Nuclei in the Cosmos School 2025

# NUCLEAR THEORY

## Equation of State from nuclear ground and excited state properties

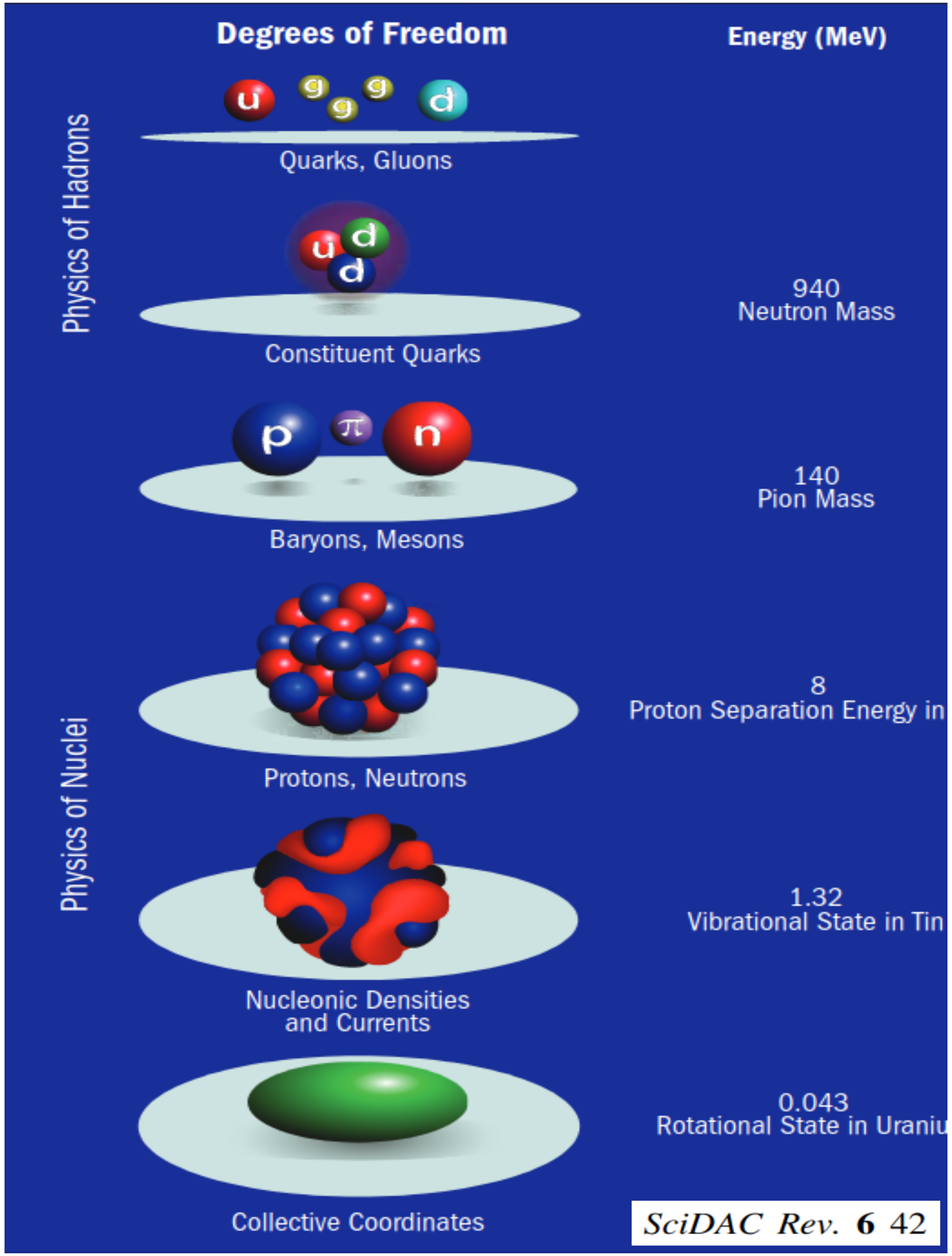
Xavier Roca-Maza
ICCUB Barcelona, Spain

NASA/Space Telescope Science Institute

# 2.-Theoretical models, associated errors and correlations

- How are we dealing with the nuclear many-body problem?

- → Ab inito methods
- → Density Functional Theory
- → …
-



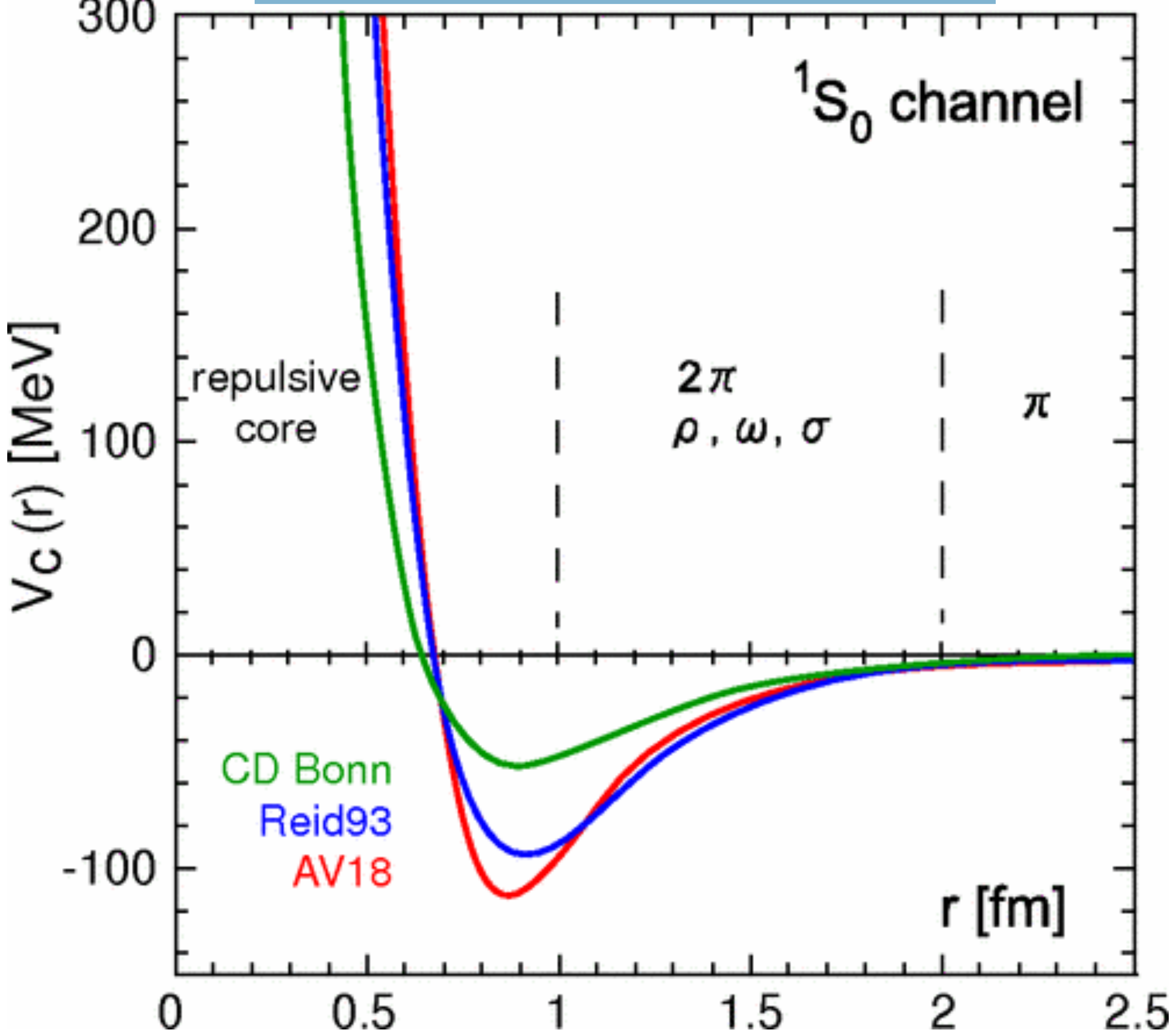
# Nuclear Many-Body Problem:

## Nuclear interaction

Underlying interaction: the "so called" **residual strong interaction** = **nuclear force** has **not** been **derived yet** (with the precision needed) from first principles as **QCD** is **non-perturbative** at the **low-energies** (~ below $m_\pi$≈140 MeV) relevant for the description of nuclei.

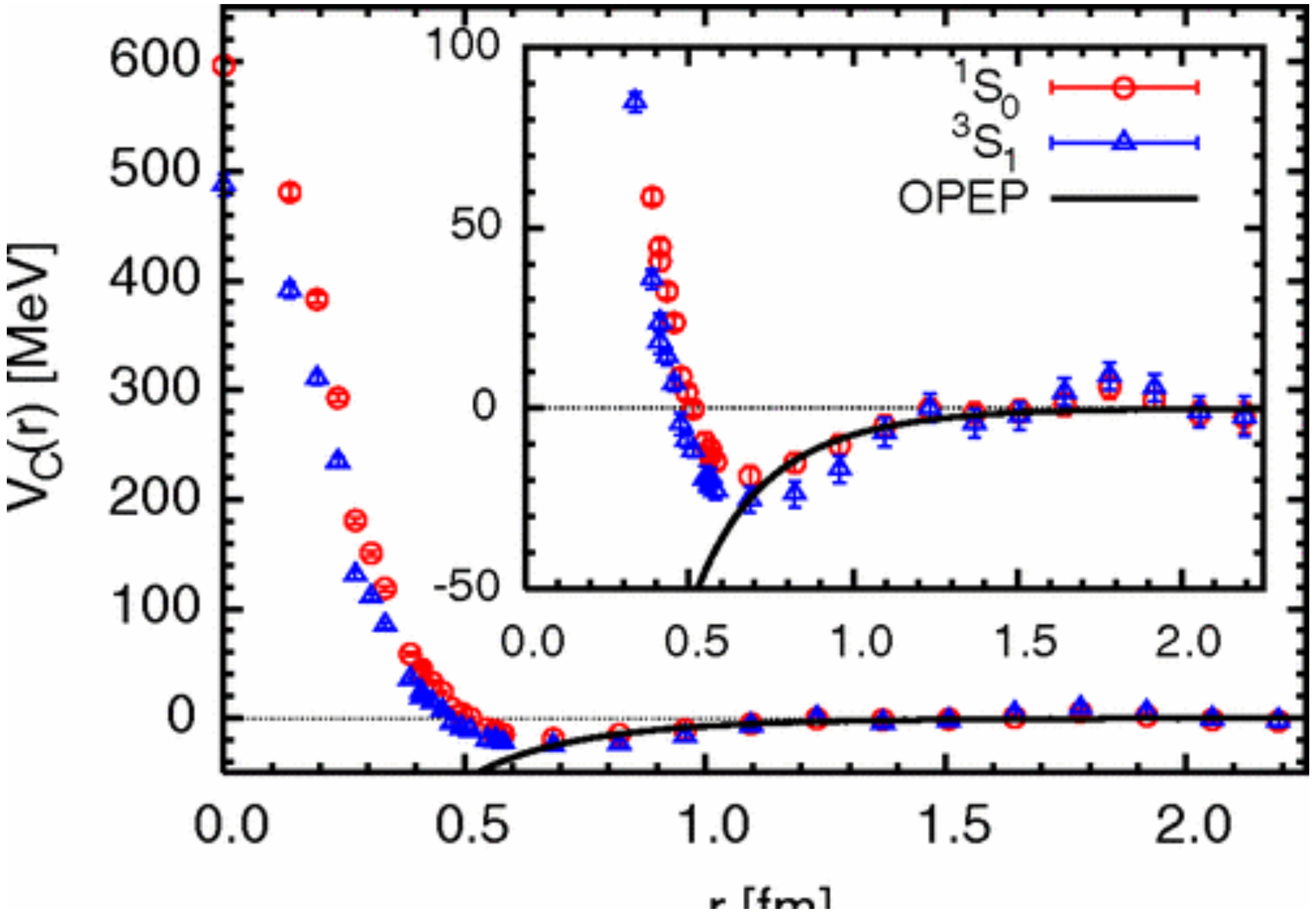
$\Delta V(r_{\min})$≈ 60 MeV !! → **different saturation energy**

