo, R. Panosso Macedo, and L. Al Sheikh, Phys. Rev. X **11**, 031003 (2021)

Covariance under scaling transformations

Scale transformation

The hyperboloidal formulation is covariant under the following scale transformation of the wave function

$$\phi(\tau,\xi) = \Omega(\xi)\tilde{\phi}(\tau,\xi)$$

$$\partial_\tau \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \end{pmatrix} = i\tilde{\mathbb{L}} \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \end{pmatrix}, \quad \tilde{\mathbb{L}} = \frac{1}{i} \begin{pmatrix} 0 & 1 \\ \tilde{\mathcal{L}}_1 & \tilde{\mathcal{L}}_2 \end{pmatrix}$$

$$\begin{cases} \tilde{\mathcal{L}}_1 \tilde{\phi} = \frac{1}{\tilde{w}} \left[\partial_\xi \left(\tilde{p} \partial_\xi \right) - \frac{\tilde{V}}{\tilde{p}} \right] \tilde{\phi} \\ \tilde{\mathcal{L}}_2 \tilde{\psi} = \frac{1}{\tilde{w}} \left(2\tilde{\gamma}\partial_\xi + \partial_\xi \tilde{\gamma} \right) \tilde{\psi} \end{cases}$$

$$\tilde{w}(\xi) = \Omega^2(\xi) w(\xi)\,, \quad \tilde{p}(\xi) = \Omega^2(\xi) p(\xi)\,, \quad \tilde{\gamma}(\xi) = \Omega^2(\xi)\gamma(\xi)$$

$$\tilde{V} = \Omega^3 p \left[\frac{\Omega V}{p} - \partial_\xi \left(p \partial_\xi \Omega \right) \right]$$

Conformal transformations of the master equation

Fix Ω by cancelling the term containing $\partial_\xi \tilde{\phi}$:

$$\frac{\Omega'}{\Omega} = \frac{g''}{2g'} \quad \longrightarrow \quad \Omega(\xi) = \Omega_o \sqrt{g'(\xi)} = \Omega_o \, p^{-1/2}(\xi)$$

Then the hyperboloidal operators reduce to:

$$\begin{cases} \mathcal{L}_1 \tilde{\phi} = \frac{1}{g'^2 - h'^2} \left(\partial_\xi^2 - \tilde{V}_\ell \right) \tilde{\phi} \\ \mathcal{L}_2 \tilde{\psi} = \frac{1}{g'^2 - h'^2} \left\{ h' \, , \, \partial_\xi \right\} \tilde{\psi} \end{cases} \qquad \tilde{V}_\ell = g'^2 V_\ell - \frac{1}{2} \mathcal{S}(g)$$

The KdV/Virasoro/Schwarzian (hidden) integrable structure is embedded in the hyperbolidal setting

Conclusions

- “Even systems which are far from integrable may have an integrable *heart* which tells one much about their behaviour”
  N.J. Hitchin, G.B. Segal and R.S. Ward, *Integrable systems: Twistors, loop groups and Riemann surfaces*

- Hidden integrable structures in BH physics provide analytic results and abstract algebraic structures

- Interplay with asymptotic dynamics and BMS symmetries

- In GW physics:
  - QNMs
  - BH spectroscopy
  - tidal Love numbers

Perturbed Einstein equations

- Full Einstein equations

$$G_{\mu\nu} + \Lambda\, g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda\, g_{\mu\nu} = 0$$

- Perturbed Einstein equations at linear order

$$g_{\mu\nu} = \widehat{g}_{\mu\nu} + h_{\mu\nu} \quad \longrightarrow \quad G_{\mu\nu} = \widehat{G}_{\mu\nu} + \delta G_{\mu\nu}$$

  - Background

$$\widehat{G}_{\mu\nu} + \Lambda\, \widehat{g}_{\mu\nu} = \widehat{R}_{\mu\nu} - \frac{1}{2} \widehat{g}_{\mu\nu} \widehat{R} + \Lambda\, \widehat{g}_{\mu\nu} = 0$$

  - Perturbations

$$-\widehat{\Box} h_{\mu\nu} + 2 h_{(\mu}{}^{\rho}{}_{;\nu)\rho} - h_{;\mu\nu} - \widehat{g}_{\mu\nu}\, h^{\rho\tau}{}_{;\rho\tau} + \widehat{g}_{\mu\nu} \widehat{\Box} h = 2\Lambda\, h_{\mu\nu}$$

Background metric

- Schwarzschild form of the metric

$$\mathrm{d}s^2 = \widehat{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2$$

where

$$f(r) = 1 - \frac{2\,M}{r} - \frac{\Lambda}{3}r^2 \quad \longrightarrow \quad \begin{cases} \text{SchdS/SchAdS} \quad \Lambda > 0/\Lambda < 0 \\ \text{Sch} \quad \Lambda = 0 \\ \text{dS/AdS} \quad M = 0\,, \Lambda > 0/\Lambda < 0 \\ \text{Minkowksi} \quad M = \Lambda = 0 \end{cases}$$

- Metric splitting in spherical symmetry

$$\widehat{g}_{\mu\nu} = \begin{pmatrix} g_{ab} & 0 \\ 0 & r^2\Omega_{AB} \end{pmatrix} \quad \longrightarrow \quad \begin{array}{l} g_{ab}\mathrm{d}x^a\mathrm{d}x^b = -f(r)\,\mathrm{d}t^2 + \mathrm{d}r^2/f(r) \\ \Omega_{AB}\mathrm{d}\Theta^A\mathrm{d}\Theta^B = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2 \end{array}$$

