Nuclei in the Cosmos
School
2025

NUCLEAR THEORY

Equation of State from nuclear ground and excited state properties

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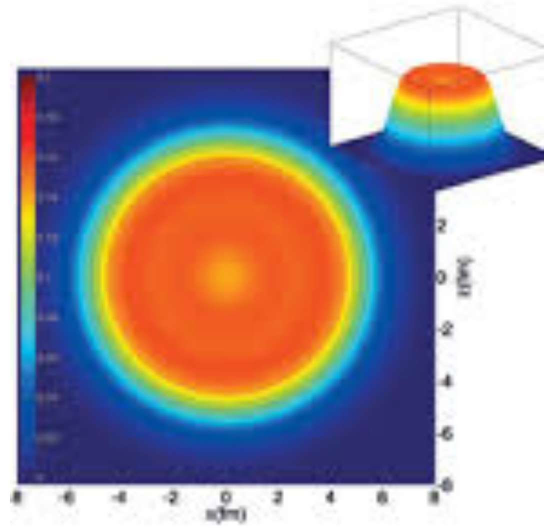
1.-Introduction and phenomenology

Where can we find neutrons and protons? And in which form? Free? In clusters?

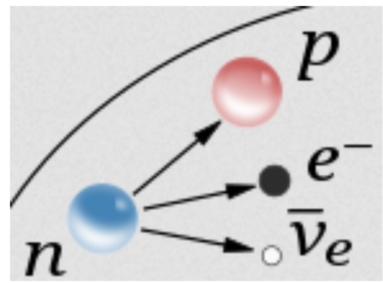


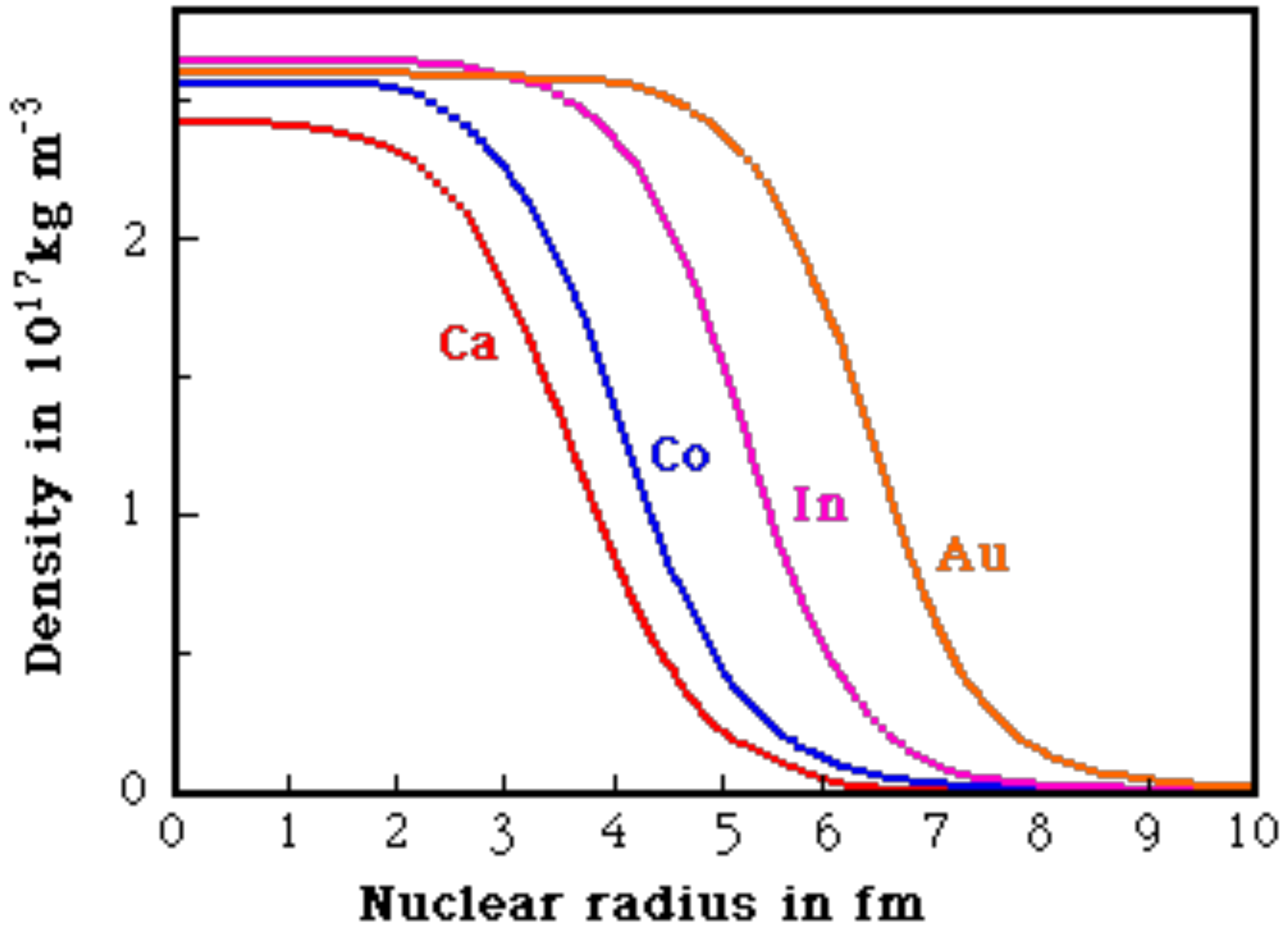
- Neutrons and protons in **Earth** are found in cluster systems: **nuclei**

  → The **interior** of all nuclei has **constant density** (10^{14} times denser than water) named **saturation density**

  → Saturation is originated from the **short range** nature of the **nuclear effective interaction**

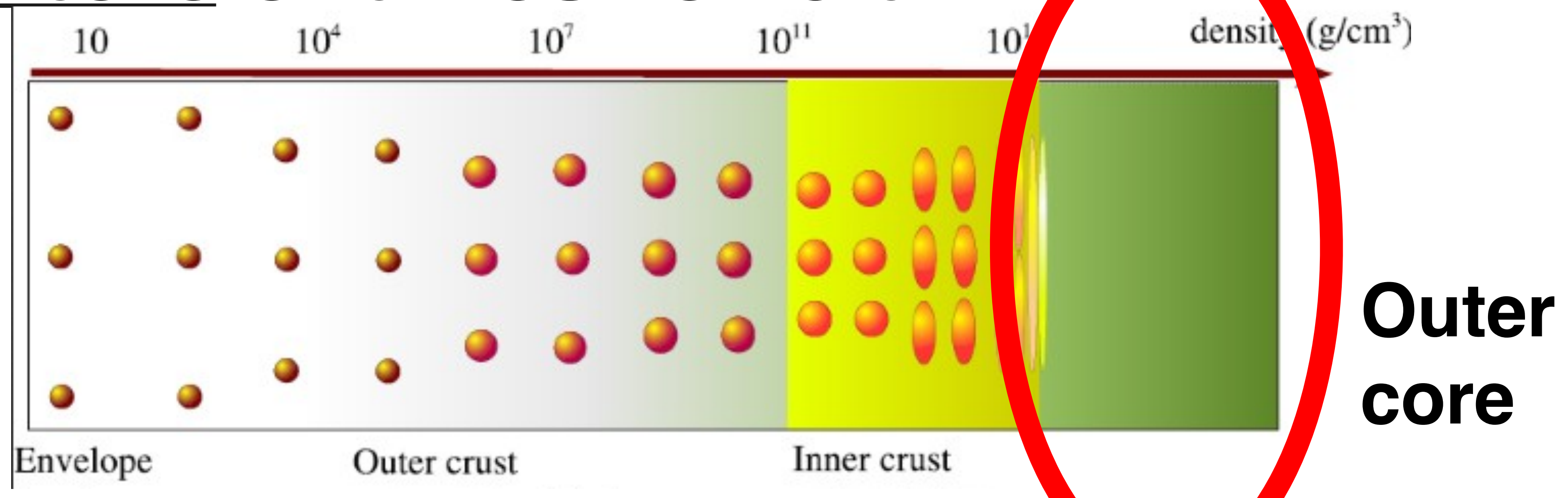
  → Neutron in 15 minutes must find a proton or …

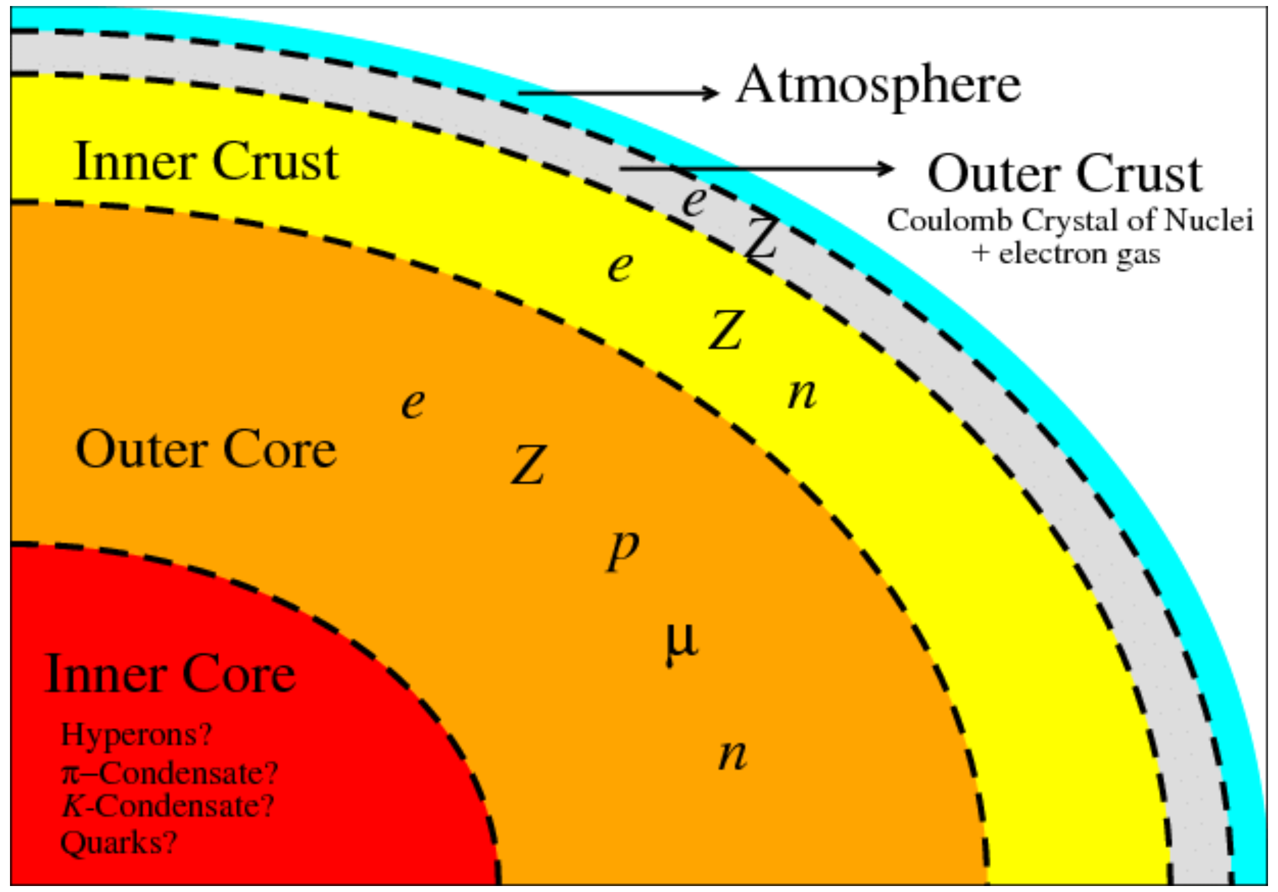




- In **heavens**, neutrons and protons can be also found as an interacting sea of fermions (Fermi liquid): **matter in the outer core of a neutron star**

  → Densities can reach several times nuclear saturation





Nuclear Equation of State (EoS)

Why???

<u>**Definition:**</u> the **energy** per **nucleon** (e=*E*/A where A=N+Z) of an **uniform system of neutrons and protons** as a function of the **neutron** (ρ_n = N/V) and **proton** (ρ_p = Z/V) **densities**, at **zero temperature, unpolarized matter**, assuming **isospin symmetry** and **neglecting Coulomb** effects among protons.

→ <u>**Zero temperature:**</u> room temperature 10^2K→10^{-8} **MeV** while "cold" neutron stars are about 10^9 K→0.1 MeV. **Separation energy** in stable nuclei (equivalent to ionization energy in atoms) is of **several MeV**.

→ <u>**Unpolarized:**</u> energy favours **couples** of neutrons and protons **occupying the same state but with opposite spins** (equivalent to electrons in atoms)

→ <u>**Isospin symmetry:**</u> neutron-neutron, proton-proton and neutron-proton **nuclear interaction** are very **similar** among them. **Masses** of neutrons and protons are almost **degenerate**. Hence neutrons and protons can be thought as **two states** of the **same particle** with different isobaric spin or **isospin** (in analogy with spin): the **nucleon.**

→ <u>**No Coulomb:**</u> **idealized uniform system (focus on strong interaction)**. Real systems are finite and frequently electrically neutral so no problems (divergences) in adding Coulomb.

