Computing magnitudes, colours, distances at any signal-to-noise level

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Having measured the parallax and its uncertainty,
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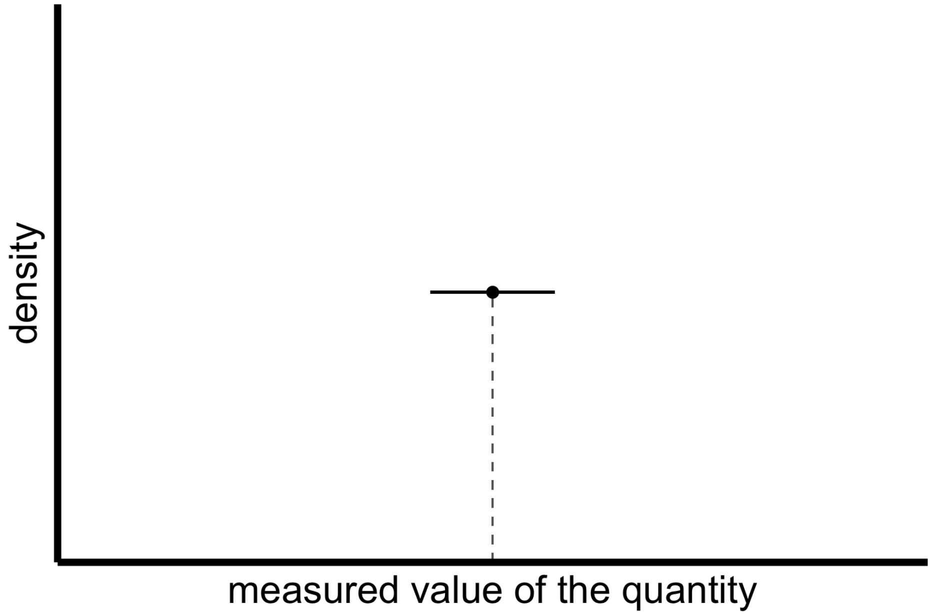

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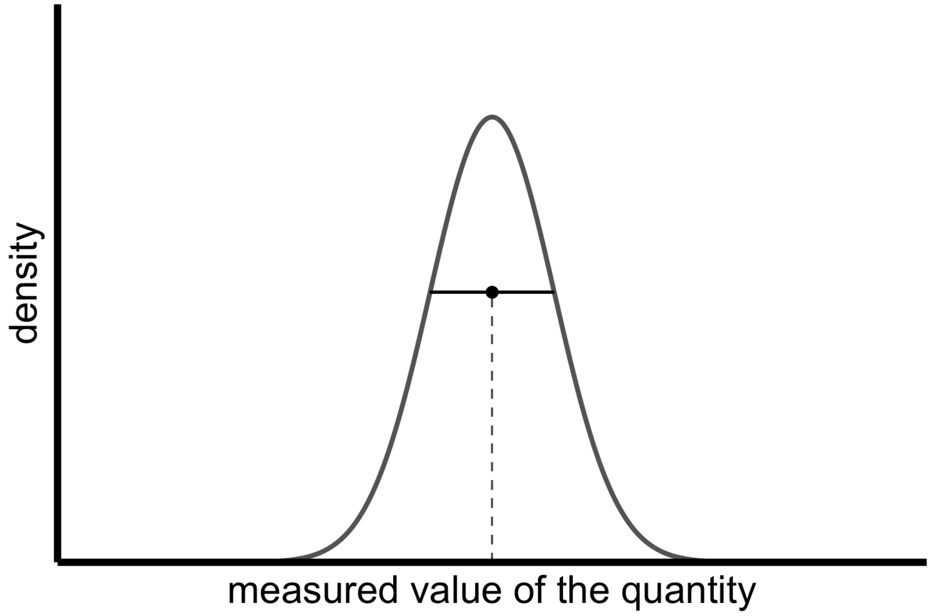