Odd and even parity spherical harmonics

- Scalar harmonics

$$\Omega^{AB} Y^{\ell m}_{|AB} = -\ell(\ell+1) Y^{\ell m}$$

- Vector harmonics

$$Y^{\ell m}_{A} \equiv Y^{\ell m}_{|A} \quad \text{even parity}$$

$$X^{\ell m}_{A} \equiv -\epsilon_{A}{}^{B} Y^{\ell m}_{B} \quad \text{odd parity}$$

- Second-rank tensor harmonic

$$T^{\ell m}_{AB} \equiv Y^{\ell m}\,\Omega_{AB} \quad \text{even parity}$$

$$Y^{\ell m}_{AB} \equiv Y^{\ell m}_{|AB} + \frac{\ell(\ell+1)}{2} Y^{\ell m}\,\Omega_{AB} \quad \text{even parity}$$

$$X^{\ell m}_{AB} \equiv X^{\ell m}_{(A|B)} \quad \text{odd parity}$$

$$\text{P}: \quad (\theta,\phi) \to (\pi-\theta, \phi+\pi) \quad \Longrightarrow \quad \begin{cases} \mathcal{O}^{\ell m} \to (-1)^{\ell} \mathcal{O}^{\ell m} & \text{even parity} \\ \mathcal{O}^{\ell m} \to (-1)^{\ell+1} \mathcal{O}^{\ell m} & \text{odd parity} \end{cases}$$

Metric multipole expansion

Splitting the metric to decouple the equations for each harmonic component

$$h_{\mu\nu} = \sum_{\ell,m} h_{\mu\nu}^{\ell m,\text{odd}} + h_{\mu\nu}^{\ell m,\text{even}}$$

- Odd parity

$$h_{\mu\nu}^{\ell m,\text{odd}} = \begin{pmatrix} 0 & h_a^{\ell m} X_A^{\ell m} \\ * & h_2^{\ell m} X_{AB}^{\ell m} \end{pmatrix}$$

- Even parity

$$h_{\mu\nu}^{\ell m,\text{even}} = \begin{pmatrix} h_{ab}^{\ell m} Y^{\ell m} & \mathrm{j}_a^{\ell m} Y_A^{\ell m} \\ * & r^2 \left(K^{\ell m} T_{AB}^{\ell m} + G^{\ell m} Y_{AB}^{\ell m} \right) \end{pmatrix}$$

Harmonic decoupled Einstein equations

$$\delta G_{ab}^{\ell m}(x^c, \Theta^A) \;=\; \mathcal{E}_{ab}^{\ell m}(x^c)\, Y^{\ell m}(\Theta^A)$$

$$\delta G_{aA}^{\ell m}(x^b, \Theta^B) \;=\; \mathcal{E}_{a}^{\ell m}(x^b)\, Y_A^{\ell m}(\Theta^B) + \mathcal{O}_a^{\ell m}(x^b)\, X_A^{\ell m}(\Theta^B)$$

$$\delta G_{AB}^{\ell m}(x^a, \Theta^C) \;=\; \mathcal{E}_{T}^{\ell m}(x^a)\, T_{AB}^{\ell m}(\Theta^C) + \mathcal{E}_{Y}^{\ell m}(x^a)\, Y_{AB}^{\ell m}(\Theta^C) + \mathcal{O}_X^{\ell m}(x^a)\, X_{AB}^{\ell m}(\Theta^C)$$

Harmonic decoupled Einstein equations

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$$\delta G_{AB}^{\ell m}(x^a, \Theta^C) = \mathcal{E}_{T}^{\ell m}(x^a)\, T_{AB}^{\ell m}(\Theta^C) + \mathcal{E}_{Y}^{\ell m}(x^a)\, Y_{AB}^{\ell m}(\Theta^C) + \mathcal{O}_{X}^{\ell m}(x^a)\, X_{AB}^{\ell m}(\Theta^C)$$

Harmonic decoupled Einstein equations

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$$\delta G_{AB}^{\ell m}(x^a, \Theta^C) = \mathcal{E}_{T}^{\ell m}(x^a)\, T_{AB}^{\ell m}(\Theta^C) + \mathcal{E}_{Y}^{\ell m}(x^a)\, Y_{AB}^{\ell m}(\Theta^C) + \mathcal{O}_X^{\ell m}(x^a)\, X_{AB}^{\ell m}(\Theta^C)$$

KdV Hamiltonian structure

- Infinite hierarchy of KdV equations

$$\partial_{\sigma_k} V = \mathcal{D} \frac{\delta \mathcal{H}_{k+1}[V]}{\delta V(x)} \,, \quad k = 0, 1, 2, \ldots \quad \mathcal{D} = \frac{\partial}{\partial x}$$

- KdV hierarchy as Hamiltonian systems: $\partial_{\sigma_k} V = \{V, \mathcal{H}_{k+1}[V]\}_{\mathrm{GZF}}$

$$\{F, G\}_{\mathrm{GZF}} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \omega(x, x', V) \frac{\delta F}{\delta V(x)} \frac{\delta G}{\delta V(x')} \,, \quad \omega = \frac{1}{2} \left(\partial_x - \partial_{x'}\right) \delta(x - x')$$

- KdV hierarchy are symmetries $\{\mathcal{H}_n[V], \mathcal{H}_k[V]\} = 0$

- KdV equation possesses a second Hamiltonian structure

$$\omega(x, x', V) = \left[-\frac{1}{2} \left(\partial_x^3 - \partial_{x'}^3\right) + 2 \left(V(x)\partial_x - V(x')\partial_{x'}\right) \right] \delta(x - x')$$