**Phenomenological**




**Similar to CD-Bonn** V($r_{\min}$)≈ −40 MeV but position of the **minimum diff. → diff. saturation density** ($m_\pi/m_\rho$ ~0.6 scaled to physical value 140/775≈0.18)

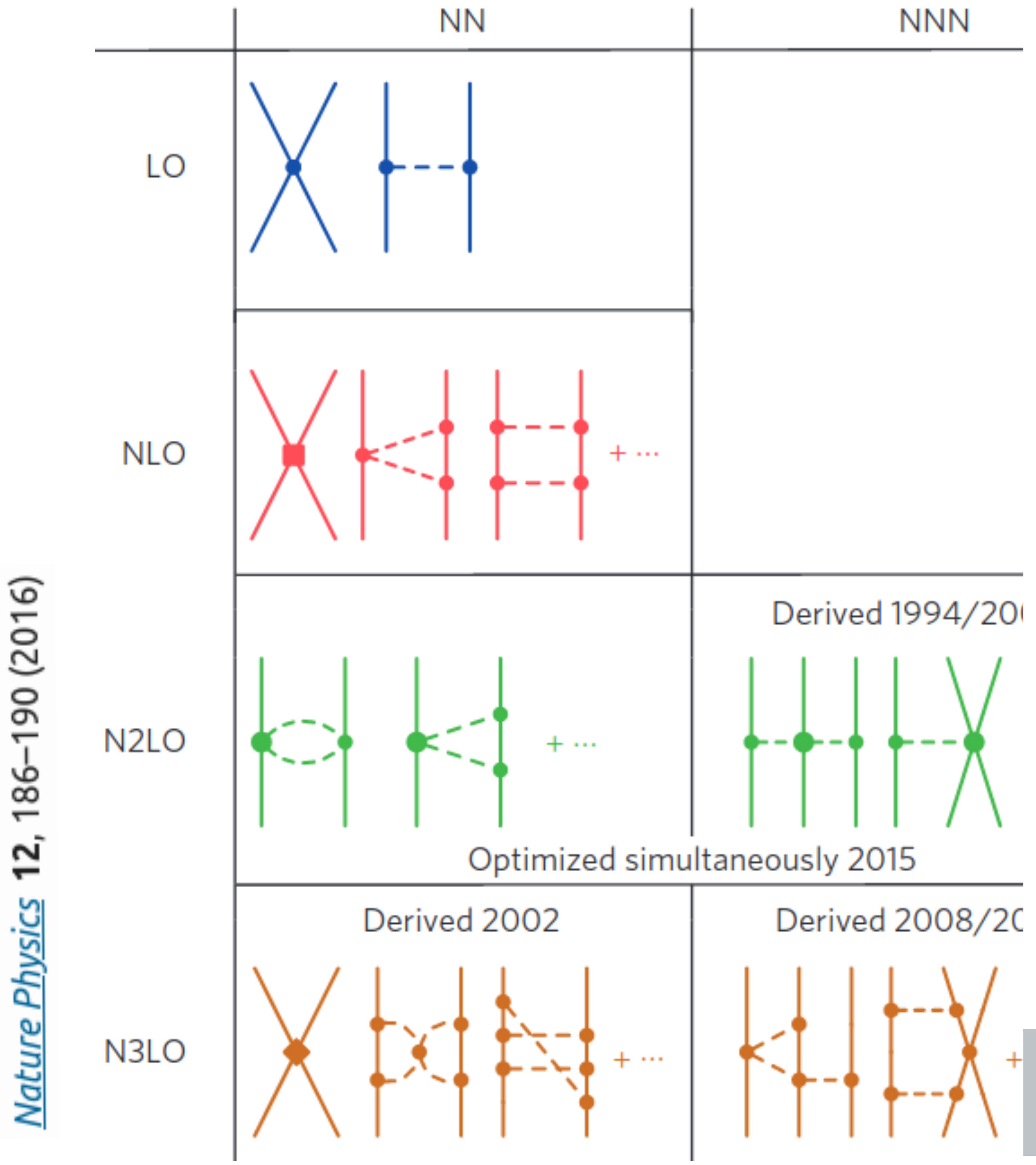
**Lattice QCD** ($m_\pi/m_\rho$~0.6)

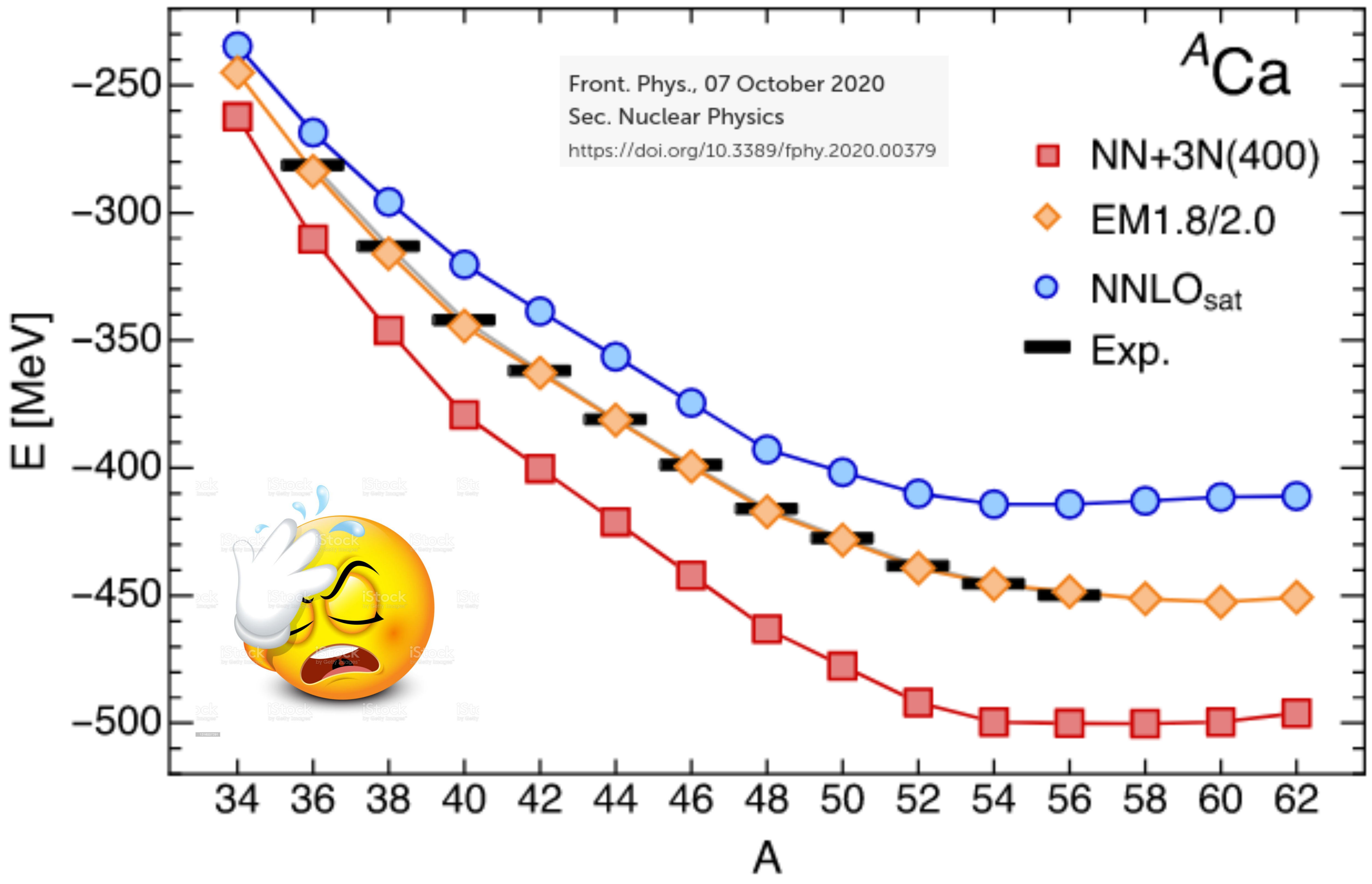


# Chiral effective field theory
## Building the interaction from QCD

H.-W. Hammer, Sebastian König, and U. van Kolck
Rev. Mod. Phys. **92**, 025004 – Published 23 June 2020

**Chiral EFT for nuclei: pions + (Delta) + nucleons** with breaking scale $\Lambda \sim 500$ MeV




Determination of the **EFT** parameters is **not unique** → different Hamiltonians that agree with experimental data on NN scattering and 3N data **do not agree** on the prediction of many body data (e.g. Ca isotopes Z=20)

# Many-body methods:

Nuclei are made from few to hundreds of nucleons!

Once the **Hamiltonian** has been built, a **many-body method** is needed to **calculate nuclei**

**Main many-body approaches seem to agree well if the same Hamiltonian is assumed:**

→ No core shell model (**NCSM**)

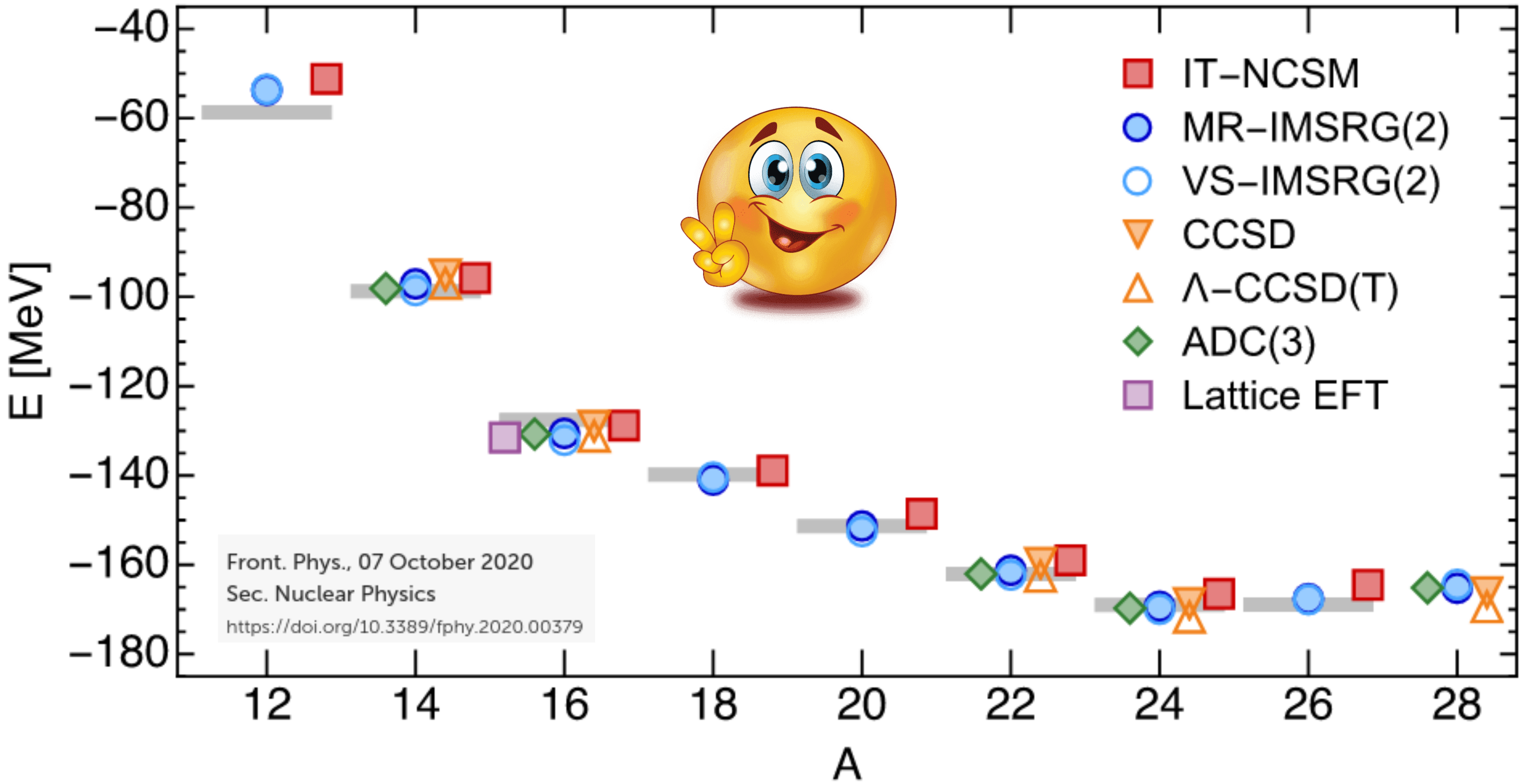
→ In medium similarity renormalization group (**IMSRG**)

→ Coupled cluster (**CC**)

→ Algebraic Diagrammatic Construction (**ADC** for <u>Self-Consistent Green's Functions</u>)

→ Quantum Monte Carlo (**QMC**)

→ Many-body perturbation theory (**MBPT**)




**Ground-state energies** of the **oxygen (Z=8)** isotopes for **various many-body approaches**, using the **same chiral NN+3N(400) Hamiltonian**. Gray bars indicate experimental data.