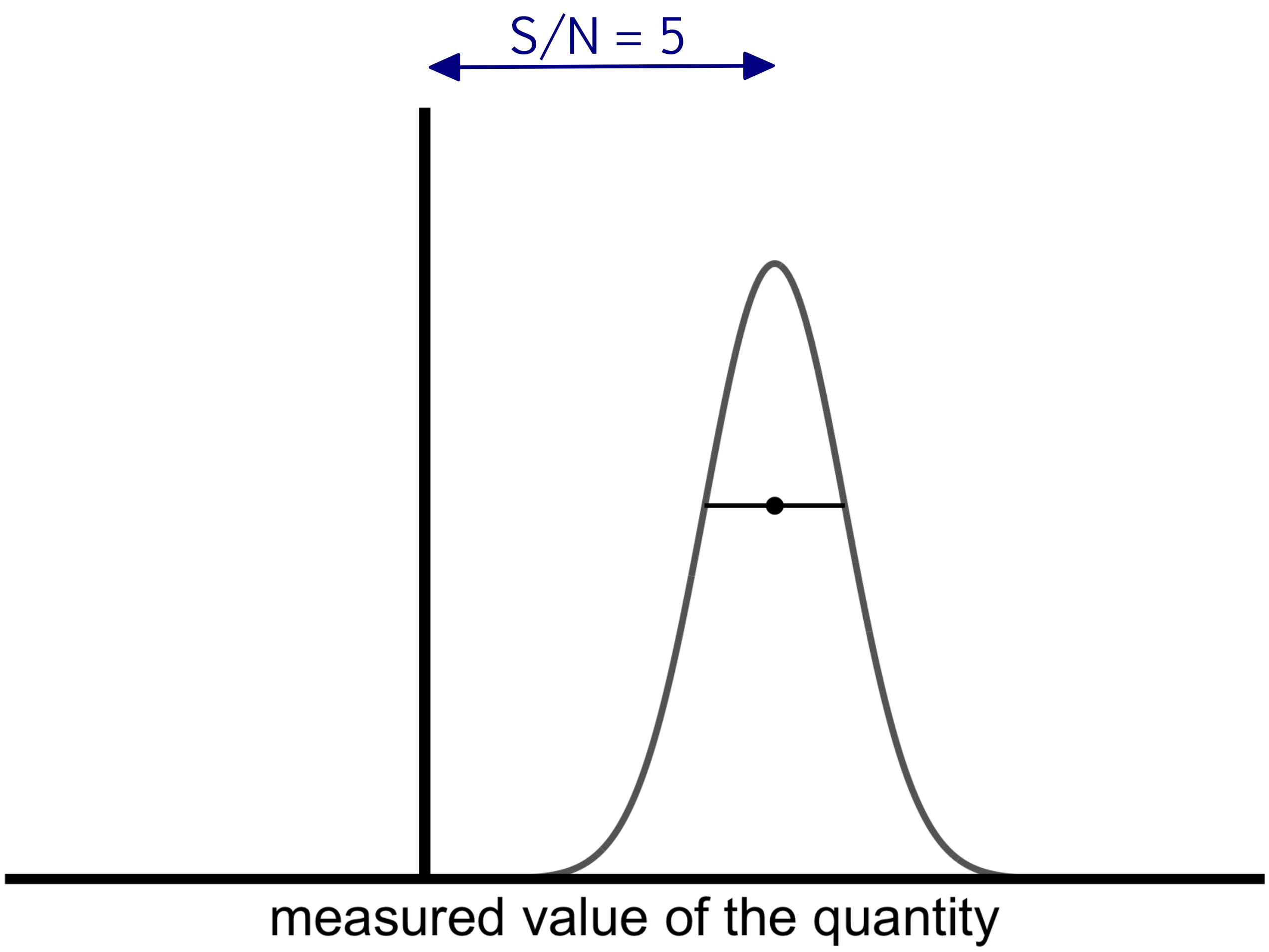
What we have measured is a pdf for the measured quantity (assumed Gaussian)

Bayes theorem (continuous form):