Nuclear Equation of State (EoS)

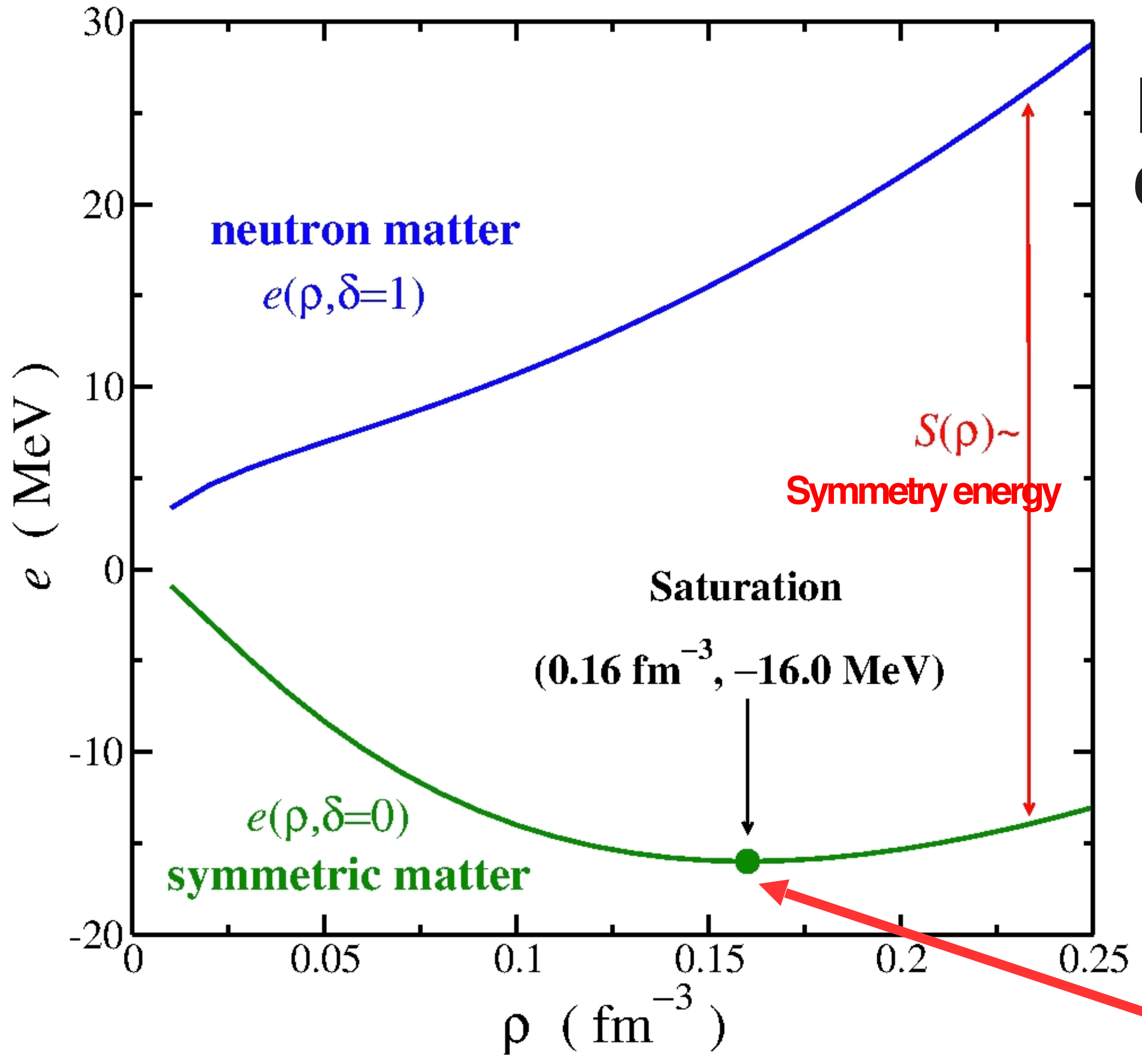
It is **convenient** to write the **energy per nucleon** (e) as a **function** of the **total density** [$\rho=\rho_n+\rho_p$] and their **relative difference** [$\delta=(\rho_n-\rho_p)/\rho$].

→ Due to isospin symmetry only even powers of δ will appear

→ Stable nuclei tend to show small values of δ

Taylor expansion for δ→ 0:

$$\boxed{e(\rho,\delta) = e(\rho,0) + S(\rho)\delta^2 + \mathcal{O}[\delta^4]}$$



It is customary to also **expand** e(ρ,0) and S(ρ) around nuclear **saturation density** $\rho_0 \sim 0.16\ \mathrm{fm}^{-3}$

$$e(\rho,0) = e(\rho_0,0) + \frac{1}{2}K_0 x^2 + \mathcal{O}[\rho^3] \text{ where } x = \frac{\rho-\rho_0}{3\rho_0}$$

$$S(\rho) = J + Lx + \frac{1}{2}K_{\mathrm{sym}}x^2 + \mathcal{O}[\rho^3,\delta^4]$$

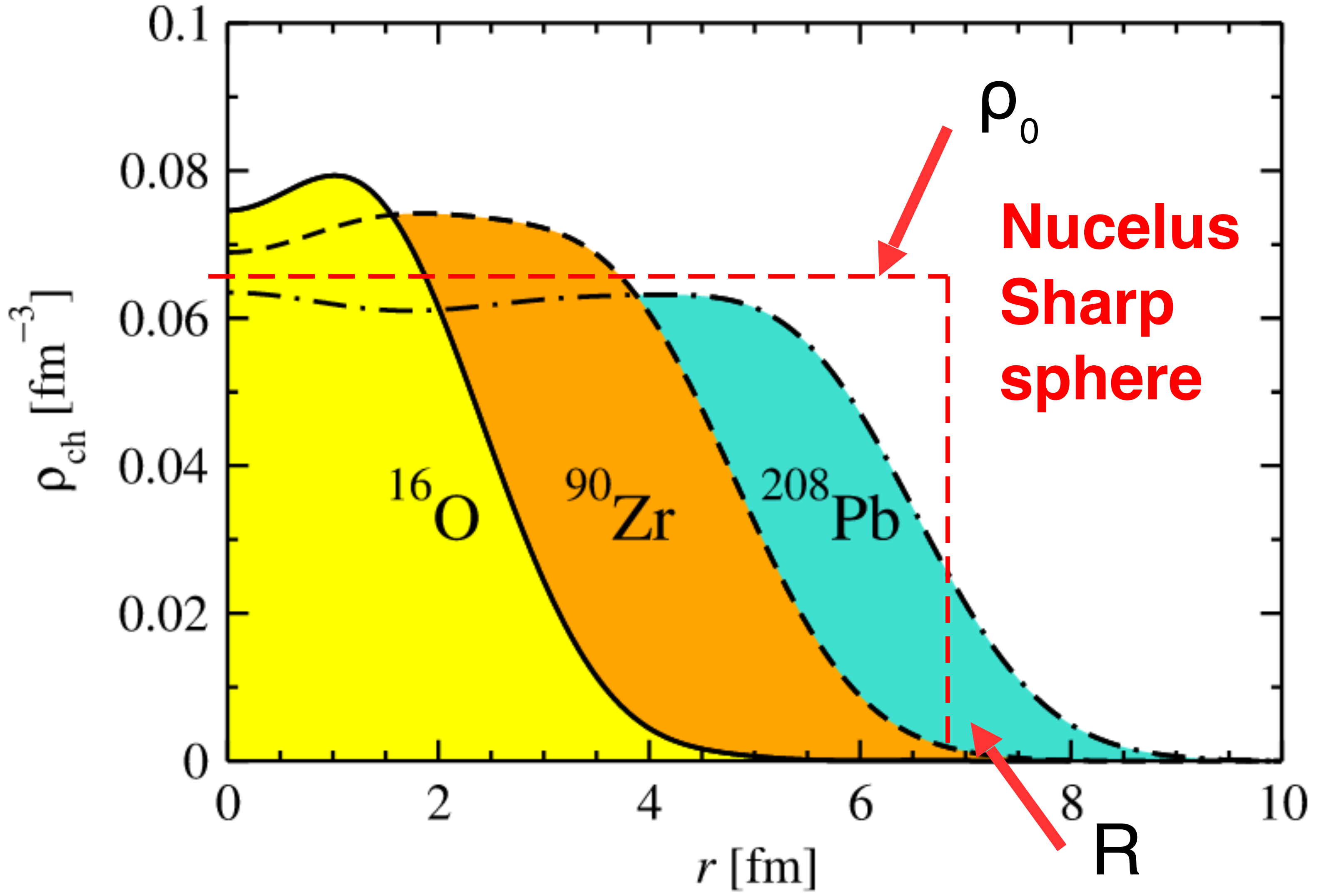
K_0 → how **compressible** is symmetric matter at ρ_0

J (a_A) → **penalty energy** for converting **protons into neutrons** for δ = 0 at ρ_0

L (a_S) → **neutron pressure** in neutron matter at ρ_0

$$P(\rho=\rho_0,\delta=0) = 0\ \ \mathrm{MeV\ fm^{-3}}$$

Saturation density $\rho_0 \approx 0.16$ fm^{-3}



→ **Range of the nuclear interaction** ($1/m_\pi \sim$ 1-2 fm) typically **shorter** than the **size** of the **nucleus.** Hence, neutrons and protons just "see" their closest neighbours.

→ **Experimental charge (Z) density** in the interior of very **different nuclei** is rather constant at around **0.06-0.08 fm^{-3}**.

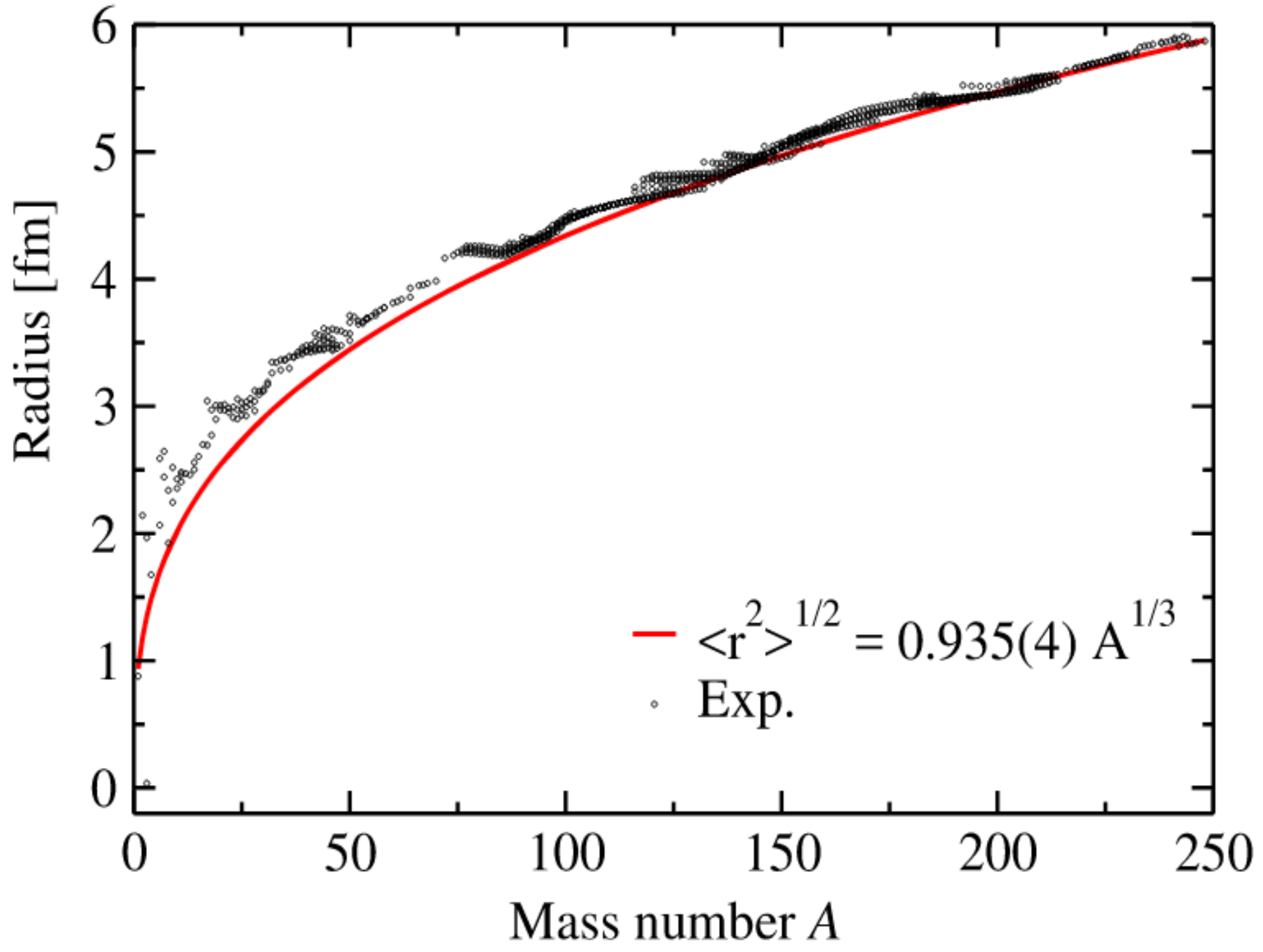
→ **Saturation mechanism** (equilibrium) that originates from the **short-range nature of the nuclear force,** much stronger than the Coulomb repulsion at the nuclear scale.