# DENSITY FUNCTIONAL THEORY
## Hohenberg-Kohn theorems
P.Hohenberg, W. Kohn, Phys. Rev. 136, B864 (1964)

→ Assuming a system of **interacting fermions** in a confining **external potential**, there exist a **universal** functional **F[ρ]** of the fermion density **ρ**:

$$E[\rho] = \langle \Psi | T + V + V_{\text{ext}} | \Psi \rangle = F[\rho] + \int V_{\text{ext}}(r)\rho(r)d\vec{r}$$

→ and it can be shown that

$$\min_{\Psi} \langle \Psi | T + V + V_{\text{ext}} | \Psi \rangle = \min_{\rho} E[\rho]$$

so **E[ρ]** has a **minimum** for the **exact ground-state density** where it assumes the **exact energy** as a value.

# Kohn-Sham realization

## $F[\rho] \rightarrow T_{non\text{-}int}[\rho]+V_{KS}[\rho]$

## In nuclei no need of external confining potential

For **any interacting system**, exists a **<u>local</u> single-particle potential $V_{ks}$(r)**, such that the **exact ground-state** density of the **interacting system** equals the **ground-state density** of the **auxiliary non-interacting system**:

$$\rho_{\text{exact}}(\vec{r}) = \rho_{\text{KS}}(\vec{r}) = \sum_{i=1}^{A} |\phi(\vec{r})|^2$$

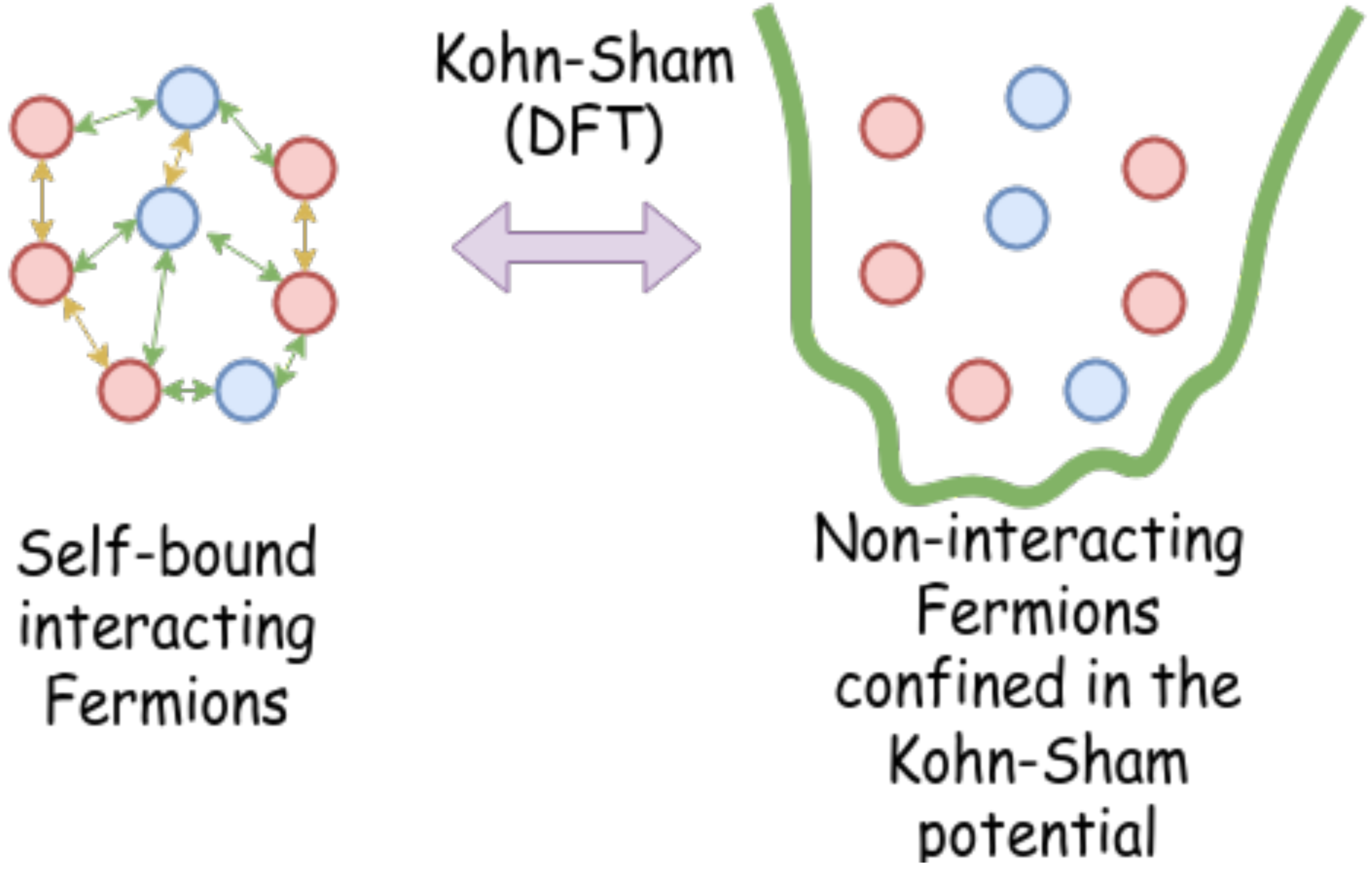
where ϕ are single-particle orbitals and the total wave-function correspond to a Slater determinant. The **E[ρ] is unique**

$$E[\rho] = T[\rho] + \int V_{\text{KS}}(\vec{r})\rho(\vec{r})d\vec{r}$$

?

where **T[ρ]** is the **kinetic energy of the non-interacting system** and for which the variational equation

$$0 = \frac{\delta E[\rho]}{\delta \rho} = \frac{\delta T[\rho]}{\delta \rho} + V_{\text{KS}}$$



# Time dependent DFT for the study of GR
## Linear Response Theory (Ring&Schuck)

**Perturbing** the initial static Hamiltonian $H_0$ with a small **time dependent operator F(t)**:

$$\mathcal{H} = \mathcal{H}_0 + F(t) \qquad F(t) = f\exp(-i\omega t) + f^\dagger \exp(i\omega t)$$

Will produce variations on the **static density $\rho_0$ linear with the external operator F(t)** in first approximation:

$$\delta\rho(t) = \delta\rho\exp(-i\omega t) + \delta\rho^\dagger \exp(i\omega t)$$

Writting the **Schroedinger equation using commutators**

$$\mathcal{H} = \sum_i h(i) \qquad h\Psi(t) = i\hbar\dot{\Psi}(t) \rightarrow [h,\rho] = i\hbar\dot{\rho}$$

$$h_0\Psi = \varepsilon\Psi \rightarrow [h_0,\rho_0] = 0 \qquad [h_0 + F(t), \rho_0 + \delta\rho(t)] = i\hbar\delta\dot{\rho}$$

# Time dependent DFT for the study of GR

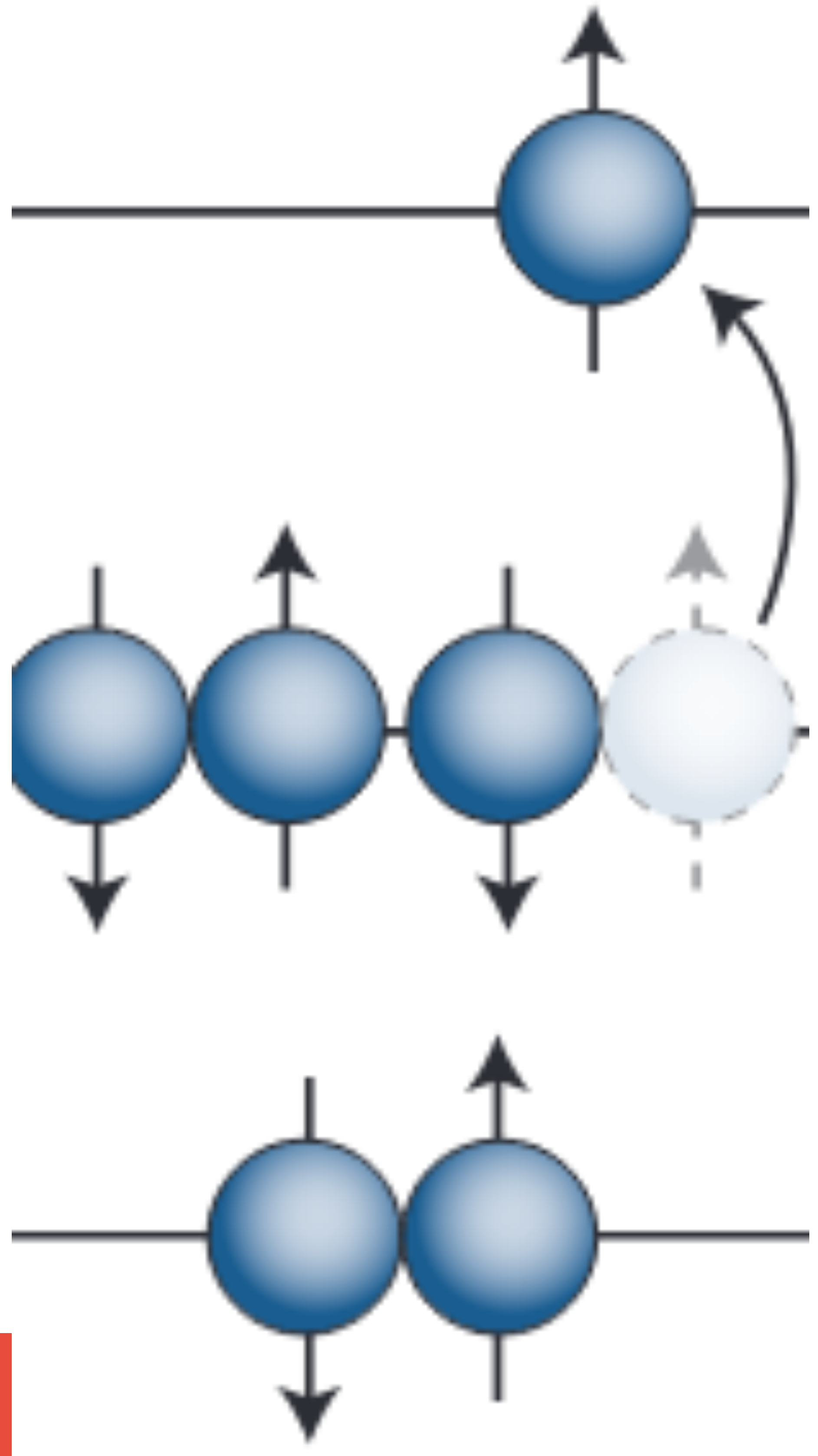
## Linear Response Theory (Ring&Schuck)