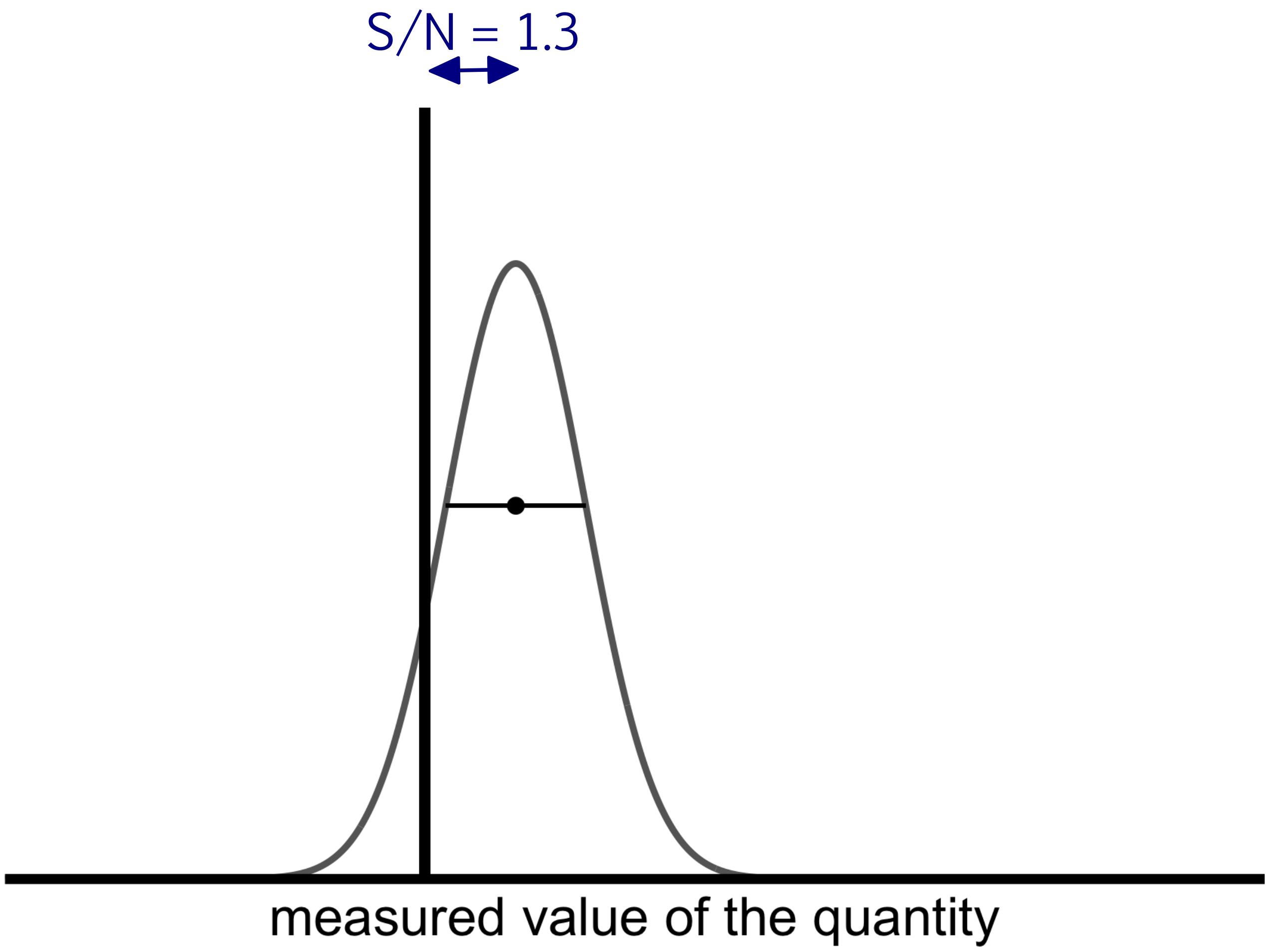
$$p(\text{true value}\,|\,\text{measured value}) = \frac{p(\text{measured value}\,|\,\text{true value}) \cdot p(\text{true value})}{Z}$$

- Relates pdf for measured value with pdf of true value
- Often the prior is neglegted (i.e. is assumed constant)
- Fine if a constant prior works over the range over which likelihood is significantly different from zero: *high signal-to-noise*

High S/N:



Low S/N:



Low S/N, non-negative *true* value

- Constant prior doesn't work, needs to take non-negativity into account:
  zero for value < 0, constant else

- This is the simplest "all purpose" approach

- Has been done before for parallaxes/distances, but there are two issues
  in what has been done (in the "EDSD estimator"):

  Issue 1: Distance-squared term in the prior for constant space density of stars

  Issue 2: Change of variable parallax – distance in pdf

Issue 1: distance-squared term

Homogeneous space density: $$\frac{\mathrm{d}N}{\mathrm{d}V} = \mathrm{const}$$

$$\mathrm{d}V = \mathrm{d}r\,\mathrm{d}A$$

In spherical coordinates: $$\mathrm{d}A = r^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$$

$$\frac{\mathrm{d}N}{\mathrm{d}r} \sim r^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$$

(Lutz & Kelker 1973)