$$N + Z \equiv A = \int d\mathbf{r}\rho(\mathbf{r}) \xrightarrow{\text{sharp sphere}} A = \frac{4}{3}\pi\rho_0 R^3 \rightarrow R = \left(\frac{3}{4\pi\rho_0}\right)^{1/3} A^{1/3}$$

$$\langle r^2\rangle^{1/2} \equiv \left(\frac{1}{A}\int d^3r\, r^2\rho\right)^{1/2} \approx \sqrt{\frac{3}{5}}R = \boxed{\sqrt{\frac{3}{5}}\left(\frac{3}{4\pi\rho_0}\right)^{1/3}} A^{1/3}$$

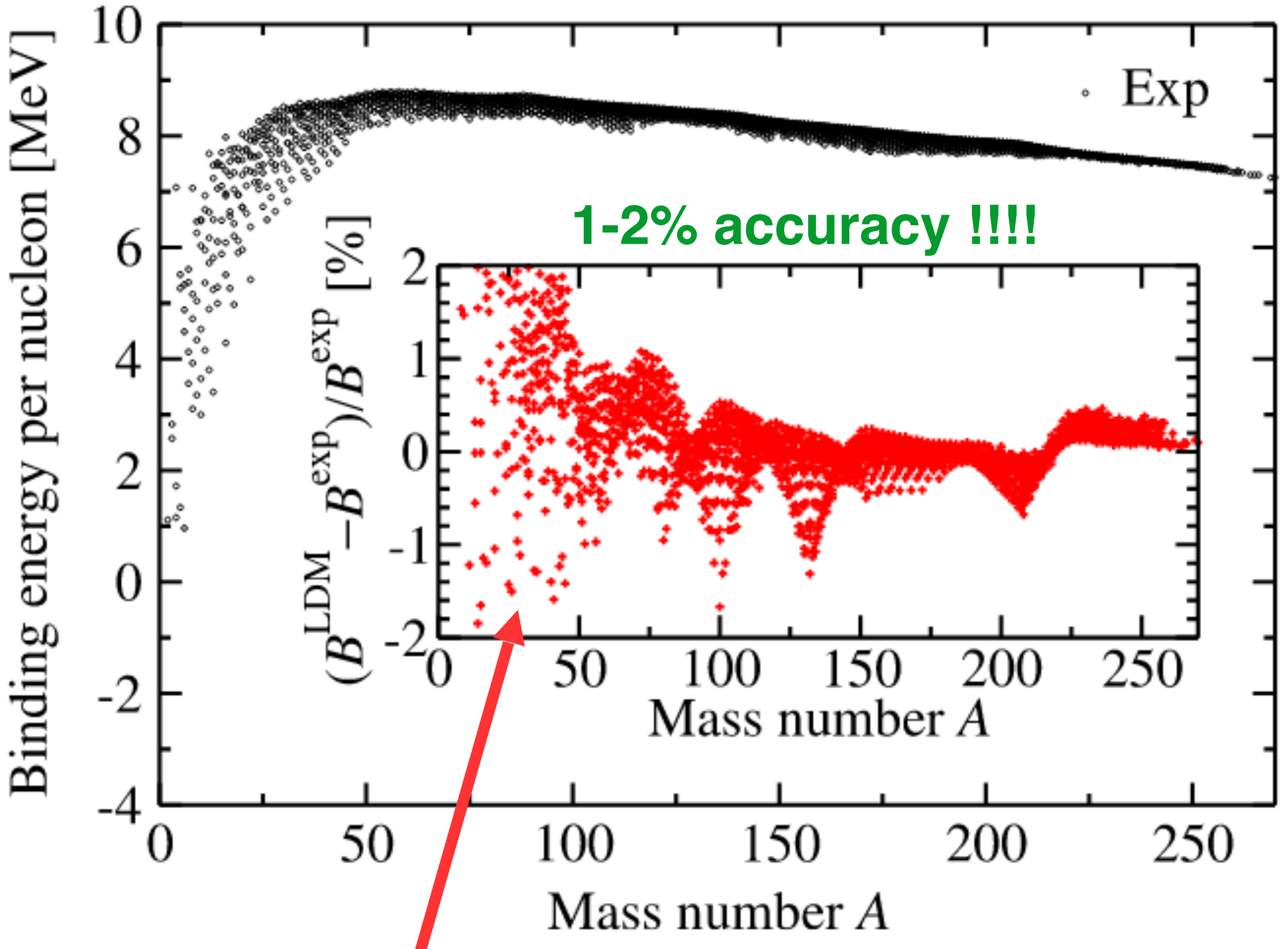
$$R \approx r_0 A^{1/3}$$

≈0.9 fm



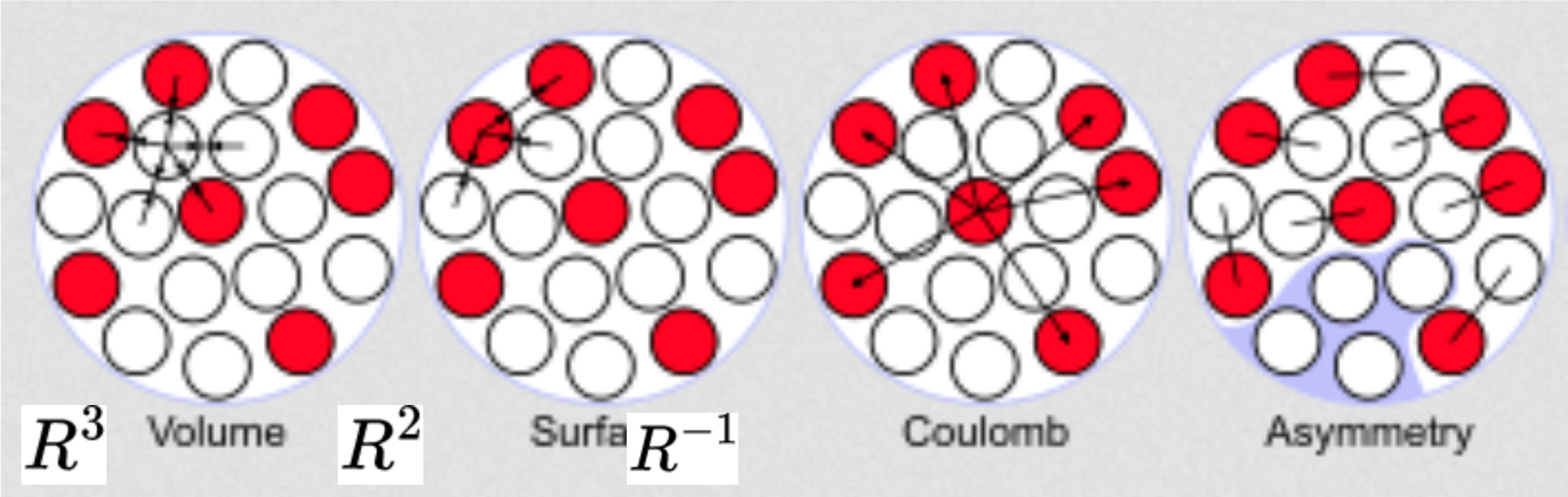
Energy at saturation density:

energy of a nucleon “far from the surface” → $a_V \approx$ 16 MeV



→ Nucleus seen as an incompressible liquid (ideal) drop: sharp sphere $R \approx r_0 A^{1/3}$

$$M(A,Z) = m_p Z + m_n(A-Z) - B(A,Z)$$



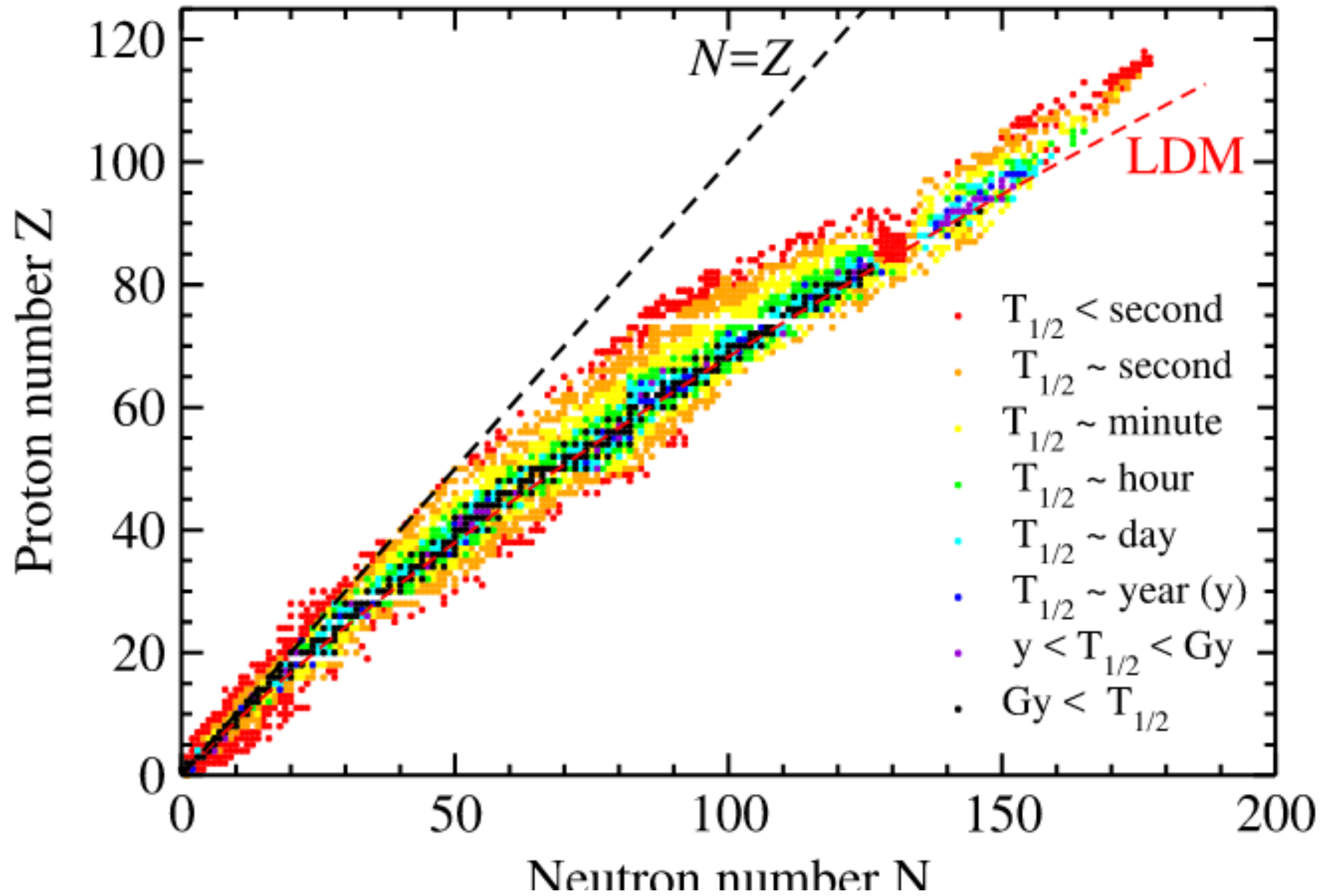
$$B(A,Z) = (a_V - a_S A^{-1/3})A - a_C \frac{Z(Z-1)}{A^{1/3}} - (a_A - a_{AS} A^{-1/3}) \frac{(A-2Z)^2}{A}$$

Arch structure in the **residuals** ↔ **shell structure** not accounted by the model, **effects** about **few %** on B

Stability of M(A,Z) with respect to **Z**

$$\frac{Z}{A} \approx \frac{1}{2} \frac{1}{1 + \frac{a_C}{4(a_A - a_{AS}A^{-1/3})} A^{2/3}}$$

Competition between Coulomb (Z→0) and asymmetry (N→Z) and surface term



Important!!

→ A small change in the saturation density will impact the size of the nucleus. Charge radii are determined to an average accuracy of 0.02 fm (Angeli 2013).

For example, if one aims at determining the **Rch = 5.5012±0.0013 fm** in ^{208}**Pb** one must be **very precise** in the determination of $\boldsymbol{\rho_0}$:

$$\boxed{\frac{\delta\rho_0}{\rho_0} = -3\frac{\delta R}{R} \quad \rightarrow \quad \frac{\delta\rho_0}{\rho_0} \lesssim 0.1\%}$$ 😱

<u>Note:</u> typical average theoretical deviation of accurate nuclear models ~ 0.02 fm → $\boldsymbol{\delta\rho_0/\rho_0}$ is determined up to about a **1% accuracy** (That is, third digit in $\rho_0 \approx 0.16$ fm^{-3}**!!**).

→ In a similar way, a **small change** in the **saturation energy** (about $e_0 \approx -16$ MeV) will **impact** on the **nuclear mass.**

For example, if one aims at determining the **B = 1636.4296±0.0012 MeV** in ^{208}**Pb** one must be **very precise** in the determination of $\mathbf{e_0}$ (changed notation!):

$$\boxed{\frac{\delta B}{B} = \frac{\delta e_0}{e_0} \rightarrow \frac{\delta e_0}{e_0} \lesssim 10^{-6}}$$ (😱)😱

<u>Note:</u> typical average theoretical deviation of accurate nuclear models ~ 1-2 MeV → $\boldsymbol{\delta e_0/\rho e_0}$ is determined up to about a **0.1% accuracy** (That is, second decimal digit in $e_0 \approx -16.0$ MeV**!!**).