Keeping the **linear terms** in the **perturbation** (F) and imposing that a Slater determinant satisfies $\boldsymbol{\rho^2=\rho}$ (only for **particle-hole** or **hole-particle** excitations [**GR→ Many coherent ph excitations!**]) one could find:

$$(\omega - \epsilon_m + \epsilon_i)\delta\rho_{mi} = f_{mi} + \sum_{m'i'} V_{mi'im'}\delta\rho_{m'i'} + V_{mm'ii'}\delta\rho_{im'}$$

$$(\omega - \epsilon_i + \epsilon_m)\delta\rho_{im} = f_{im} + \sum_{m'i'} V_{imi'mm'}\delta\rho_{m'i'} + V_{im'mi'}\delta\rho_{im'}$$

$$V_{kl'lk'} = \sum_{kk'} \left.\frac{\partial h_{kl}}{\partial \rho_{k'l'}}\right|_{\rho^{(0)}} = \left.\frac{\partial^2 E}{\partial \rho_{lk}\partial \rho_{k'l'}}\right|_{\rho^{(0)}}$$

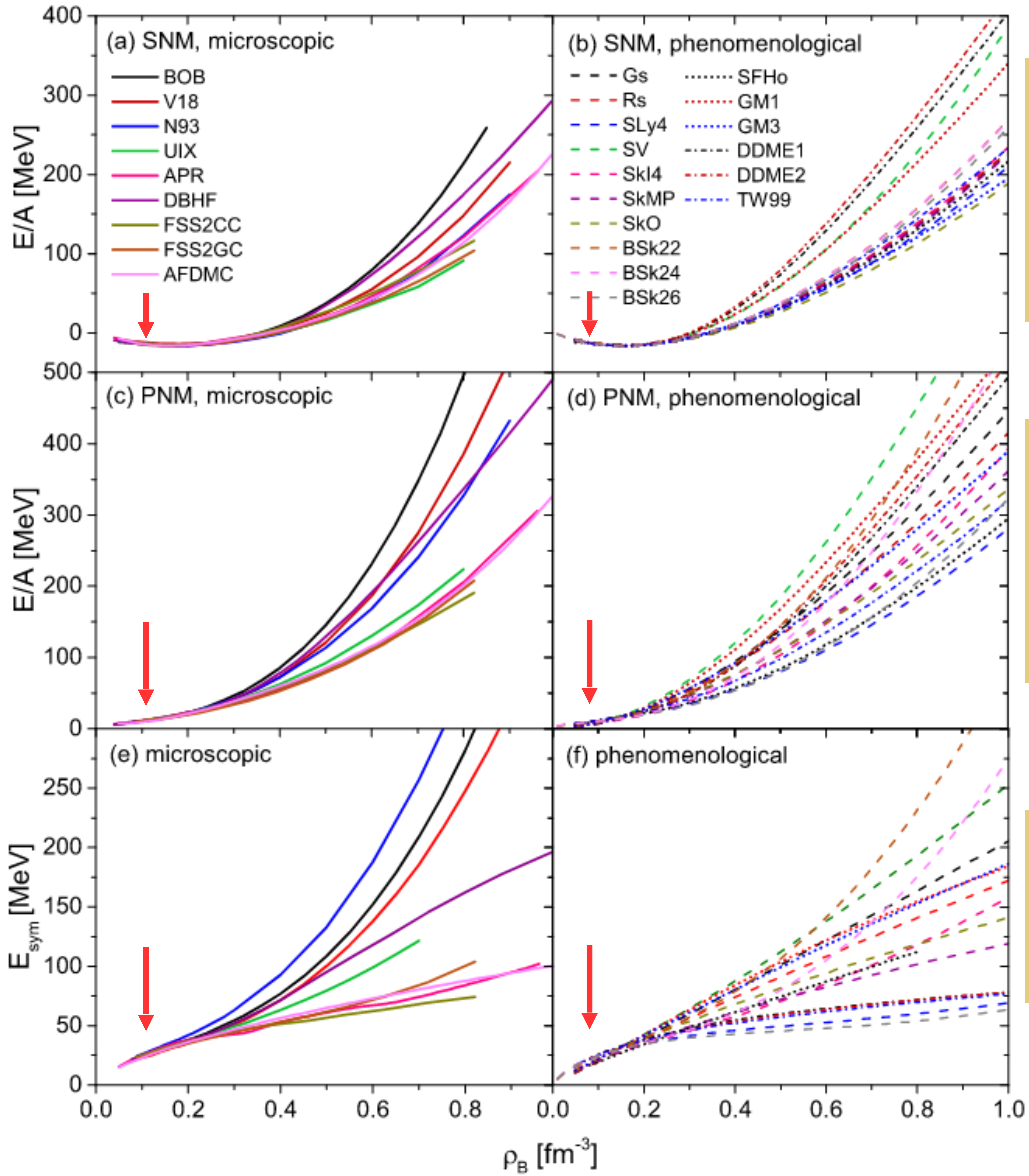


For **F→ 0** and solving the Eqs. for **δρ** one finds the **Random Phase Approximation** where the knowledge of <u>**E[ρ] is suffiient**</u>, no need to impose H.

# Nuclear EoS as predicted by modern nuclear models

**Discrepancies among models, not only for large densities**

$$e(\rho,1) \approx e(\rho,0) + S(\rho)$$
$$S(\rho) \approx e(\rho,1) - e(\rho,0)$$




**Symmetric nuclear matter**
$e(\rho,0)$

**Pure neutron matter**
$e(\rho,1)$

**Symmetry energy**
$e(\rho,1) - e(\rho,0)$

# Determination of the parameters; theoretical (statistical) errors and correlations

→ **Chiral EFT** expansion **allows** for the **estimation** of the **errors** associated to a given truncation in the determination of the Hamiltonian.

→ Most **many-body techniques** (except EDF) **allow** to **estimate** the **error** associated to the method by evaluating the following (and more complex) terms. Analogous to the expansion in the interaction, think about (many-body) perturbation theory.

→ **All nuclear models are effective** and, thus, **parameters must be determined (<u>fitted to experiment</u>).**

→ **Two statistic approaches to this problem in the literature:**

**Frequentists** concentrate on having methods guaranteed to work most of the time, given minimal assumptions. Based on the ratio of times we expect an event to occur (#successes / #experiment )

**Bayesians** try to make inferences that take into account all available information and answer the question of interest given the particular data set. Based on individual's degree of belief of the occurrence of an event

# Systematic uncertainties (EXAMPLE):

**Beyond statistical errors there exist other types of errors!**

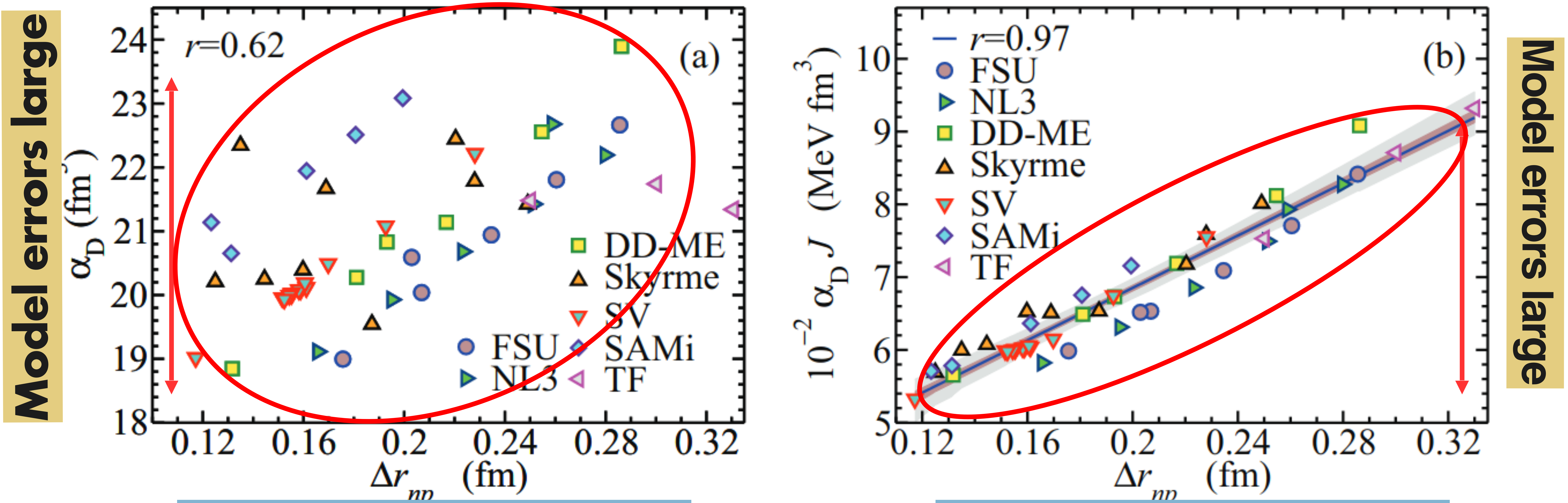
**Differences among equally “good” models**

* **Up to now statistical errors from the fit. Is that the whole story?**

$$\sigma^2 = \sigma^2_{\mathrm{stat}} + \sigma^2_{\mathrm{syst}}$$

* **Differences between theory and experiment:** model error or systematic theoretical error → not allways possible.

* **Differences among (reasonable) models → proxy to model error**




**Correlation between models low**

**Correlation between models high**

# 3.-What can we learn from the Earth and the Heavens about the Nuclear Equation of State?

(some examples)