Integrating over solid angle, normalising: *radial pdf for stars*

For normalisation –

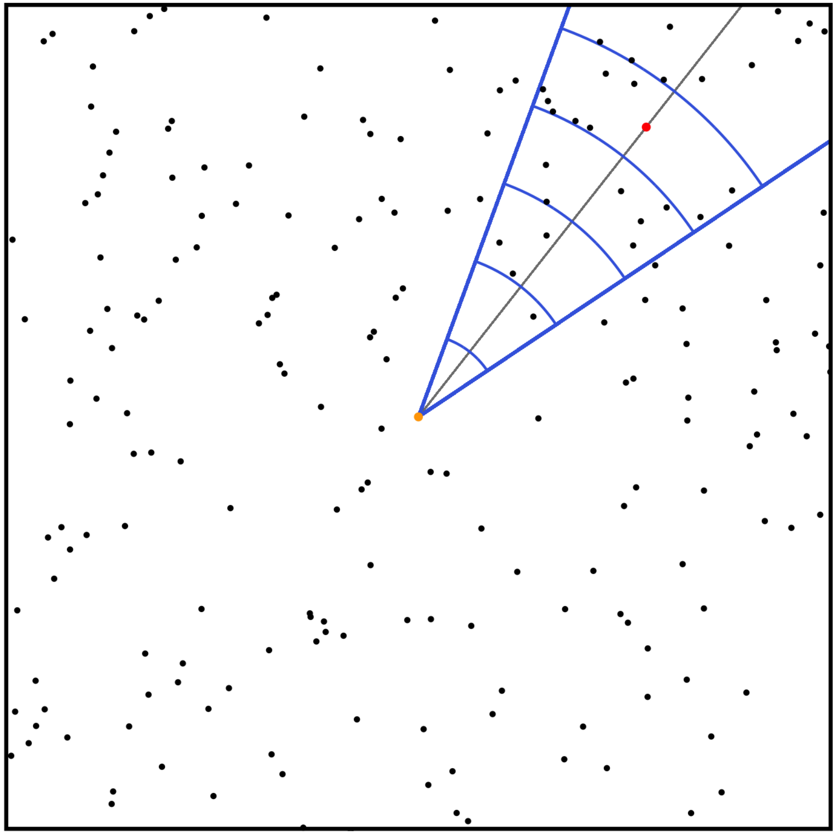
truncation at R: $$f_r(r) = \frac{3}{R^3}\,r^2$$

exponential decay: $$f_r(r) = \frac{1}{2\,L^3}\,r^2\,\mathrm{e}^{-\frac{r}{L}}$$

(Bailer-Jones 2015)

Use priors with distance square dependency, *not* a constant one?

Homogeneous space density



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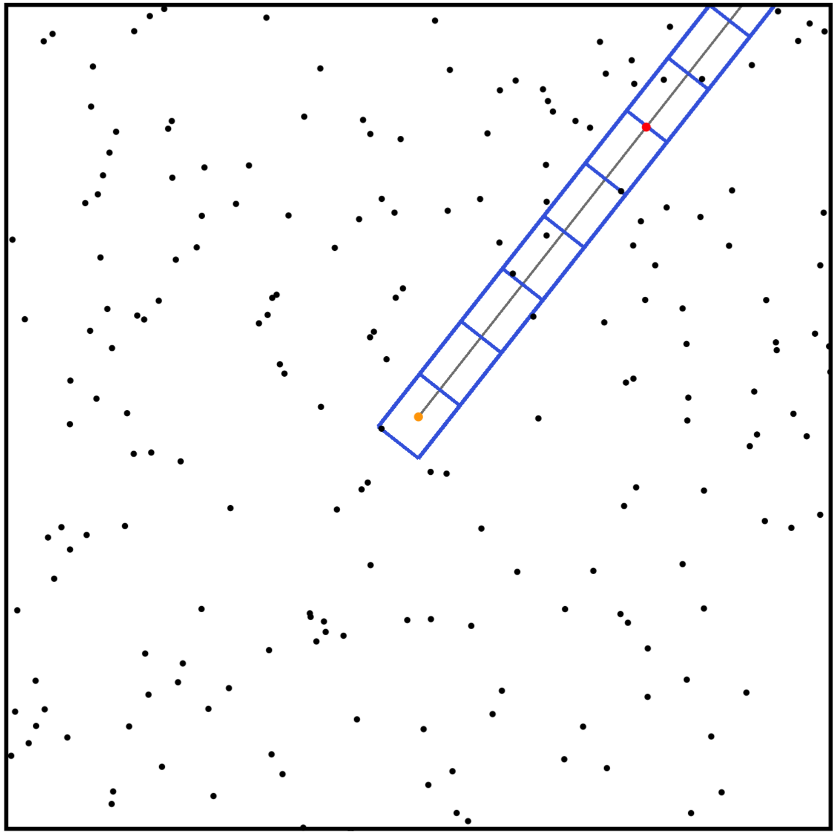
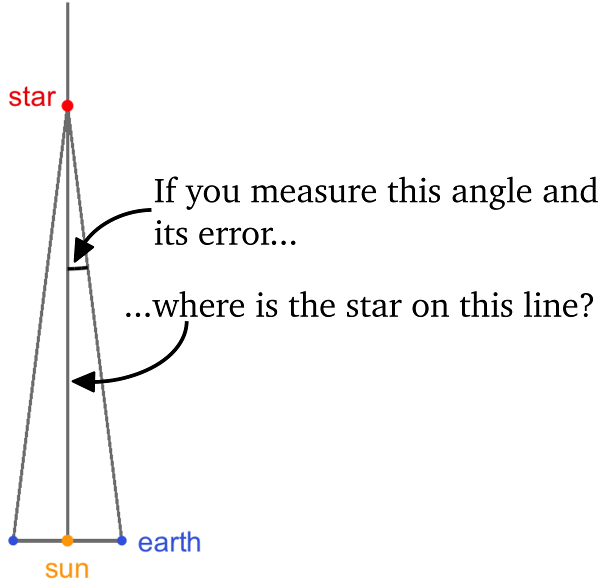