Neutron and proton radii difference

essentially due to the difference between N and Z

$$\Delta r_{np} \equiv \langle r_n^2 \rangle^{1/2} - \langle r_p^2 \rangle^{1/2}$$

- Elastic electron scattering → electromagnetic size of the nucleus ↔ $\boldsymbol{\rho_p}$
- We have mostly indirect measurements on $\boldsymbol{\rho_n}$ (weakly interacting probes difficult)
- In nuclei with **different** number of **neutrons** and **protons**, we expect $\mathbf{R_n}$ could be different from $\mathbf{R_p}$:

$$\frac{\langle r_n^2 \rangle^{1/2} - \langle r_p^2 \rangle^{1/2}}{\langle r^2 \rangle^{1/2}} \approx \frac{N^{1/3} - Z^{1/3}}{A^{1/3}} \xrightarrow[I \ll 1]{I \equiv (N-Z)/A} \frac{\langle r_n^2 \rangle^{1/2} - \langle r_p^2 \rangle^{1/2}}{\langle r^2 \rangle^{1/2}} \propto \frac{N-Z}{A}$$

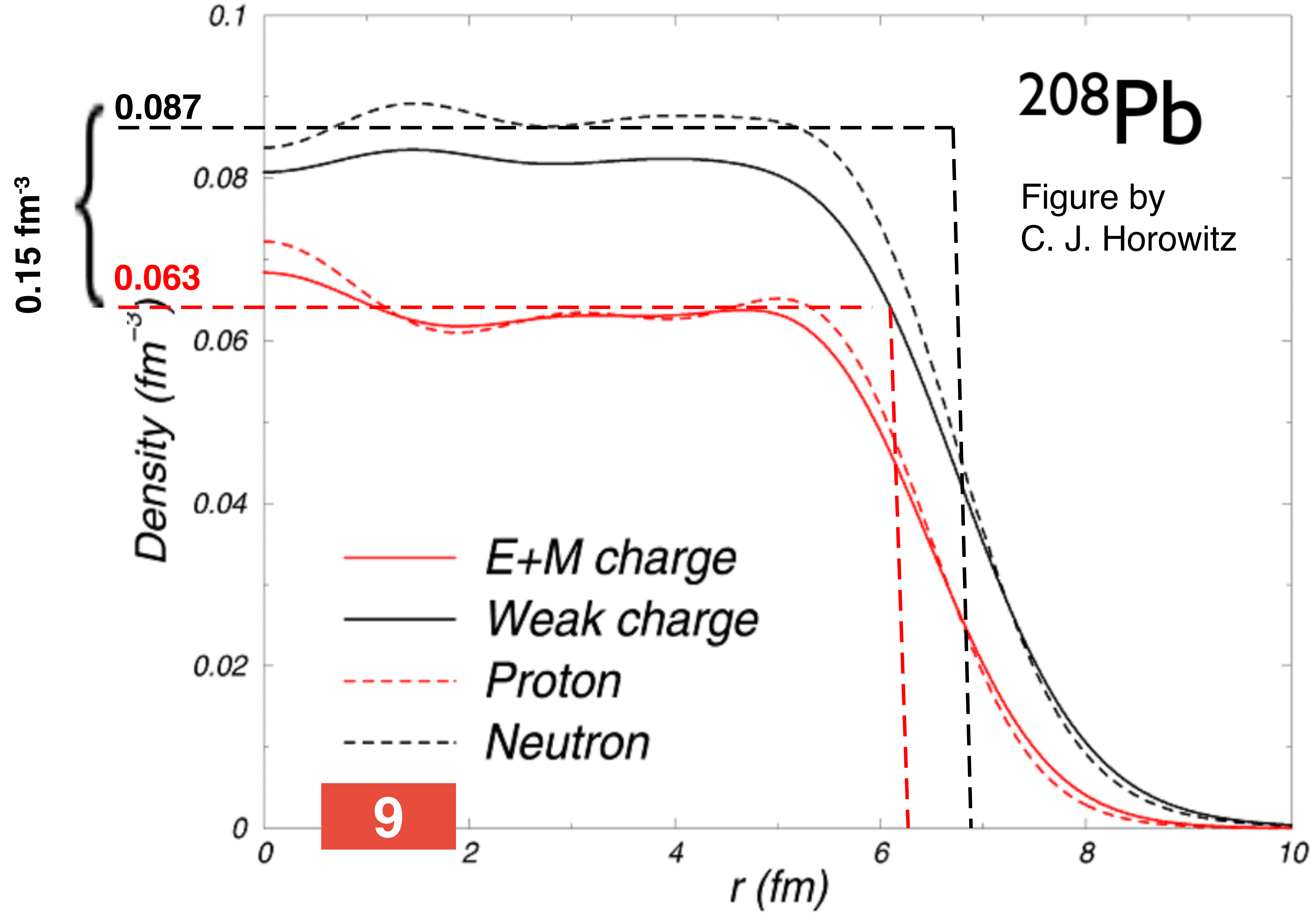
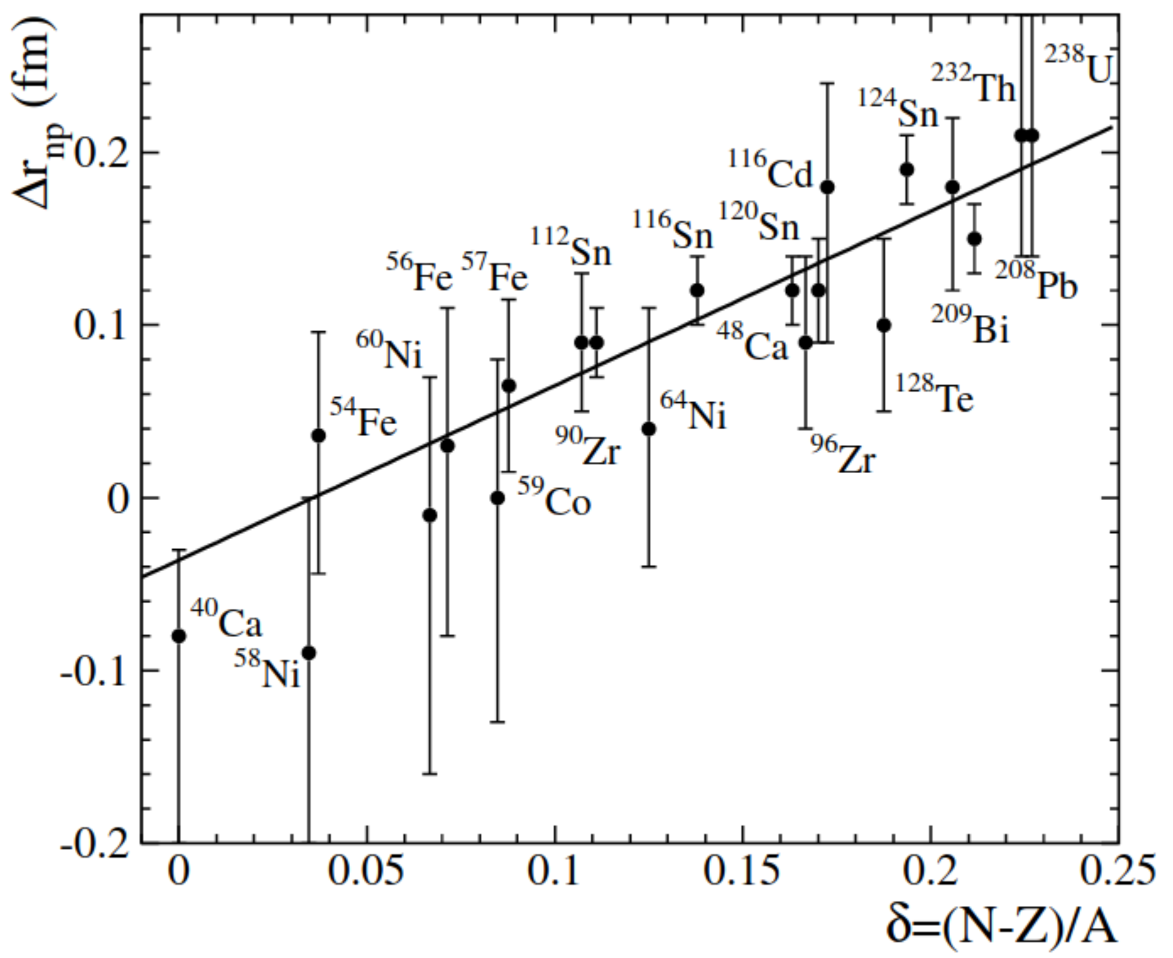


Figure by C. J. Horowitz

Coulomb repulsion → R_p larger than R_n



$$\Delta r_{np} = -0.04(3) + 1.01(15)\frac{N-Z}{A}\ \text{fm}$$

A. Trzcińska, J. Jastrzębski, P. Lubiński, F. J. Hartmann, R. Schmidt, T. von Egidy, and B. Kłos Phys. Rev. Lett. **87**, 082501 – Published 2 August 2001

Neutron skin thickness ($\Delta r_{np} := r_n - r_p$) and neutron pressure

For a fixed **(N-Z)/A**, one must **expect** that the **larger the pressure felt by nucleons, the larger the skin**

$$P = -\frac{\partial E}{\partial V}\bigg|_A = \rho^2 \frac{\partial e(\rho,\delta)}{\partial \rho}\bigg|_\delta =$$

$$\rho^2 \frac{\partial}{\partial \rho}\left[e(\rho,0) + S(\rho)\delta^2\right] =$$

$$\rho^2\delta^2 \frac{\partial S(\rho)}{\partial \rho} = \frac{1}{3}\rho\delta^2 L$$

→ **From the Droplet Model:**

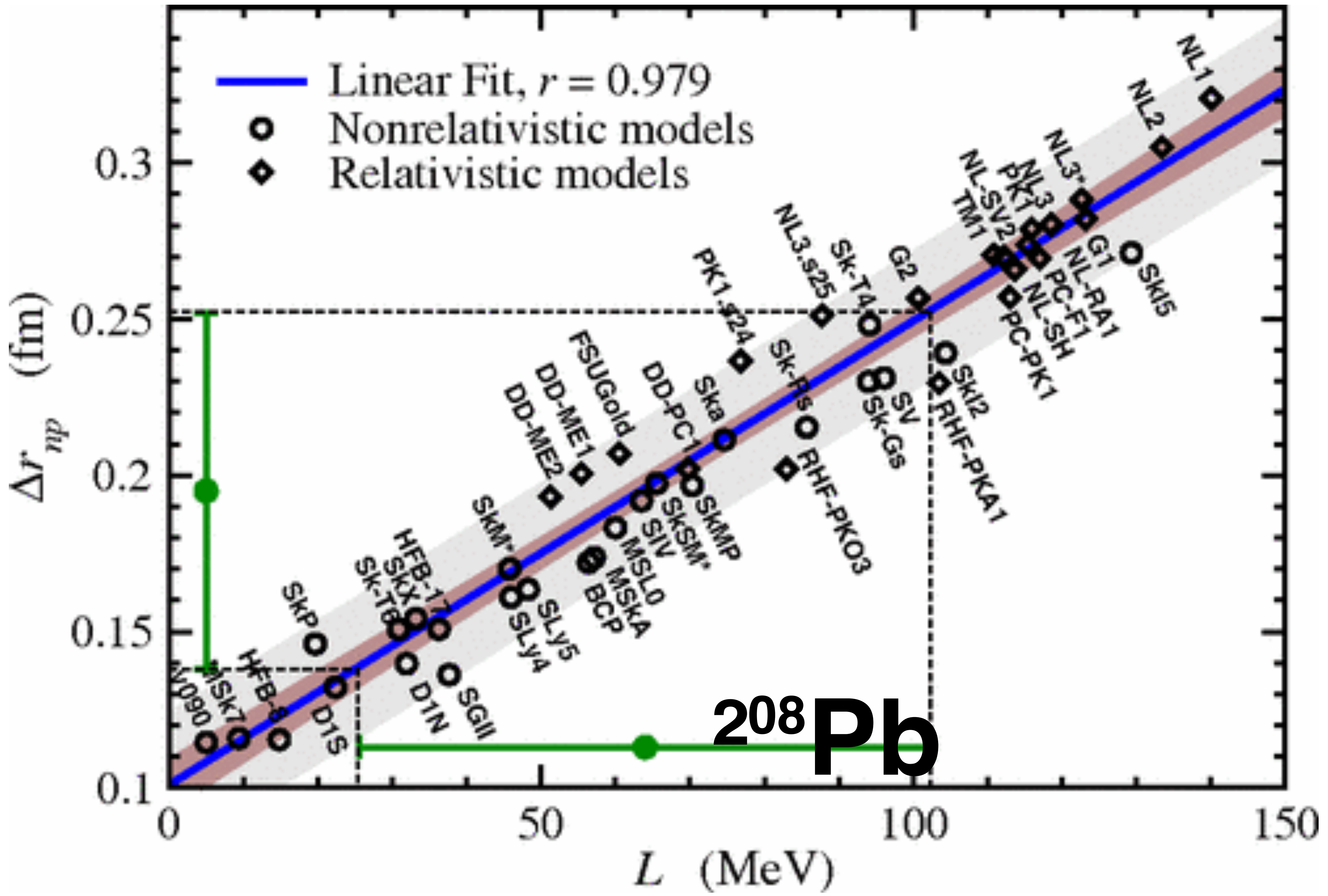
$$\Delta r_{np} \approx \frac{1}{12}\frac{N-Z}{A}\frac{R}{J}L$$