# From Heaven: Neutron Star Mass

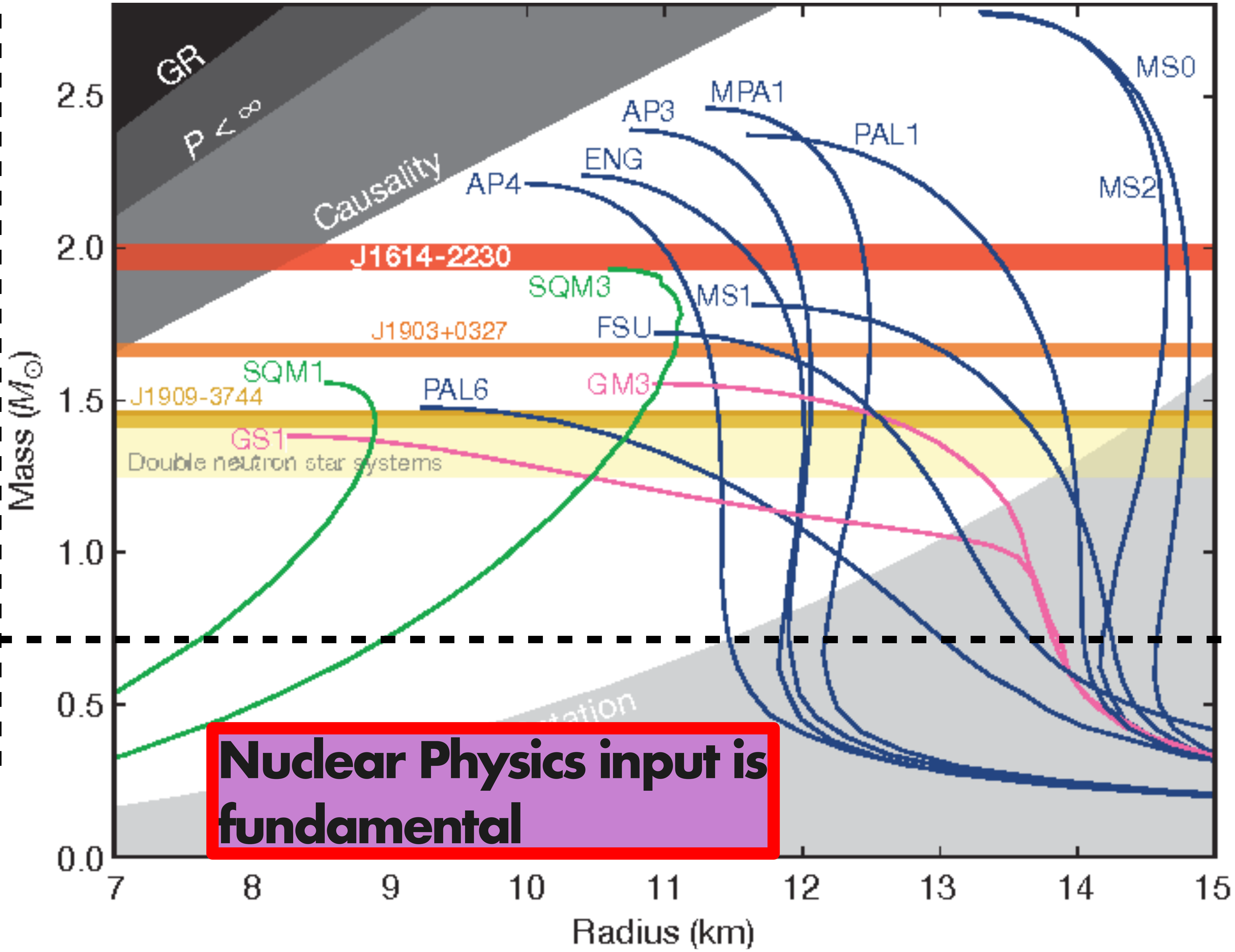
**Nuclear models** that account for different nuclear properties on **Earth** predict a large **variety** of **Neutron Star Mass-Radius** relations → **Observation of a 2$M_{sun}$ has constrained nuclear models.**

**Tolman–Oppenheimer–Volkoff equation (sph. sym.):**

$$\frac{dM(r)}{dr} = 4\pi r^2 \mathcal{E}(r);$$

$$\frac{dP}{dr} = -G\frac{\mathcal{E}(r)M(r)}{r^2}\left[1+\frac{P(r)}{\mathcal{E}(r)}\right]\left[1+\frac{4\pi r^3 P(r)}{M(r)}\right]\left[1-\frac{2GM(r)}{r}\right]^{-1}$$

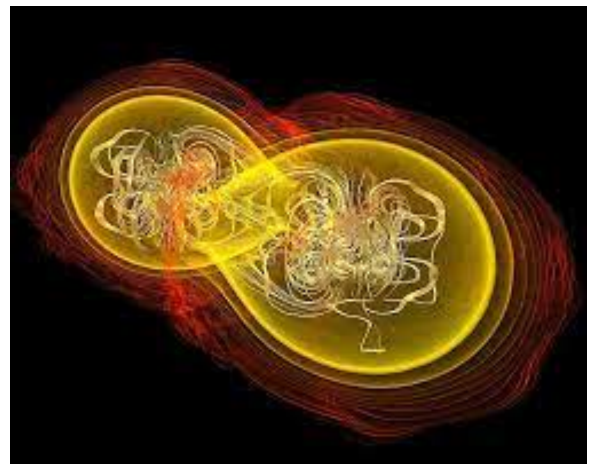
$\mathcal{E}(r) \to$ degeneracy pressure from neutrons $\to M_{max} = 0.7 M_{sun}$



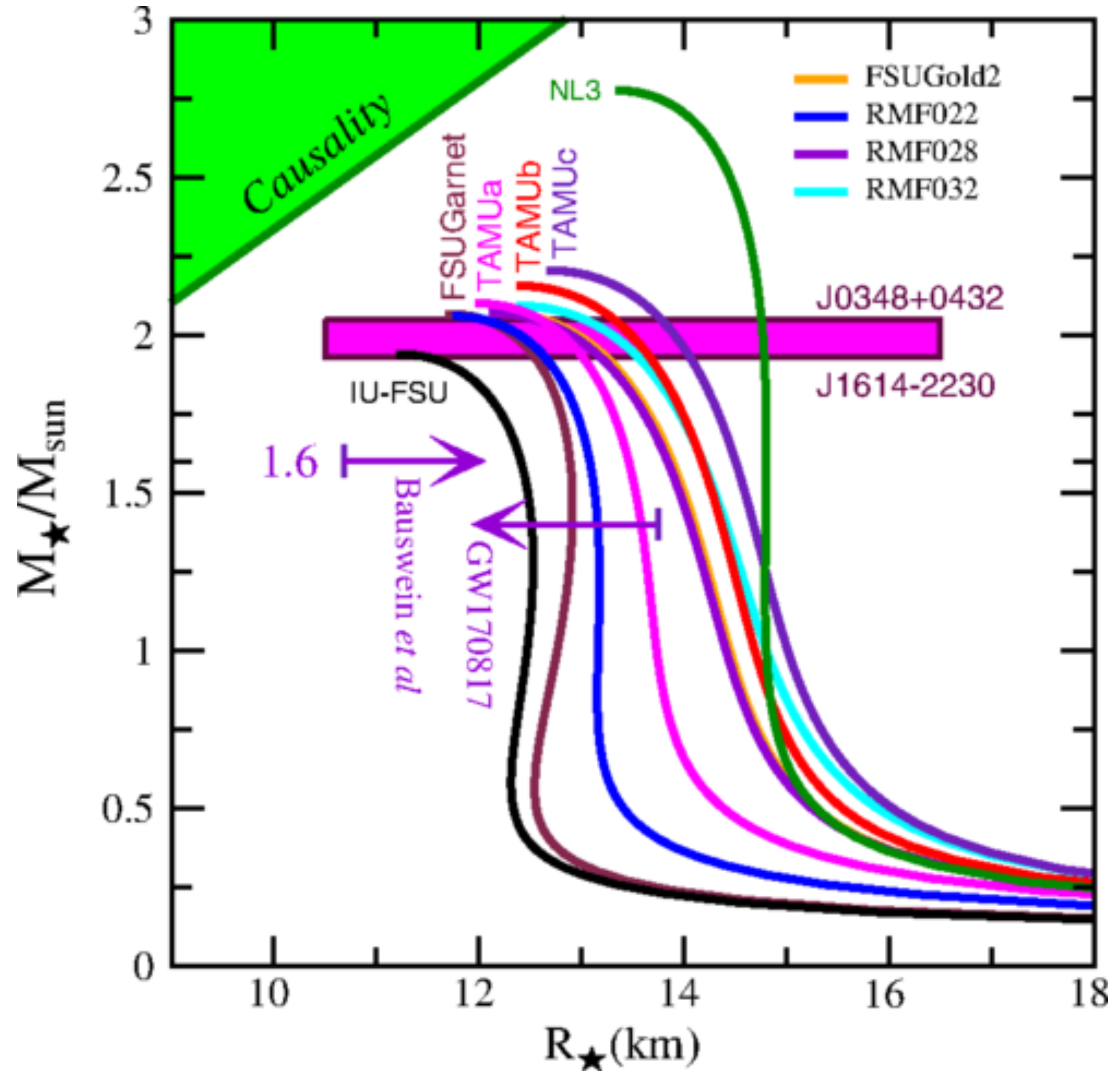

**Nuclear Physics input is fundamental**

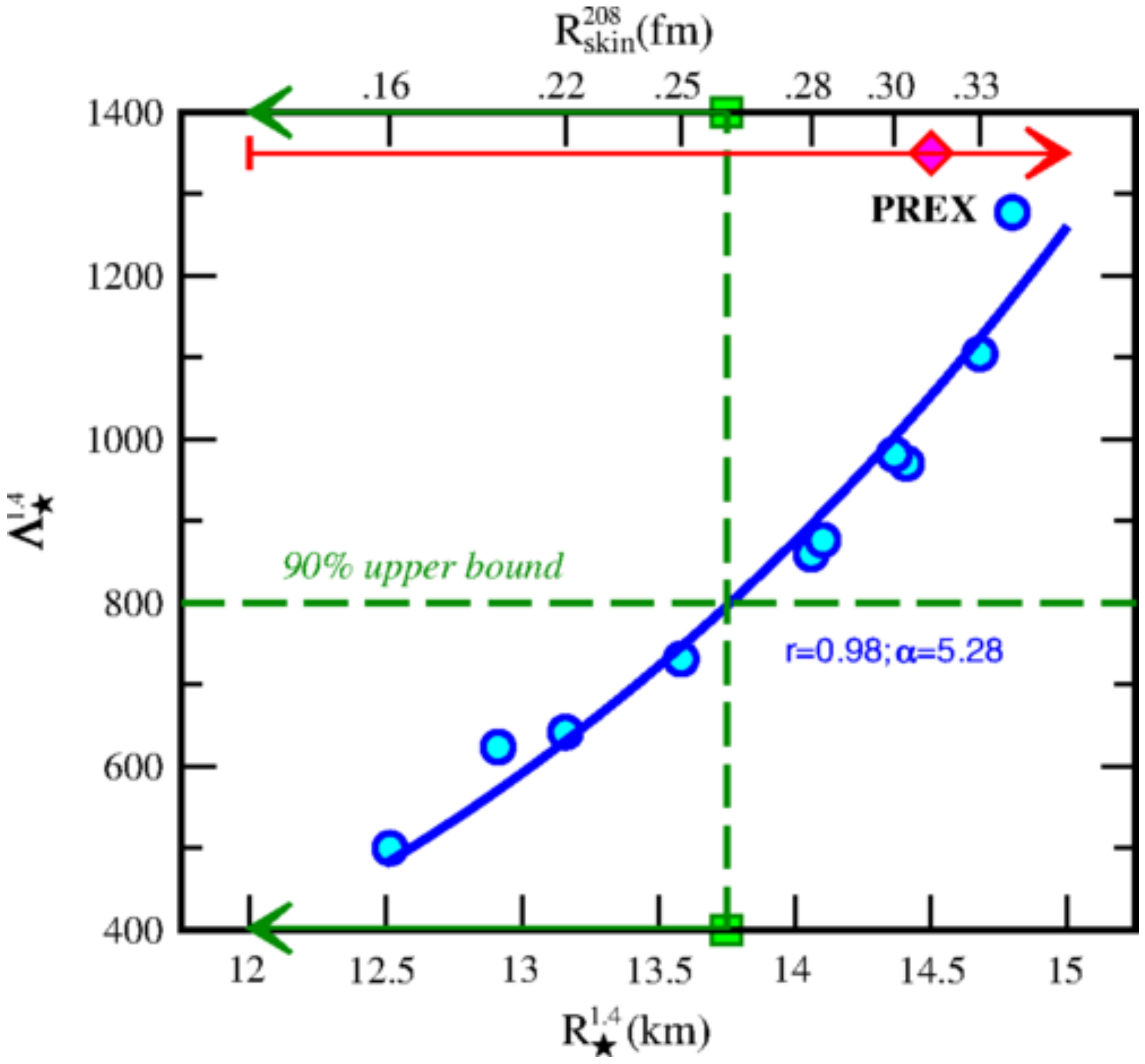
*A two-solar-mass neutron star measured using Shapiro delay - P. B. Demorest, T. Pennucci, S. M. Ransom, M. S. E. Roberts & J. W. T. Hessels - Nature volume 467, 1081–1083(2010)*

# From Heaven: Gravitational wave signal from a binary neutron star merger



**GW170817** from the binary neutron star merger → **constraint** neutron star **radius** and, thus, the **nuclear EoS**




**Tidal deformability (Λ)** is a quadrupole deformation inferred from **GW signal** → proportional to **restoring force.** Hence, sensitive to the **nuclear EoS**