From parallax to distance in principle:



- ▷ No solid angle involved, no distance-squared term.

Issue 2: Change of variable

From Bailer-Jones, PASP, 127, 994 (2015):

$$P(\varpi|r,\sigma_\varpi) = \frac{1}{\sqrt{2\pi}\sigma_\varpi}\exp\left[-\frac{1}{2\sigma_\varpi^2}\left(\varpi - \frac{1}{r}\right)^2\right] \quad \text{where } \sigma_\varpi \geq 0 \tag{1}$$

$$P^*_{iu}(r|\varpi,\sigma_\varpi) = \begin{cases} P(\varpi|r,\sigma_\varpi) & \text{if } r > 0 \\ 0 & \text{otherwise} \end{cases}. \tag{7}$$

Examples of this posterior are shown in Figure 1 for $\varpi = 1/100$ and various values of f. This demonstrates the skewness discussed in § 2.2.

Inspection of equation (1) shows that

$$\lim_{r\to\infty} P^*_{iu}(r|\varpi,\sigma_\varpi) = \text{const.} \tag{8}$$

The posterior does not converge, has an infinite area, and so cannot be normalized. Consequently, it has no mean, no standard deviation, no median, and no quantiles. The only plausible

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Probabilities, *not* probability *densities* need to be equal when changing variables:

$$P(r\,|\,\varpi)\,\mathrm{d}r = P(\varpi\,|\,r)\,\mathrm{d}\varpi$$

$$P(r\,|\,\varpi) = P(\varpi\,|\,r)\,\frac{\mathrm{d}\varpi}{\mathrm{d}r}$$

Cf. standard transformation rule for pdfs:

$$y = g(x)$$

$$f_y(y) = f_x\left(g^{-1}(y)\right)\cdot\left|\frac{\mathrm{d}}{\mathrm{d}y}g^{-1}(y)\right|$$

This gives a $1/r^2$ term, and everything is nicely normalised...

From parallax to distance in practice:

- pdf for measured parallax is a normal distribution
- pdf for true parallax is a truncated normal distribution:

$$f_T(x) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\sigma}\,\frac{1}{1+\mathrm{erf}\left(\frac{\mu}{\sqrt{2}\sigma}\right)}\,\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-t^2}\,\mathrm{d}t$$

- pdf for the distance D is:

$$f_D(D) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\sigma_\varpi}\,\frac{1}{1+\mathrm{erf}\left(\frac{\mu_\varpi}{\sqrt{2}\sigma_\varpi}\right)}\,\frac{1}{D^2}\,\mathrm{e}^{-\frac{\left(\frac{1}{D}-\mu_\varpi\right)^2}{2\sigma_\varpi^2}}$$

for $D > 0$, *0* else

Distance distribution

Probability density function:

$$f_D(D) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\sigma_\varpi}\,\frac{1}{1+\mathrm{erf}\left(\frac{\mu_\varpi}{\sqrt{2}\,\sigma_\varpi}\right)}\,\frac{1}{D^2}\,\mathrm{e}^{-\frac{\left(\frac{1}{D}-\mu_\varpi\right)^2}{2\,\sigma_\varpi^2}}$$