The nuclear droplet model for arbitrary shapes

W.D Myers, W.J Swiate

Annals of Physics

Volume 84, Issues 1–2, 15 May 1974, Pages 186-210



Neutron Skin of 208Pb, Nuclear Symmetry Energy, and the Parity Radius Experiment

X. Roca-Maza, M. Centelles, X. Viñas, and M. Warda Phys. Rev. Lett. 106, 252501 (2011)

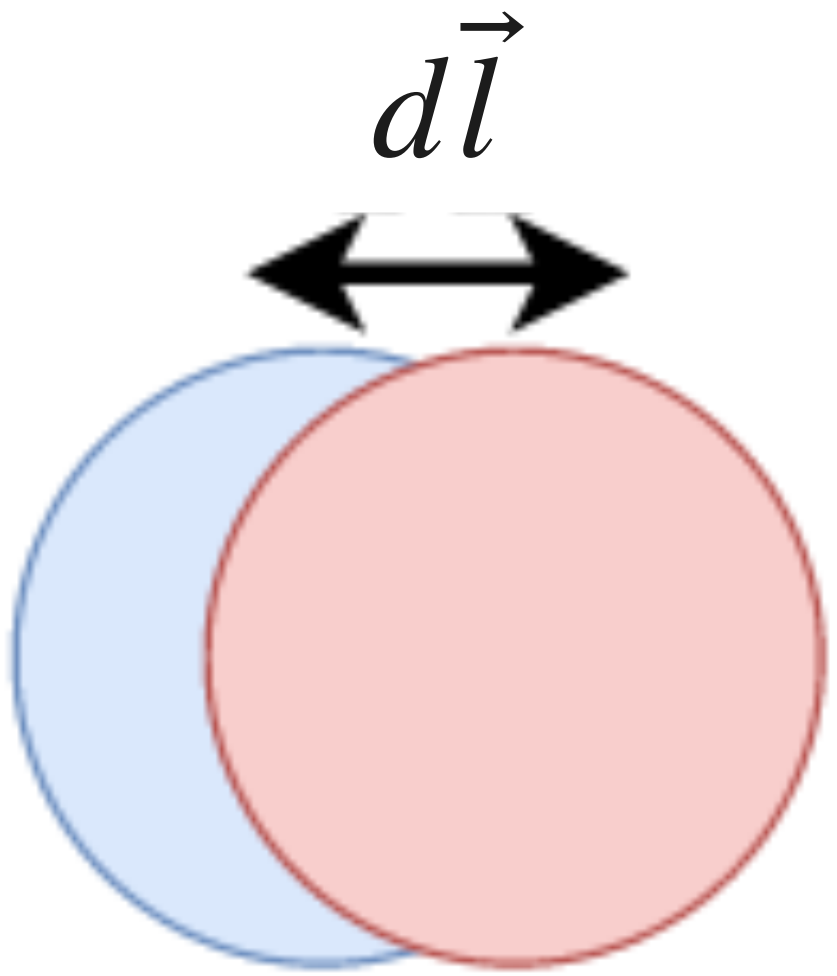
What happens if we now perturb the ground state densities?

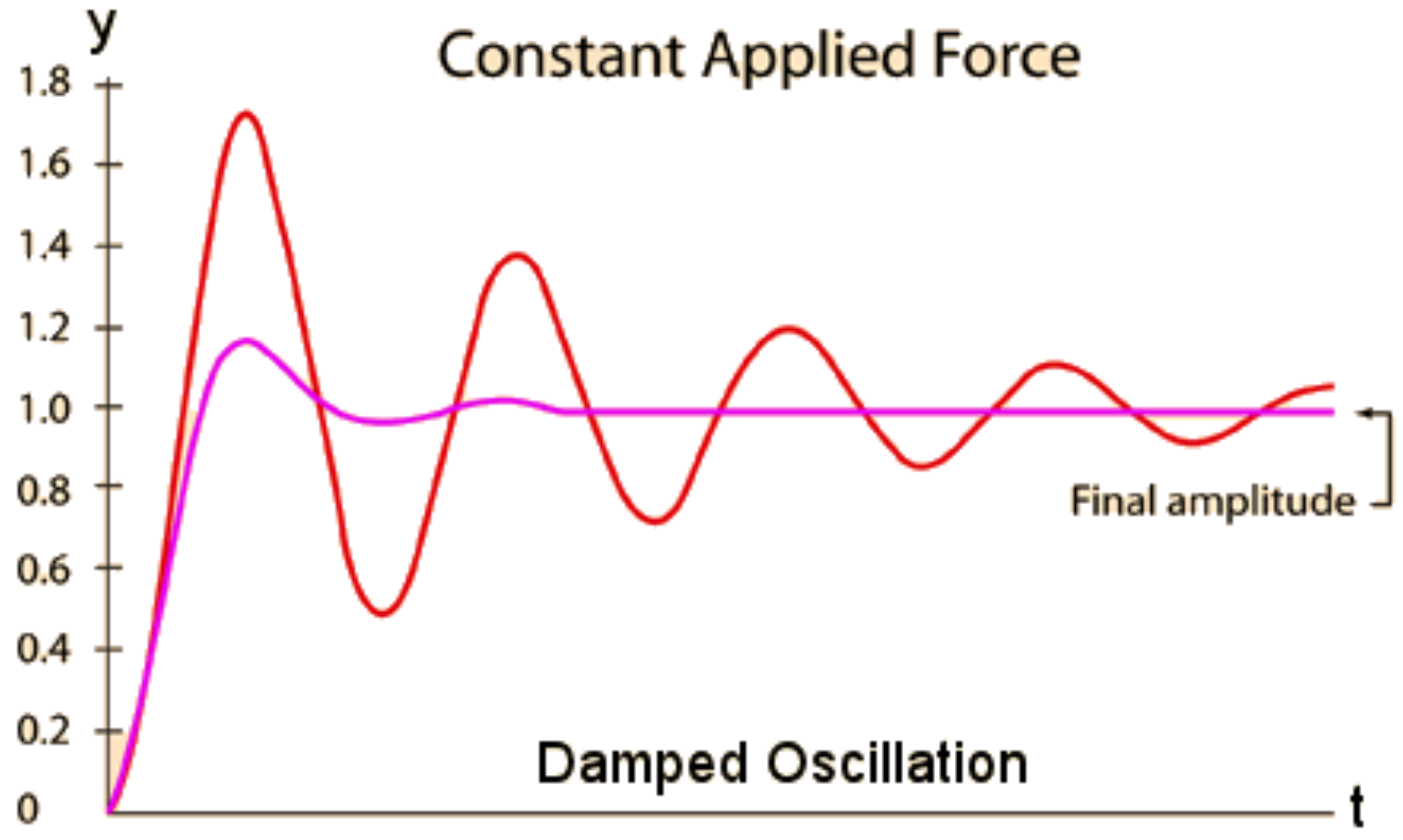
Produce a **small displacement (dl)** between **neutron and proton densities** (drops)

$$\rho = \rho_0 + \delta\rho_0 \approx$$

$$\rho_0 + d\vec{l} \cdot \vec{\nabla}\rho_0$$

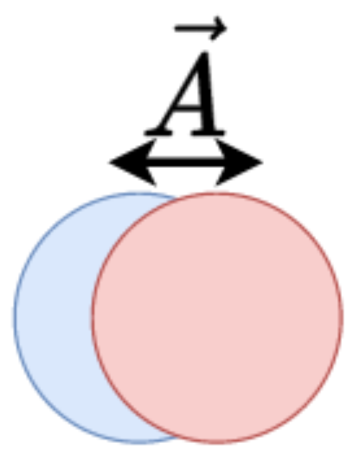
(Linear response theory)





Under different types of perturbations, **nuclei use to show resonant behaviors** where all nucleons oscillate coherently and the nucleus as a whole vibrate at an specific resonant energy → known as **Giant Resonances**

Giant resonances: the IVGDR



→ The **Isovector Giant Dipole Resonance** was the first resonance measured (**photo-absorption** exp.)

→ The **cross section** for the **excitation** of the nucleus to a **final state** $|\nu\rangle$ with energy E_ν from the **ground state** $|0\rangle$ with energy E_0 by a **photon** at a given energy E can be written as

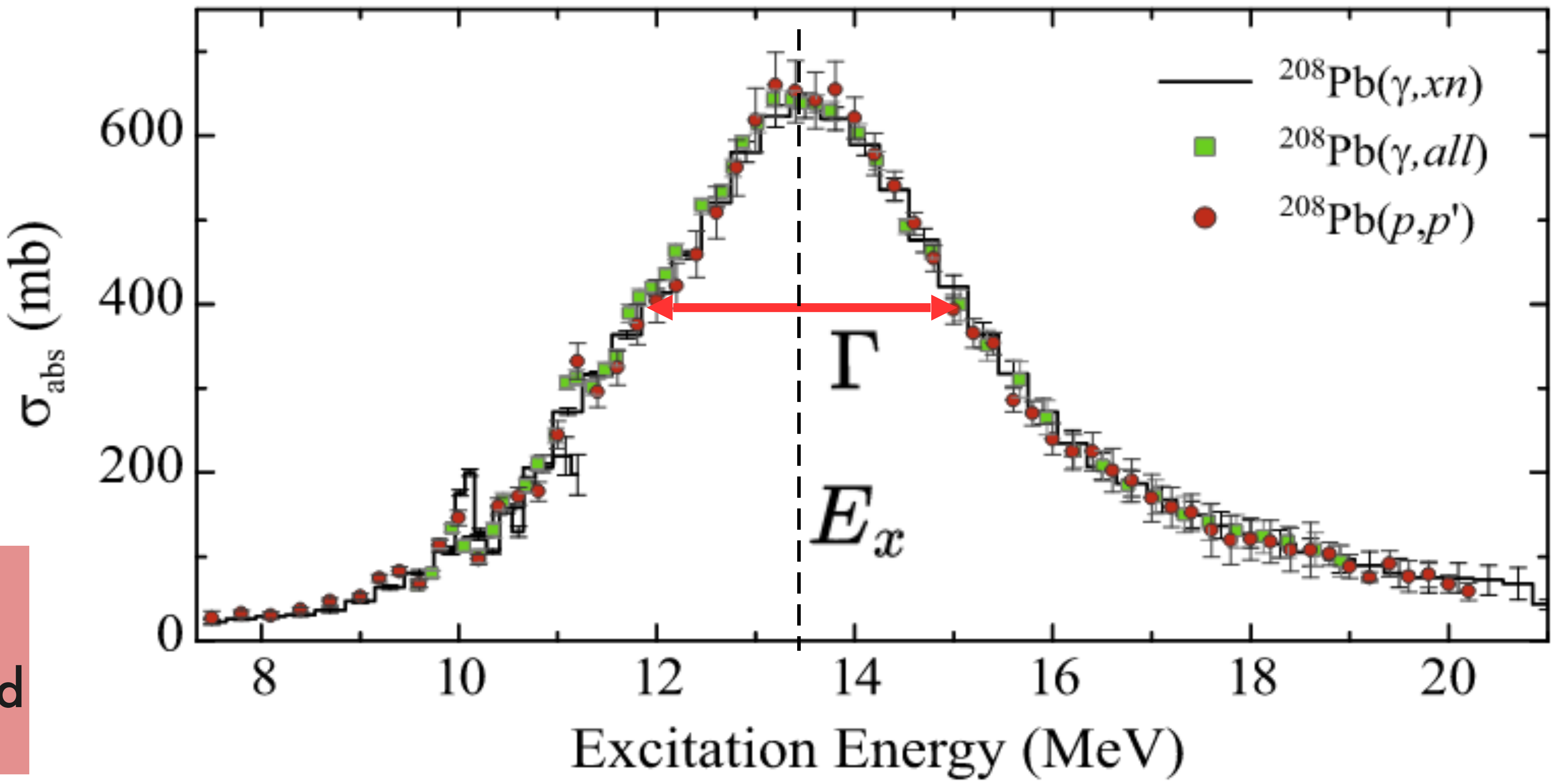
$$\sigma_\nu(E) = 4\pi^2\alpha(E_\nu - E_0)|\langle\nu|F_{\mathrm{dipole}}|0\rangle|^2\delta(E - E_\nu + E_0)$$

Convenient operator $[\sim r\,Y_{10}(\Omega)]$ produces dipole transitions and subtract CM motion