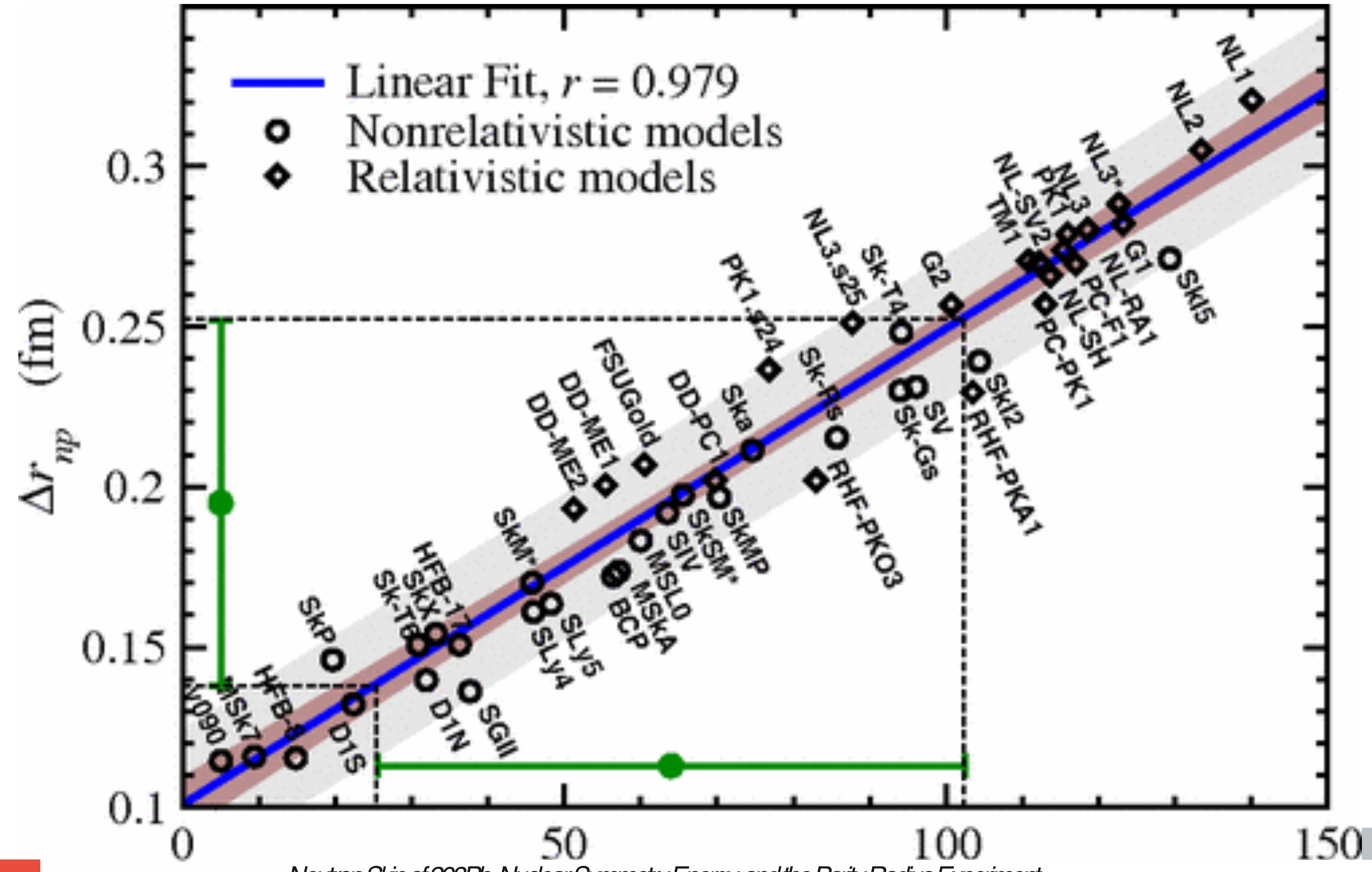
*Neutron Skins and Neutron Stars in the Multimessenger Era*

*F.J. Fattoyev, J. Piekarewicz, and C.J. Horowitz Phys. Rev. Lett. 120, 172702 (2018)*

# From Heaven & Earth: neutron skin and the Radius of a Neutron Star

Both, the **neutron skin thickness** ($\Delta r_{np}$=$r_n$-$r_p$) in neutron rich nuclei and the **radius** of a **neutron star** are related to the **neutron pressure** in infinite matter. The former around $\rho_0$ (L) while the latter in a broad range of densities.

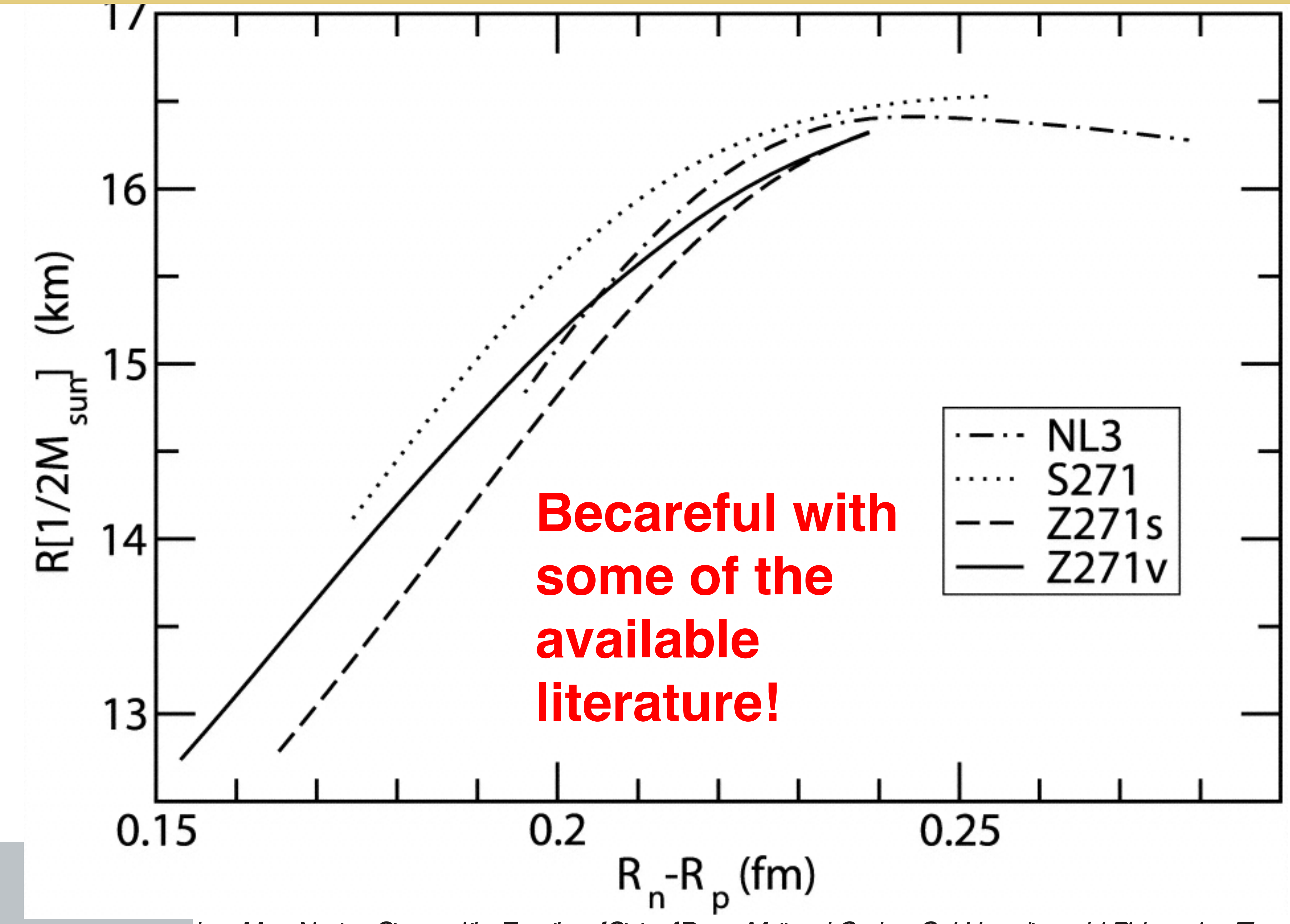
$$\Delta r_{np} \approx \frac{1}{12}\frac{N-Z}{A}\frac{R}{J}L$$




Neutron Skin of 208Pb, Nuclear Symmetry Energy, and the Parity Radius Experiment

X. Roca-Maza, M. Centelles, X. Viñas, and M. Warda Phys. Rev. Lett. 106, 252501 (2011)

→ Only for <u>unrealistically</u> **small neutron stars**, that is, for small central densities ($\rho_c \sim \rho_0$): nuclear models predict a **linear** relation between **R** and **$\Delta r_{np}$**



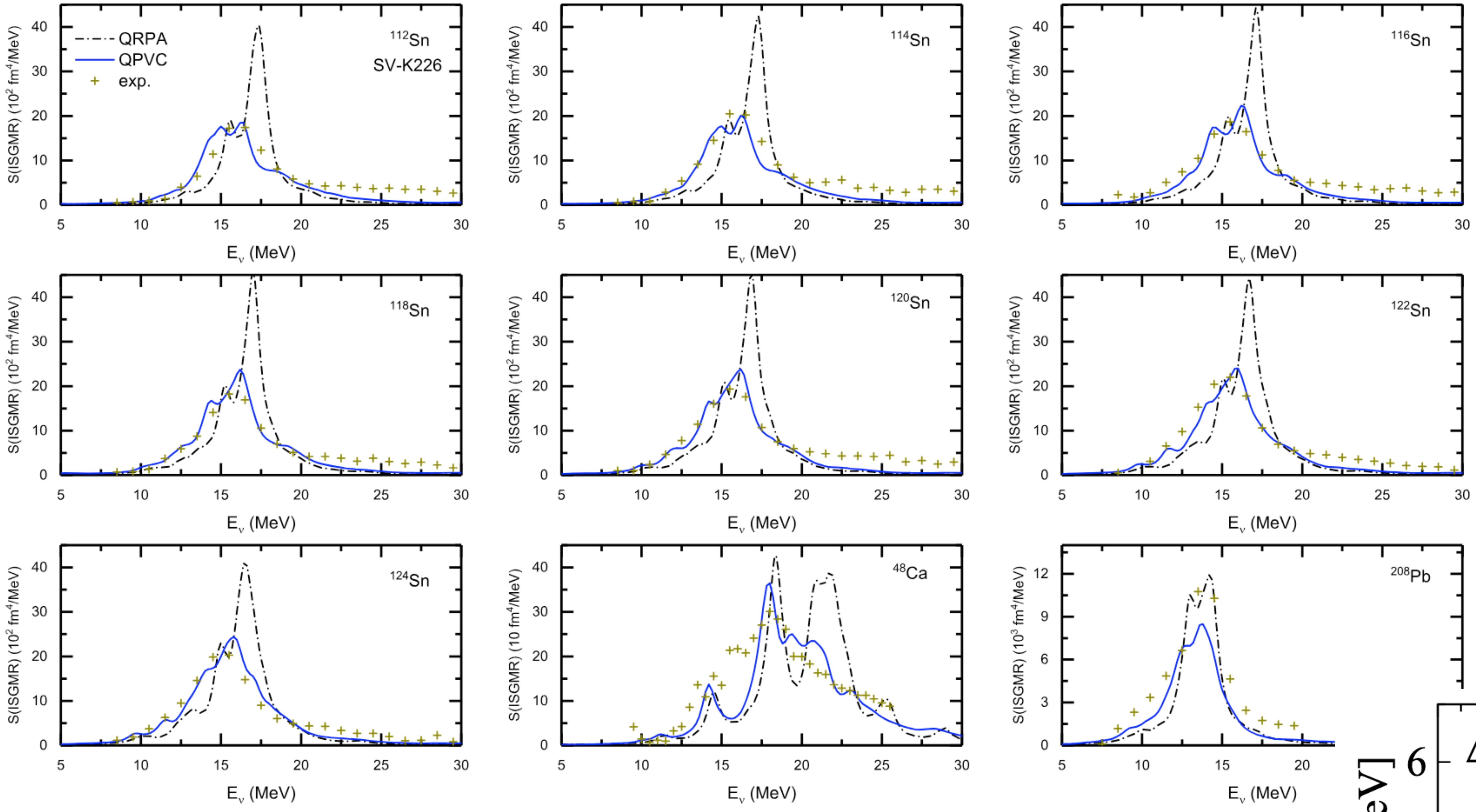

**Becareful with some of the available literature!**

Low-Mass Neutron Stars and the Equation of State of Dense Matter - J. Carriere, C. J. Horowitz, and J. Piekarewicz - The Astrophysical Journal, 593 (2003) 463