→ The **total cross-section** will be

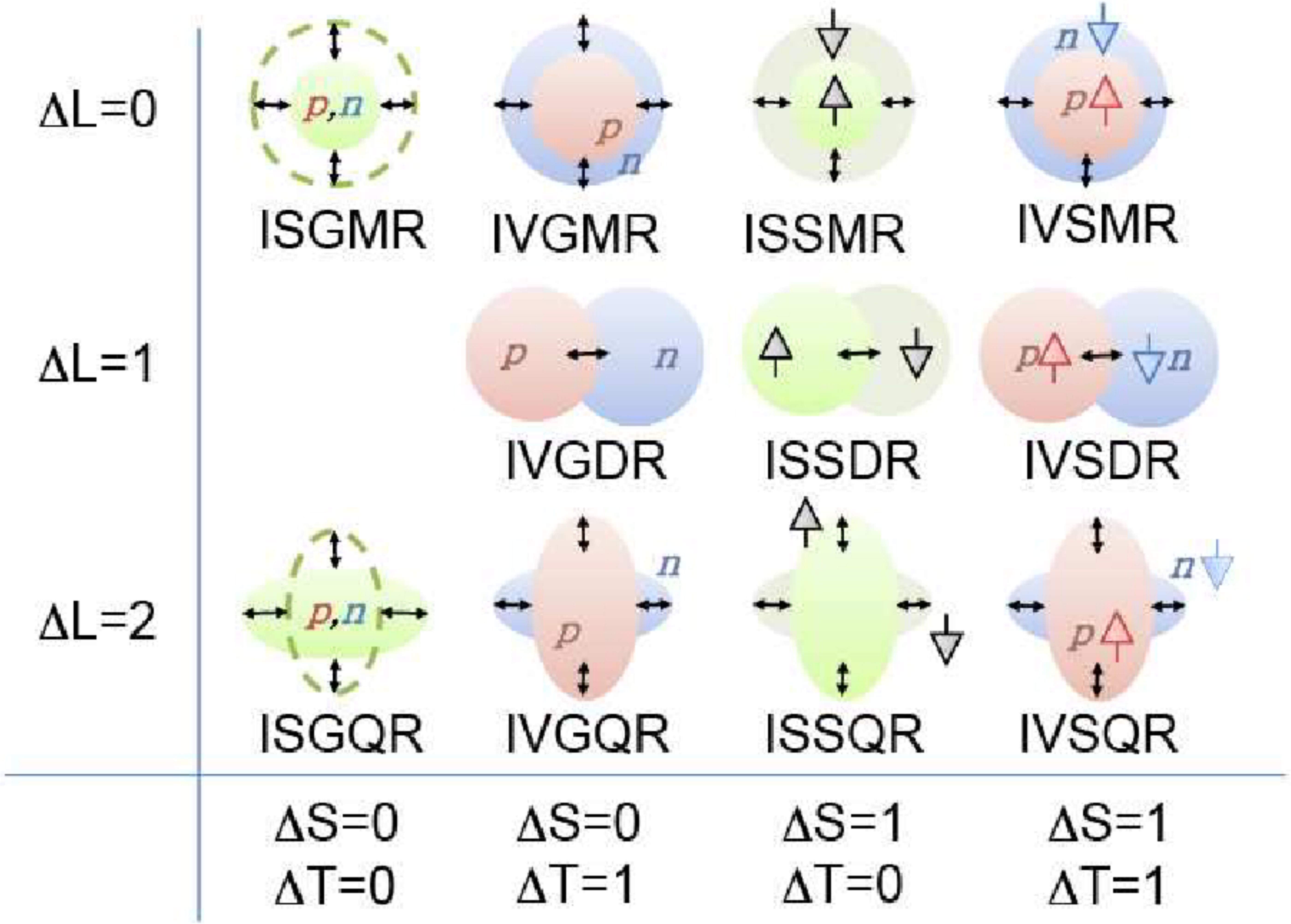
$$\sigma_{\gamma-\mathrm{abs}} = 4\pi^2\alpha\sum_\nu(E_\nu - E_0)|\langle\nu|F_{\mathrm{dipole}}|0\rangle|^2$$

$$S(E) \equiv \sum_\nu |\langle\nu|F|0\rangle|^2\delta(E - E_\nu + E_0)$$



The strength function S(E) is used to **characterize** the nuclear **response** (experimentally, it is commonly parametrized as a Lorentzian function with energy E_x and width Γ)

Giant Resonances



http://www.majimak.com/wordpress/

Giant Resonances: Harmonic oscillator

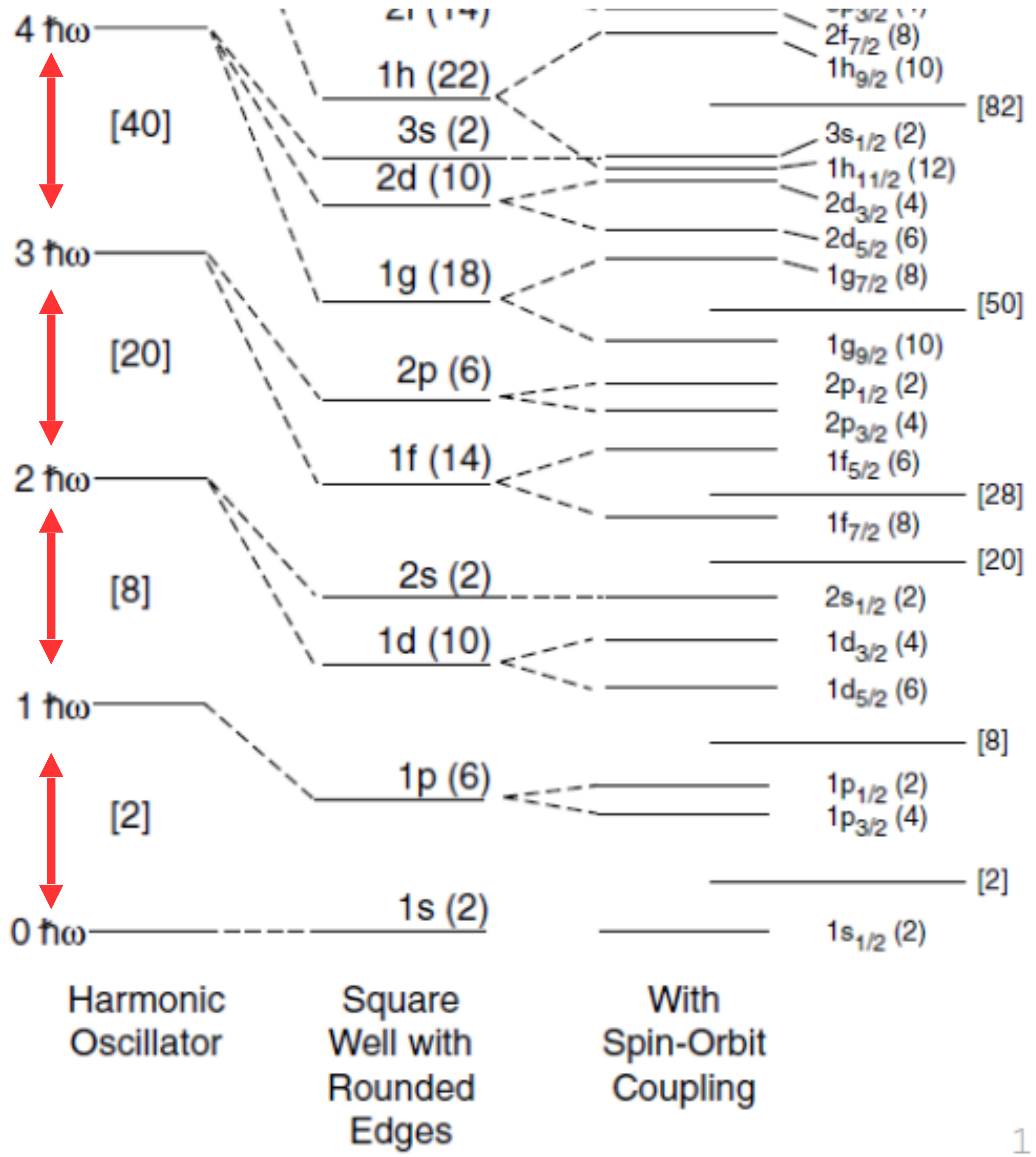
Assuming **nucleons** as **non-interacting** fermions confined in an **Harmonic Oscillator** (HO) trap with suitable **hω_0** that **preserves** the main features of the g.s. **energy spectra**

Shell gaps equal in HO

$$E \sim \frac{1}{R} \sim A^{-1/3}$$

$$\hbar\omega_0 \sim A^{-1/3}$$

$$\hbar\omega_0 \approx 41A^{-1/3}\ \mathrm{MeV}$$



Giant Resonances: Harmonic oscillator

The **HO Hamiltonian**, in terms of the conjugate variables **a** (or r) and da/dt (or v) and the ($C_0 \leftrightarrow m\omega^2$) and ($B_0 \leftrightarrow$ m) parameters, could be written as (Bohr&Mottelson):

$$\mathcal{H}_0 = \frac{1}{2}B_0\dot{\alpha}^2 + \frac{1}{2}C_0\alpha^2 \rightarrow E_0 = \hbar\left(\frac{C_0}{B_0}\right)^{1/2}$$

$$E = \left(\frac{1}{B_0}\frac{\partial^2 U}{\partial \alpha^2}\right)^{1/2}$$

Assume harmonic perturbation (restoring force)

$$F = -\kappa\alpha \rightarrow V = \frac{1}{2}\kappa\alpha^2$$

$$\mathcal{H} = \mathcal{H}_0 + V \rightarrow \mathcal{H} = \frac{1}{2}B_0\dot{\alpha}^2 + \frac{1}{2}(C_0 + \kappa)\alpha^2$$

Depending on the type of perturbation

$$E = E_0\left(1 + \frac{\kappa}{C_0}\right)^{1/2}$$

Restoring force

$$\boxed{\begin{aligned} E &\propto \left(\frac{\partial^2 e}{\partial \delta^2}\right)^{1/2} = [S(\rho)]^{1/2} \\ E &\propto \left(\frac{\partial^2 e}{\partial \rho^2}\right)^{1/2} = [K]^{1/2} \end{aligned}}$$

Sum rules:
Ground state gives access to excited state properties!!

→ **Sum Rules** or **moments** of the **strength function S(E)**

$$S(E) = \sum_\nu |\langle \nu || \hat{F}_J || \tilde{0} \rangle|^2 \delta(E - E_\nu + E_0)$$

k-moment of S(E):

$$m_k = \int dE\, E^k S(E) = \sum_\nu |\langle \nu || \hat{F}_J || \tilde{0} \rangle|^2 (E_\nu - E_0)^k$$

Example: Energy Weighted Sum Rule (EWSR)

$$m_1 = \sum_\nu (E_\nu - E_0) |\langle \nu | F | 0 \rangle|^2 = \langle 0 | F^\dagger [\mathcal{H}, F] | 0 \rangle$$

Written in terms of a **commutator** with the **Hamiltonian** evaluated in the **G.S. !!**

→ [F,V]=0, if the excitation **operator commutes** with the **interaction** the **sum rule** will be **model independent!!**

$$\hat{F}_J^{(IS)} = \sum_{i=1}^{A} f_J(r_i) Y_{JM}(\hat{r}_i) \qquad m_1^{(IS)}(J) = \frac{\hbar^2}{2m} \frac{A}{4\pi} (2J+1) \langle g_J \rangle,$$

$$\langle g_J \rangle = \frac{1}{A} \int g_J(r) \rho(r) d^3 r,$$

$$g_J(r) = \left(\frac{df_J}{dr}\right)^2 + J(J+1) \left(\frac{f_J}{r}\right)^2$$

→ [F,V] different form 0, if the excitation **operator does not commute** with the **interaction** the **sum rule** will be **model dependent** but still can be **used** to better **understand** nuclear **phenomenology**

$$\hat{F}_J^{(IV)} = \sum_{i=1}^{A} f_J(r_i) Y_{JM}(\hat{r}_i) \tau_z(i) \qquad m_1^{(IV)}(J) = m_1^{(IS)}(J)(1 + \tilde{\kappa})$$

Model dependent term

Dielectric theorem:

Inverse Energy Weighted Sum Rule m_{-1}

Ground state |0> **perturbed** by an **external field** λF (λ → 0) so that perturbation theory holds
→ The **expectation value** of the **Hamiltonian** <H> and of the **operator** <F> can be written:

$$\delta\langle\mathcal{H}\rangle = \lambda^2 \sum_{\nu\neq 0} \frac{|\langle\nu|F|0\rangle|^2}{E_\nu - E_0} + \mathcal{O}(\lambda^3) = \lambda^2 m_{-1} + \mathcal{O}(\lambda^3)$$

$$\delta\langle F\rangle = -2\lambda \sum_{\nu\neq 0} \frac{|\langle\nu|F|0\rangle|^2}{E_\nu - E_0} + \mathcal{O}(\lambda^2) = -2\lambda m_{-1} + \mathcal{O}(\lambda^2)$$

$$\boxed{m_{-1} = \frac{1}{2}\frac{\partial^2\langle\mathcal{H}\rangle}{\partial\lambda^2}\bigg|_{\lambda=0} = -\frac{1}{2}\frac{\partial\langle F\rangle}{\partial\lambda}\bigg|_{\lambda=0}} \longrightarrow \frac{1}{m_{-1}} = 2\frac{\partial^2\langle\mathcal{H}\rangle}{\partial\langle F\rangle^2}$$

Use of sum rules:
Giant Monopole Resonance

The operator leading to monopole transitions (isotropic changes in the volume if we think about a liquid drop) cannot depend on the orbital angular momentum or spin (r will produce a translation, so r^2):

$$F = \sum_{i=1}^{A} r_i^2 \quad (*)$$

Isotropic harmonic perturbation!

The m_1 and m_{-1} moments are:

$$m_1 = \frac{2\hbar^2}{m}\langle r^2\rangle \qquad \frac{1}{m_{-1}} = 2\frac{\partial^2\langle\mathcal{H}\rangle}{\partial\langle r^2\rangle^2}$$

Therefore,

$$\boxed{(E_x^{\mathrm{ISGMR}})^2} = \frac{m_1}{m_{-1}} = 4\frac{\hbar^2}{m}\langle r^2\rangle\frac{\partial^2 E}{\partial\langle r^2\rangle^2} = 4A\frac{\hbar^2}{m\langle r^2\rangle}\langle r^2\rangle^2\frac{\partial^2(E/A)}{\partial\langle r^2\rangle^2} \equiv \boxed{K_A\frac{\hbar^2}{m\langle r^2\rangle}}$$

We have now defined the **incompressibilty** of a finite nucleus and **connected** it to an **experimentally measurable quantity**. Can we say something about the EoS?

(*) other definitions could differ by a factor 4π depending if $Y_{00} = 1/\mathrm{sqrt}(4\pi)$ included or not

Use of sum rules:
Giant Monopole Resonance

Nuclear compressibilities
J.P. Blaizot
Physics Reports
Volume 64, Issue 4, September 1980, Pages 171-248

Assuming a **Liquid Drop Model** like expansion for K_A one can connect it to the bulk incompressibility K_0 (also named "leptodermus" expansion) of the **nuclear EoS**

$$K_A = K_0 + K_s A^{-1/3} + K_\tau \left(\frac{N-Z}{A}\right)^2 + K_C \frac{Z(Z-1)}{A^{4/3}} + \dots$$

Fitting to the excitation energy of the ISGMR one would obtain the coefficients of this formula. Among them K_0 (recent estimated accuracy over 10%)

This formula is **qualitative** since misses **shell effects** and **pairing** as well as terms in the **expansion** that goes as powers of **A** and **(N-Z)/A**. Very much like the LDM. Hence the estimation of K_0 would have large systematic (theoretical) errors

For the description of ^{208}Pb (E_x=13.6±0.5 MeV), K_0 must be determined at about **7%** accuracy or better

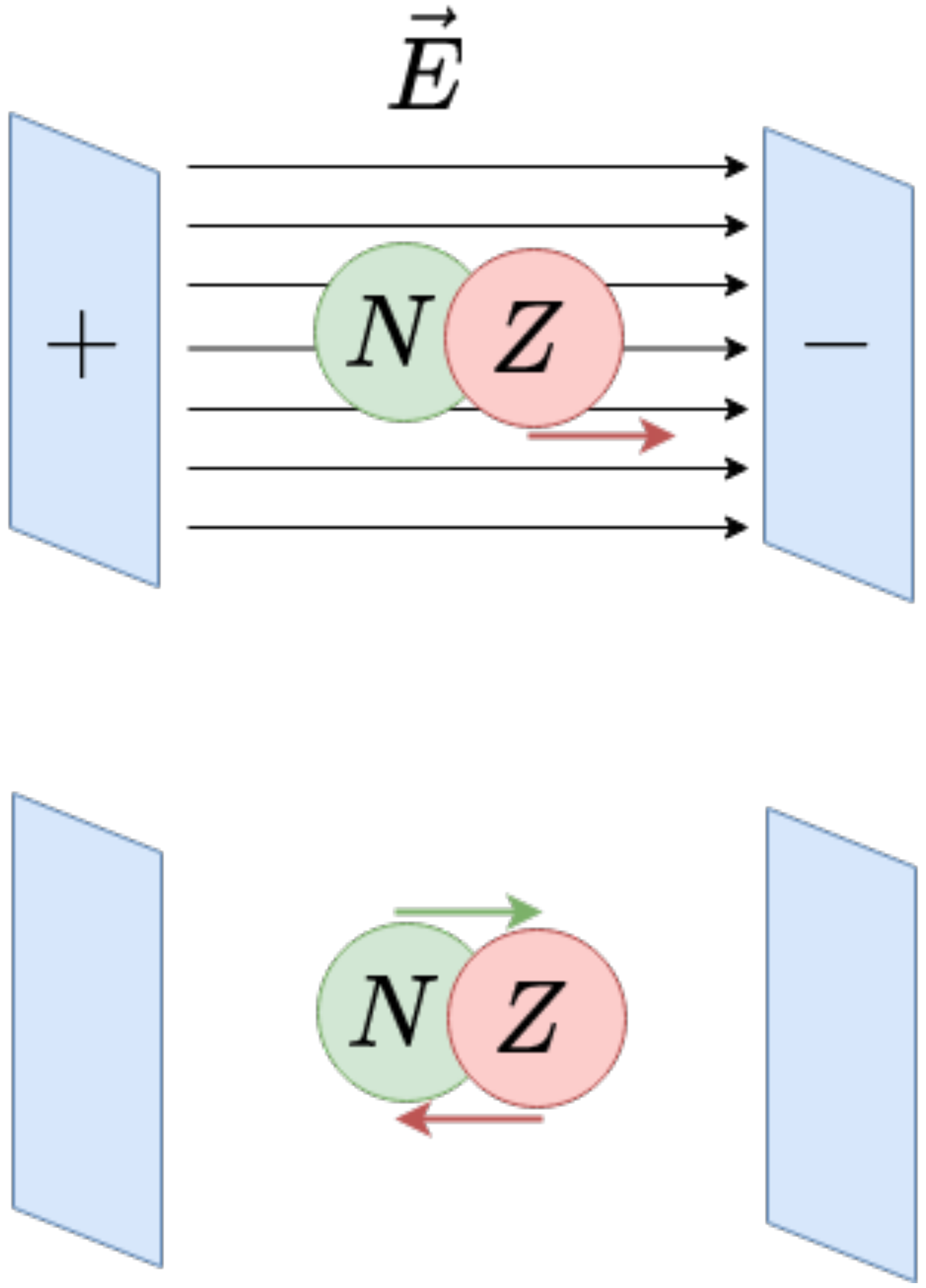
$$\frac{\delta K_0}{K_0} \sim 2\frac{\delta E_x}{E_x} \sim 7\,\%$$

Use of sum rules:

Dipole polarizability (Giant Dipole Resonance)

As in Electromagnetism course in the Physics degree, the **electric polarizability** measures the **tendency** of the nuclear **charge distribution** to be **distorted**

$$\alpha = \frac{\text{electric dipole moment}}{\text{external electric field applied}}$$



Polarizability is **proportional** to the inverse energy weighted sum rule $\mathbf{m_{-1} = \Sigma\, S(E)/E}$ (response function theory)

How easy is to separate neutrons from protons? Symmetry energy will tell. Remember the HO model?

$$e(\rho, \delta) = e(\rho, 0) + S(\rho)\delta^2$$

$$E_x \sim \sqrt{\frac{\partial^2 e(\rho,\delta)}{\partial \delta^2}} \sim \sqrt{S(\rho)}$$

Use of sum rules:
Dipole polarizability (Giant Dipole Resonance)

→ Calculate the polarizability (α), **proportional to** m_{-1} from the **dielectric theorem** and assuming the **Droplet Model** ($J \sim a_A$)

$$\alpha_D = \frac{8\pi e^2}{9} m_{-1}(E1)$$

$$m_{-1} \approx \frac{A\langle r^2\rangle^{1/2}}{48J}\left(1 + \frac{15}{4}\frac{J}{Q}A^{-1/3}\right)$$

J. Meyer, P. Quentin, and B. Jennings, Nucl. Phys. A **385**, 269

$$a_{sym}(A) = \frac{J}{1 + x_A}, \quad \text{with} \quad x_A = \frac{9J}{4Q}A^{-1/3} \qquad \Delta r_{np}^{DM} = \frac{2r_0}{3J}[J - a_{sym}(A)]A^{1/3}(I - I_C)$$

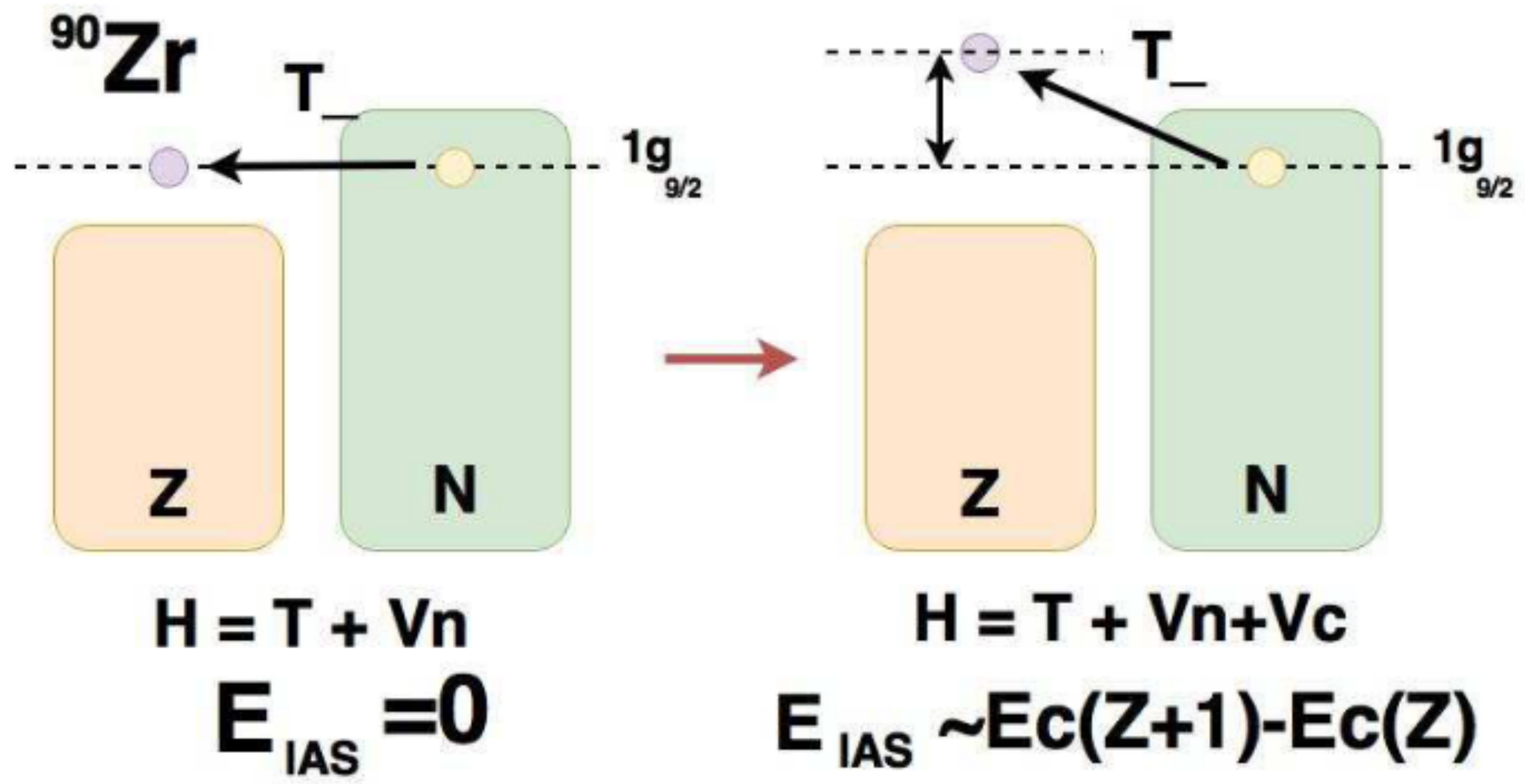
$$\alpha_D \approx \frac{A\langle r^2\rangle}{12J}\left[1 + \frac{5}{2}\frac{\Delta r_{np} + \sqrt{\frac{3}{5}}\frac{e^2 Z}{70J} - \Delta r_{np}^{surface}}{\langle r^2\rangle^{1/2}(I - I_C)}\right]$$

Polarizability must increase with the mass (for the dipole $A^{5/3}$, for the quadrupole $A^{7/3}$ and so on) and **surface symmetry energy** and **decrease** with the **bulk symmetry energy**

Use of sum rules:

Fermi or Isobaric Analog Resonance

$$F = T_{\pm} = \sum_{i}^{A} t_{\pm}(i)$$



Isospin algebra analogous to spin algebra
s→ t and τ → σ (Pauli matrices with t=τ/2)

$$t_-|n\rangle = \frac{1}{2}|p\rangle$$
$$t_+|p\rangle = -\frac{1}{2}|n\rangle$$

$$T_+^\dagger = T_- \qquad T_-^\dagger = T_+$$
$$[T_z, T_\pm] = \pm T_\pm$$
$$[T_+, T_-] = 2T_z$$

→ non-energy weighted sum rule:

$$m_0^- - m_0^+ = \langle 0|T_+T_-|0\rangle - \langle 0|T_-T_+|0\rangle$$
$$= \langle 0|[T_+, T_-]|0\rangle = \langle 0|2T_z|0\rangle$$
$$= N - Z$$

→ energy weighted sum rule:

$$m_1 = \sum_\nu (E_\nu - E_0)|\langle \nu|F|0\rangle|^2 = \langle 0|T_+[\mathcal{H}, T_-]|0\rangle$$

$[H, T\_]$ **different from zero** only if **H** contains terms that **breaks isospin symmetry**

Use of sum rules:

Fermi or Isobaric Analog Resonance

$$F = T_{\pm} = \sum_{i}^{A} t_{\pm}(i)$$

→ Hence, the **centroid energy** m_1/m_0:

$$E_{\mathrm{IAS}} = \frac{\langle 0|T_+[\mathcal{H}, T_-]|0\rangle}{\langle 0|T_+T_-|0\rangle} = \frac{1}{N-Z}\langle 0|T_+[\mathcal{H}, T_-]|0\rangle$$