# From Earth: Giant Monopole Resonance

## do we understand it?

*U. Garg, G. Colò / Progress in Particle and Nuclear Physics 101 (2018) 55–95*



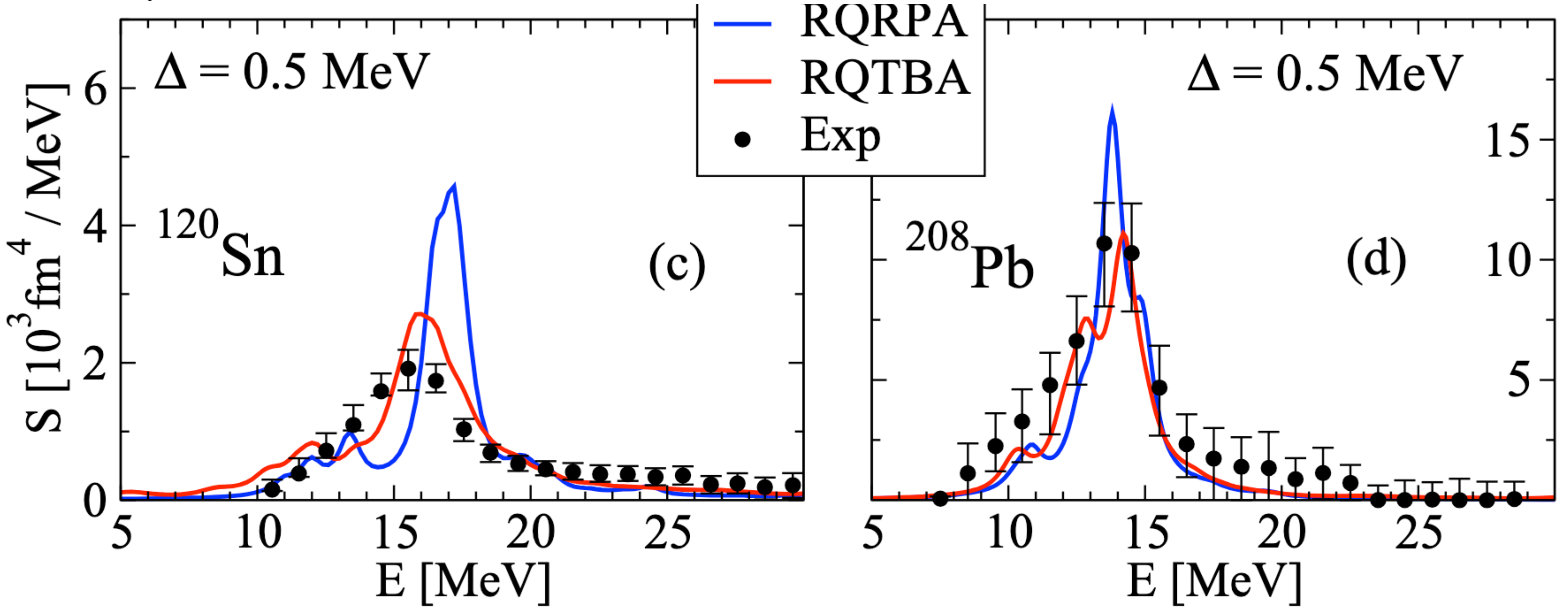

**Toward a Unified Description of Isoscalar Giant Monopole Resonances in a Self-Consistent Quasiparticle-Vibration Coupling Approach**

Z. Z. Li (李征征)[1,2,3], Y. F. Niu (牛一斐)[1,2,*], and G. Colò[3,4,†]

Only **very recently**, two works describe the **ISGMR** in different nuclei but with $K_0$ ranging from **226 MeV** (non-relativistic approach) to **260 MeV** (relativistic approach)

**Relativistic approach to the nuclear breathing mode**

Elena Litvinova



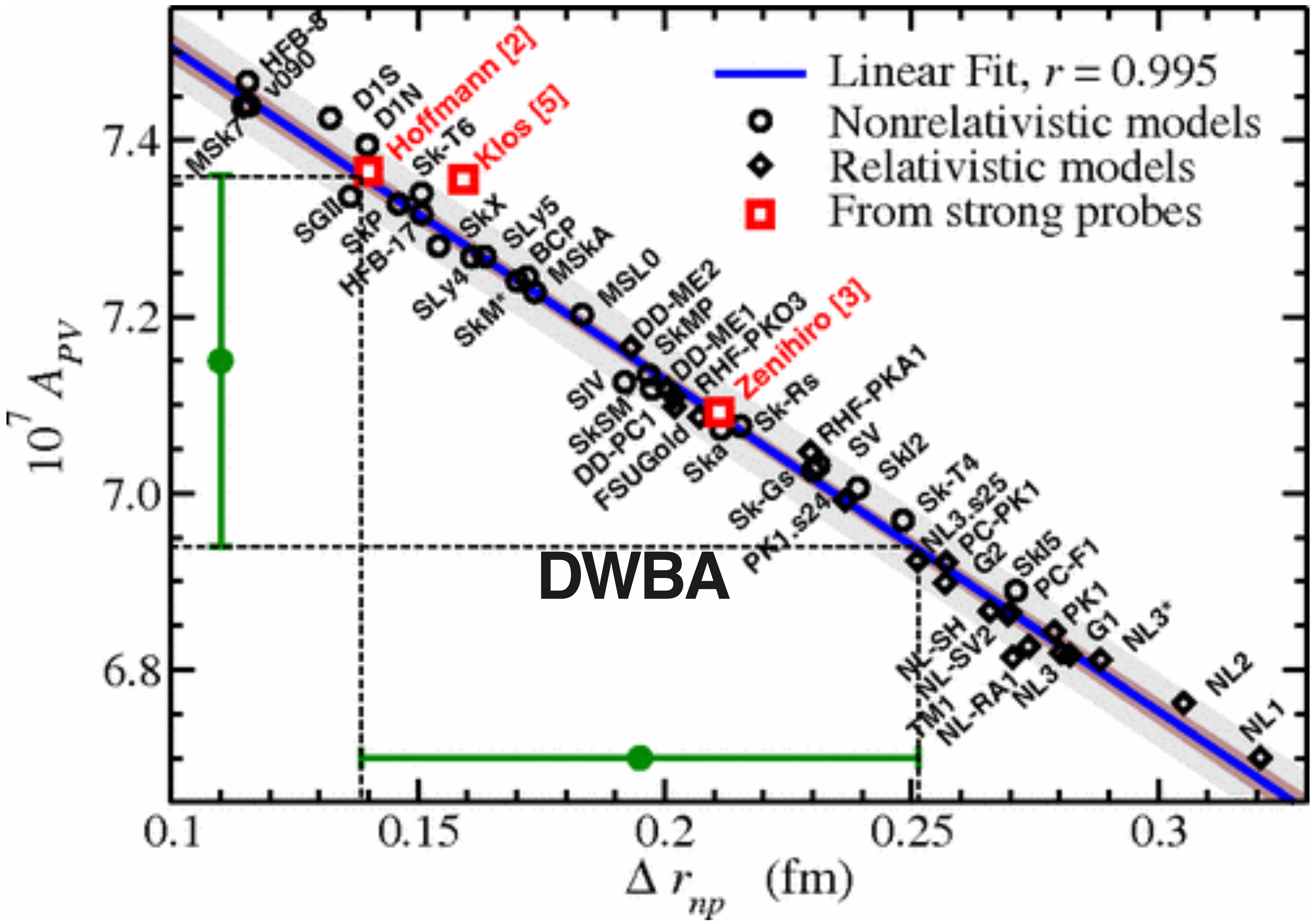

These calculations points towards a **plausible estimate** on K = 220-260 MeV. **Is that the final word?** Further experiments are planned.

# From Earth: Parity violating electron scattering and the neutron skin

**Polarized electron-Nucleus scattering:**

→ In good approximation, **the weak interaction** probes the **neutron distribution** in nuclei while **Coulomb interaction** probes the **proton distribution**

→ **Different experimental efforts @ Jlab (USA) & MAMI (Germany)**




→ **Electrons** interact by **exchanging** a **γ** (couples to **p**) or a $\mathbf{Z_0}$ boson (couples to **n**)

→ **Ultra-relativistic electrons**,depending on their helicity (±), will interact with the nucleus seeing a slightly different potential: **Coulomb** ± **Weak**

$$A_{pv} = \frac{d\sigma_+/d\Omega - d\sigma_-/d\Omega}{d\sigma_+/d\Omega + d\sigma_-/d\Omega} \sim \frac{\text{Weak}}{\text{Coulomb}}$$

→ Main **unknown** is $\boldsymbol{\rho_n}$

→ In **PWBA** for small momentum transfer **q**:

$$A_{pv} = \frac{G_F q^2}{4\sqrt{2}\pi\alpha}\left(1 - \frac{q^2 r_p^2}{3F_p(q)}\right)\Delta r_{np}$$