→ Assuming a simple model: **indepenent particle** model with only **Coulomb breaking isospin symmetry** (neglect exchange effects)

$$E_{\mathrm{IAS}}^{\mathrm{C,direct}} = \frac{1}{N-Z}\int \left[\rho_n(\vec{r}) - \rho_p(\vec{r})\right] U_{\mathrm{C}}^{\mathrm{direct}}(\vec{r})d\vec{r}\,,$$

$$U_{\mathrm{C}}^{\mathrm{direct}}(\vec{r}) = \int \frac{e^2}{|\vec{r}_1 - \vec{r}|}\rho_{\mathrm{ch}}(\vec{r}_1)d\vec{r}_1$$

→ Assuming sharp sphere to describe ρ_n and ρ_p and $\rho ch = \rho_p$

$$U_{\mathrm{C}}^{\mathrm{direct}}(\vec{r}) = \begin{cases} \frac{Ze^2}{2R_p}\left(3 - \frac{r^2}{R_p^2}\right) & \text{for } r < R_p \\ \frac{Ze^2}{r} & \text{for } r > R_p \end{cases}$$

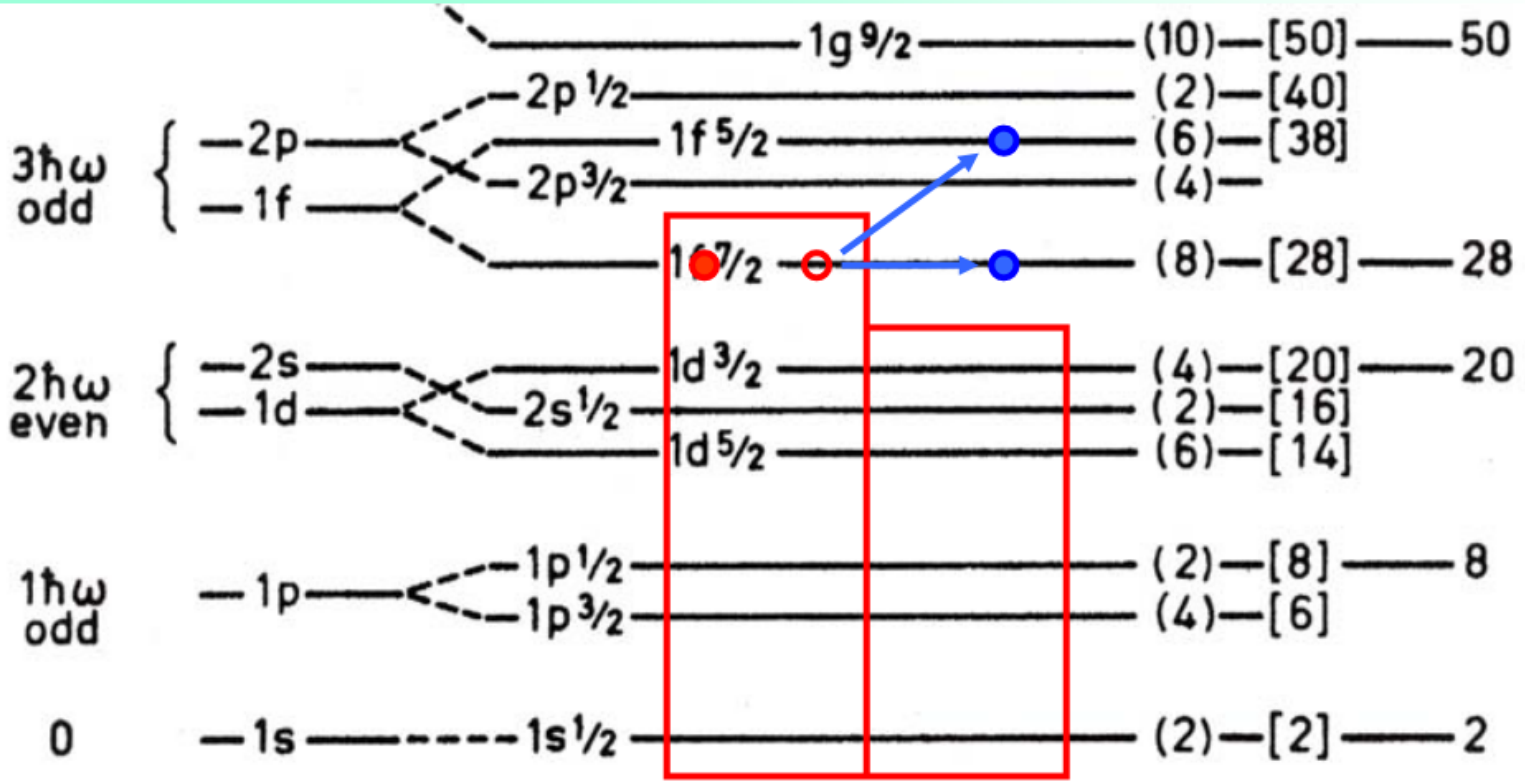
$$\begin{aligned} E_{\mathrm{IAS}} &\approx E_{\mathrm{IAS}}^{\mathrm{C,direct}} \\ &\approx \frac{6}{5}\frac{Ze^2}{R_p}\left(1 - \frac{1}{2}\frac{N}{N-Z}\frac{R_n - R_p}{R_p}\right) \\ &\approx \frac{6}{5}\frac{Ze^2}{r_0A^{1/3}}\left(1 - \sqrt{\frac{5}{12}}\frac{N}{N-Z}\frac{\Delta R_{np}}{r_0A^{1/3}}\right) \end{aligned}$$

Use of sum rules:

Gamow-Teller Resonance

$$F = O_{\pm}^{\mathrm{GT}} = \sum_{i=1}^{3}\sum_{j=1}^{A} \sigma_i(j) t_{\pm}(j)$$

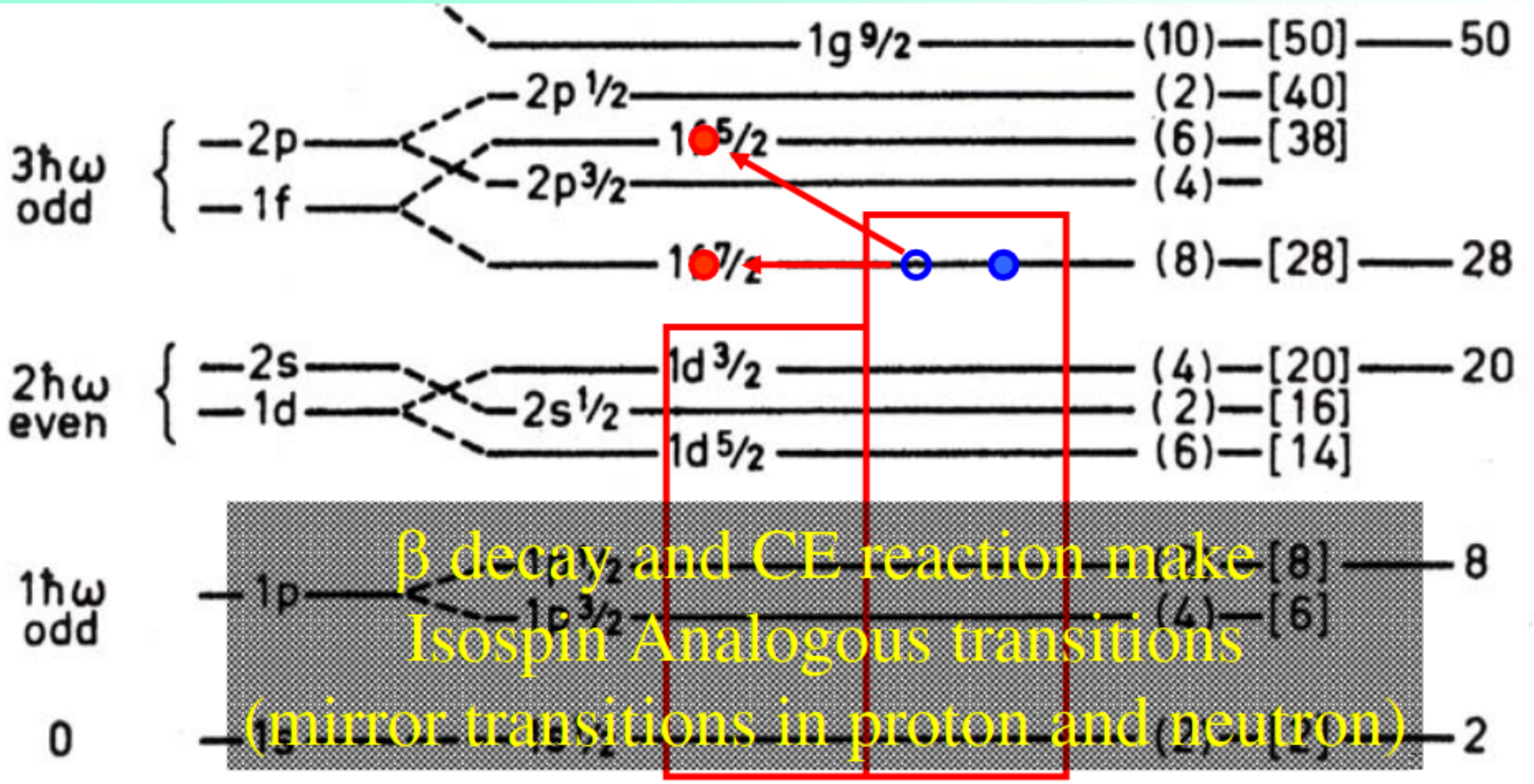
Transitions from ^{42}Ti : β decay



proton: $f_{7/2}$ → neutron $f_{7/2}$

proton: $f_{7/2}$ → neutron $f_{5/2}$

Transitions from ^{42}Ca : CE Reaction



neutron: $f_{7/2}$ → proton $f_{7/2}$

neutron: $f_{7/2}$ → proton $f_{5/2}$

Use of sum rules:

Gamow-Teller Resonance

$$F = O_{\pm}^{\mathrm{GT}} = \sum_{i=1}^{3}\sum_{j=1}^{A}\sigma_i(j)t_{\pm}(j)$$

→ Non-energy weighted sum-rule (model independent):

$$S_- - S_+ = \sum_f |\langle f|O_-^{\mathrm{GT}}|0\rangle|^2 - \sum_f |\langle f|O_+^{\mathrm{GT}}|0\rangle|^2 = \langle 0|[O_+^{\mathrm{GT}}, O_-^{\mathrm{GT}}]|0\rangle = 3(N-Z)$$

→ Centroid energy m_1/m_0 (model dependent):

spin-orbit

isospin

$$V = \sum_i^A \kappa_{ls}\boldsymbol{l}(i)\cdot\boldsymbol{s}(i) + \frac{1}{2}\frac{\kappa_\tau}{A}\sum_{i\neq j}^A \boldsymbol{\tau}(i)\cdot\boldsymbol{\tau}(j)$$

spin

$$+ \frac{1}{2}\frac{\kappa_\sigma}{A}\sum_{i\neq j}^A \boldsymbol{\sigma}(i)\cdot\boldsymbol{\sigma}(j)$$

spin-isospin

$$+ \frac{1}{2}\frac{\kappa_{\sigma\tau}}{A}\sum_{i\neq j}^A (\boldsymbol{\sigma}(i)\cdot\boldsymbol{\sigma}(j))(\boldsymbol{\tau}(i)\cdot\boldsymbol{\tau}(j)),$$

$$E_{\mathrm{GT}} - E_{\mathrm{IAS}} = \frac{\langle 0|[O_+,[V,O_-]]|0\rangle}{(N-Z)}$$

$$= -\frac{4}{3}\frac{\kappa_{ls}}{N-Z}\langle 0|\sum_i^A \boldsymbol{l}(i)\cdot\boldsymbol{s}(i)|0\rangle + 2(\kappa_{\sigma\tau}-\kappa_\tau)\frac{N-Z}{A}.$$

Use of sum rules:

Spin Dipole Resonance

$$\sum_{i=1}^{A} \sum_{M} t_{\pm}(i) r_i^L \left[Y_L(\hat{r}_i) \otimes \boldsymbol{\sigma}(i)\right]_{JM}$$

$$m_0(t_-) - m_0(t_+) = \langle 0|O_-^{\rm SD}|0\rangle - \langle 0|O_+^{\rm SD}|0\rangle$$

→ Non-energy weighted sum-rule (model independent):

$$= \langle 0|[O_-^{\rm SD}, O_+^{\rm SD}]|0\rangle = \frac{9}{4\pi}\sum_{i=1}^{A}\langle 0|r_i^2[t_-(i), t_+(i)]|0\rangle$$

$$= 2\frac{9}{4\pi}\sum_{i=1}^{A}\langle 0|r_i^2 t_z(i)|0\rangle = \boxed{\frac{9}{4\pi}\left(N\langle r_n^2\rangle - Z\langle r_p^2\rangle\right)}$$

→ Rewritting it in terms of the neutron skin thickness: $\Delta r_{np} = \langle r_n^2\rangle^{1/2} - \langle r_p^2\rangle^{1/2}$

A model independent sum rule that gives information on the skin!!

$$m_0(t_-) - m_0(t_+) = \frac{9}{4\pi}(N-Z)\langle r_p^2\rangle\left[1 + \frac{2N}{N-Z}\frac{\Delta r_{np}}{\langle r_p^2\rangle^{1/2}} + \frac{N}{N-Z}\left(\frac{\Delta r_{np}}{\langle r_p^2\rangle^{1/2}}\right)^2\right]$$

$$\approx \frac{9}{4\pi}(N-Z)\langle r_p^2\rangle\left(1 + \frac{2N}{N-Z}\frac{\Delta r_{np}}{\langle r_p^2\rangle^{1/2}}\right)$$