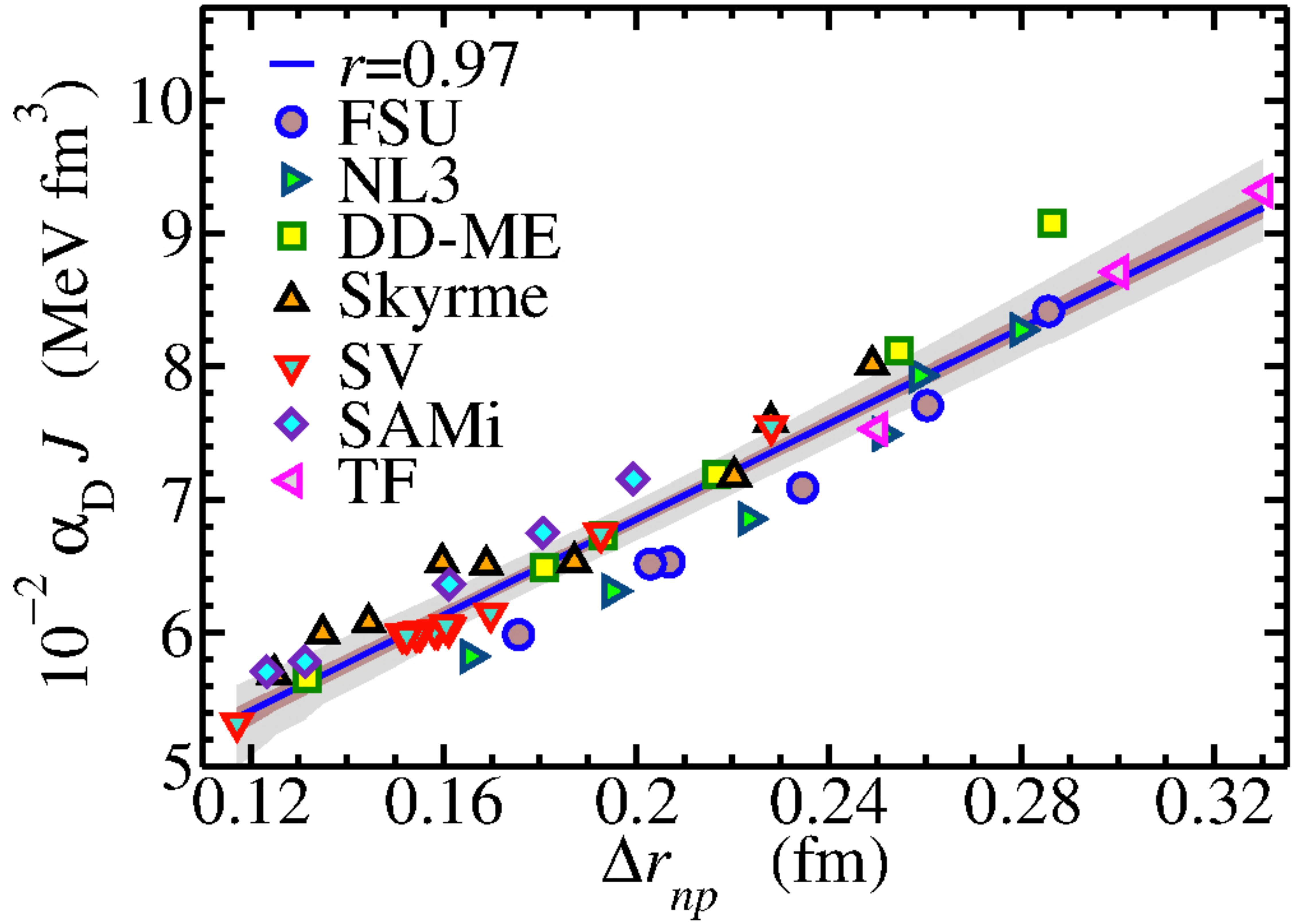


Neutron Skin of 208Pb, Nuclear Symmetry Energy, and the Parity Radius Experiment

X. Roca-Maza, M. Centelles, X. Viñas, and M. Warda Phys. Rev. Lett. 106, 252501 (2011)

# From Earth: dipole polarizability and neutron skin

The dipole **polarizability** measures the **tendency** of the nuclear **charge** distribution to be **distorted.**
From a macroscopic point of view **α ~ (electric dipole moment)/(external electric field)**




→ Using the **dielectric theorem**: the polarizability can be computed from the expectation value of the Hamiltonian in the constrained ground state H′=H+λD

→ For <u>guidance</u> assuming the **Droplet model** for H, one would find:

$$\alpha_D \approx \frac{\pi e^2}{54}\frac{\langle r^2\rangle}{J}A\left(1+\frac{5}{2}\frac{\Delta r_{np}-\Delta r_{np}^{\rm surf}-\Delta r_{np}^{\rm Coul}}{\langle r^2\rangle^{1/2}(I-I_{\rm Coul})}\right)$$

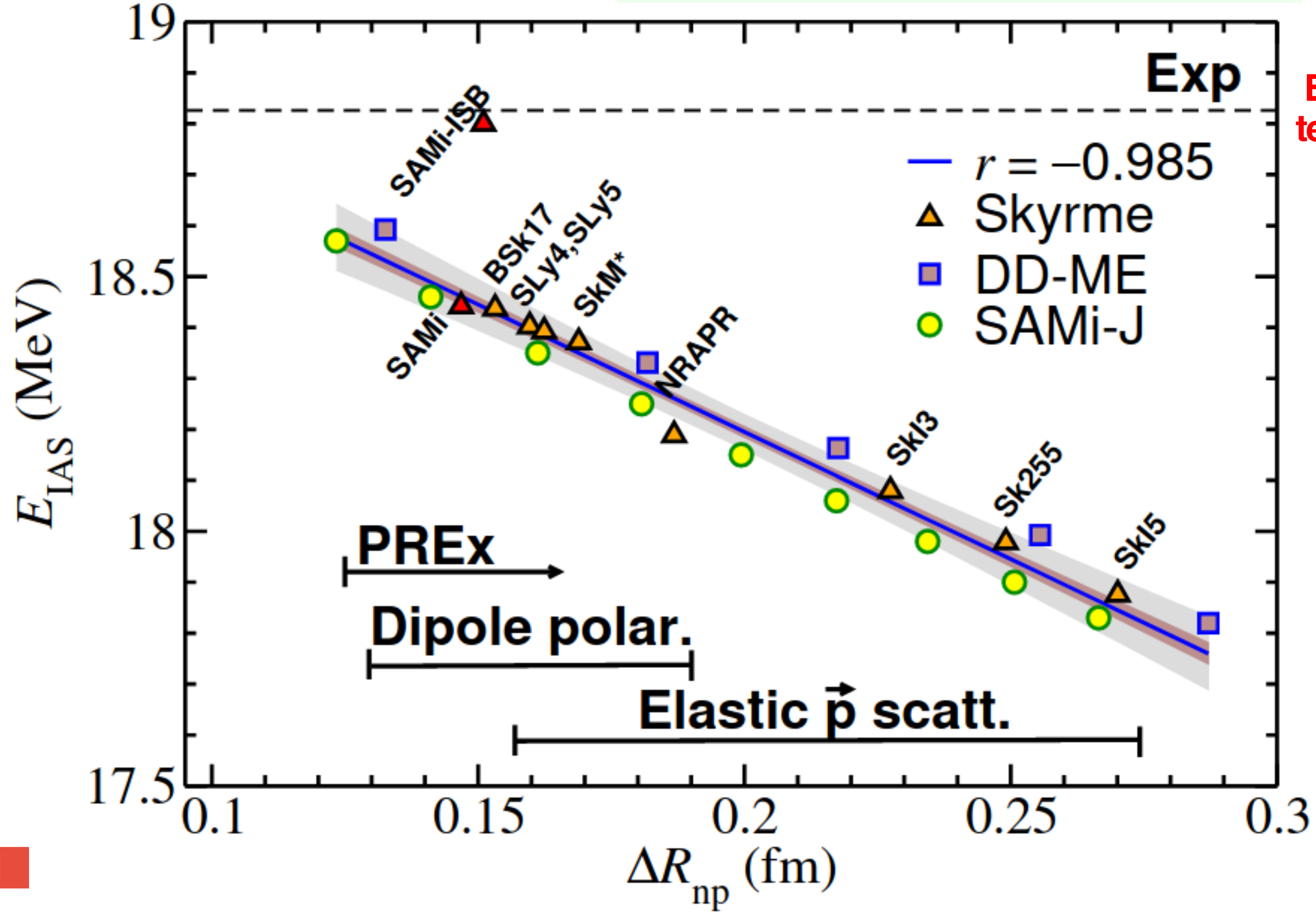


Electric dipole polarizability in 208Pb: Insights from the droplet model - X. Roca-Maza, M. Brenna, G. Colò, M. Centelles, X. Viñas, B. K. Agrawal, N. Paar, D. Vretenar, and J. Piekarewicz

Phys. Rev. C 88, 024316 (2013)

# From Earth: Isobaric Analog State and the breaking of isospin symmetry

$$E_{IAS} \approx E_{IAS}^{C,direct} \approx \frac{6}{5}\frac{Ze^2}{R_p}\left(1 - \sqrt{\frac{5}{12}\frac{N}{N-Z}\frac{\Delta r_{np}}{R_p}}\right)$$




Exp errors in IAS ~ tens of keV

(or smaller)

# Summary

with qualitative indication of accuracy needed to describe experiment (note that absolute values might be subject to systematics)

→ $\rho_0 \in$ [0.154,0.159] fm$^{-3}$ → relative accuracy 2%

→ needed to describe experiment (Rch) ≤0.1%

→ $e_0 \in$ [15.6,16.2] MeV→ relative accuracy 4%

→ needed to describe experiment (B) ≤0.0001%

→ $K_0 \in$ [200,260] MeV→ relative accuracy 25%

→ needed to describe experiment ($E_x^{GMR}$) ≤7%

→ $J \in$ [30,35] MeV → relative accuracy (α) 15%

→ needed to describe experiment ≤15%

→ $L \in$ [20,120] MeV → relative accuracy (α) 150%

→ needed to describe experiment ≤50%

→ ...

# Summary from Progress in Particle and Nuclear Physics 101 (2018) 96–176

| EoS par. | Observable | Range | Comments |
|---|---|---|---|
| $\rho_0$ | $\langle r^2_{ch}\rangle^{1/2}$ | 0.154–0.159 | Most accurate EDFs on $M(N,Z)$ and $\langle r^2_{ch}\rangle^{1/2}$ (see Section 5) |
| $e_0$ | $M(N,Z)$ | −16.2 to −15.6 | Most accurate EDFs on $M(N,Z)$ and $\langle r^2_{ch}\rangle^{1/2}$ (see Section 5) |
| $K_0$ | $M(N,Z)$ | 220–245 | Most accurate EDFs on $M(N,Z)$ and $\langle r^2_{ch}\rangle^{1/2}$ (see Section 5) |
| | ISGMR | 220–260 | From EDFs in closed shell nuclei [116] |
| | ISGMR | 250–315 | Blaizot's formula [Eq. (32)] [51] |
| | ISGMR | ∼200 | EDF describing also open shell nuclei [118] |
| $J$ | $M(N,Z)$ | 29-35.6 | Most accurate EDFs on $M(N,Z)$ and $\langle r^2_{ch}\rangle^{1/2}$ (see Section 5) |
| | IVGDR | ∼24.1(8) + L/8 | From EDF analysis [$S(\rho = 0.1\ \mathrm{fm}^{-3}) = 24.1(8)$ MeV] [273] |
| | PDS | 30.2–33.8 | From EDF analysis [370] |
| | PDS | 31.0–33.6 | From EDF analysis [371] |
| | $\alpha_D$ | $24.5(8) + 0.168(7)L$ | From EDF analysis $^{208}$Pb [96] |
| | $\alpha_D$ | 30–35 | From EDF analysis [179] |
| | IAS and $\Delta r_{np}$ | 30.2–33.7 | From EDF analysis [325] |
| | AGDR | 31.2–35.4 | From EDF analysis [401] |
| | PDS, $\alpha_D$, IVGQR, AGDR | 32–33 | From EDF analysis [508] |
| | compilation | 29.0–32.7 | [106] |
| | compilation | 30.7–32.5 | [107] |
| | compilation | 28.5–34.9 | [3] |
| $L$ | $M(N,Z)$ | 27–113 | Most accurate EDFs on $M(N,Z)$ $\langle r^2_{ch}\rangle^{1/2}$ (see Section 5) |
| | $\rho_n$ | 40–110 | proton-$^{208}$Pb scattering [24] |
| | $\rho_n$ | 0–60 | $\pi$ photoproduction ($^{208}$Pb) [181] |
| | $\rho_n$ | 30–80 | antiprotonic at. (EDF analysis) [102,509] |
| | $\rho_{weak}$ | >20 | Parity violating scattering [27] |
| | PDS | 32–54 | From EDF analysis [370] |
| | PDS | 49.1–80.5 | From EDF analysis [371] |
| | $\alpha_D$ | 20–66 | From EDF analysis [179] |
| | IVGQR and ISGQR | 19–55 | From EDF analysis [101] |
| | IAS and $\Delta r_{np}$ | 35–75 | From EDF analysis [325] |
| | AGDR | 75.2–122.4 | From EDF analysis [401] |
| | PDS, $\alpha_D$, IVGQR, AGDR | 45.2–54.6 | From EDF analysis [508] |
| | compilation | 40.5–61.9 | [106] |
| | compilation | 42.4–75.4 | [107] |
| | compilation | 30.6–86.8 | [3] |

**Some alternative compilations:**

M. Oertel, M. Hempel, T. Klähn, S. Typel, Rev. Modern Phys. 89 (2017) 015007.

M.B. Tsang, et al., Phys. Rev. C 86 (2012) 015803.

Bao-An Li, Xiao Han, Phys. Lett. B 727 (1) (2013) 276–281.