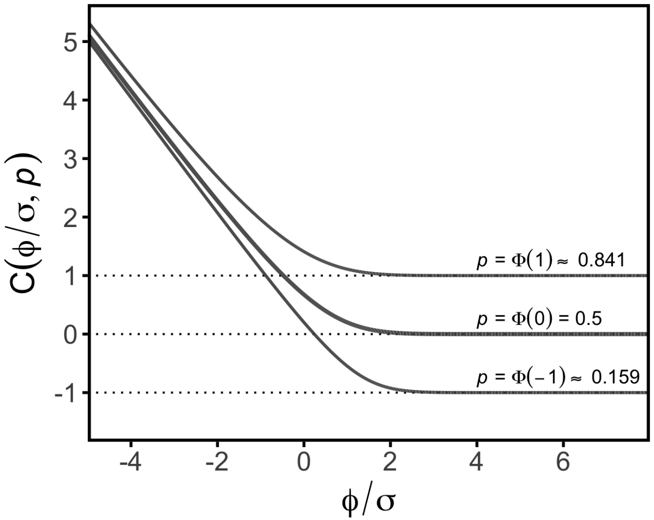
Cumulative distribution function:

$$F_D(D) = \frac{1}{1+\mathrm{erf}\left(\frac{\mu_\varpi}{\sqrt{2}\,\sigma_\varpi}\right)}\left[1-\mathrm{erf}\left(\frac{1}{\sqrt{2}\,\sigma_\varpi\,D}-\frac{\mu_\varpi}{\sqrt{2}\,\sigma_\varpi}\right)\right]$$

Quantile function:

$$D(F_D) = \frac{1}{\mu_\varpi+\sigma_\varpi\,C\left(\frac{\mu_\varpi}{\sigma_\varpi},F_D\right)}$$

with $C(x,p) = \sqrt{2}\cdot\mathrm{erf}^{-1}\left(1-p\cdot\left[1+\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right]\right)$



Distance estimator

From the measured parallax $\varpi \pm \sigma_\varpi$ estimate the distance as:

$$D = \frac{1}{\varpi + \sigma_\varpi C\left(\frac{\varpi}{\sigma_\varpi}, \frac{1}{2}\right)}$$

The lower and upper error intervals (consistent with std. dev of a normally distributed distance) are:

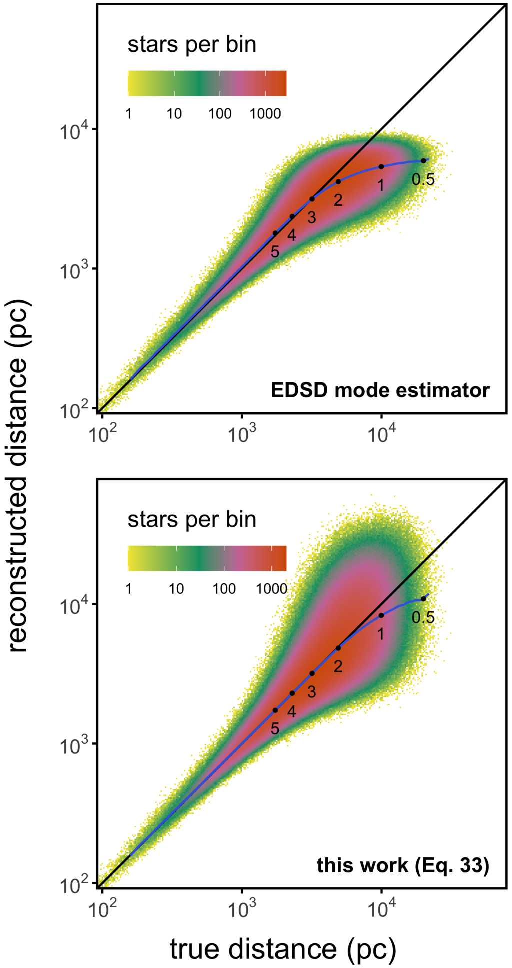
$$\sigma^- = D - \frac{1}{\varpi + \sigma_\varpi C\left(\frac{\varpi}{\sigma_\varpi}, \Phi(-1)\right)}$$

$$\sigma^+ = \frac{1}{\varpi + \sigma_\varpi C\left(\frac{\varpi}{\sigma_\varpi}, \Phi(1)\right)} - D$$

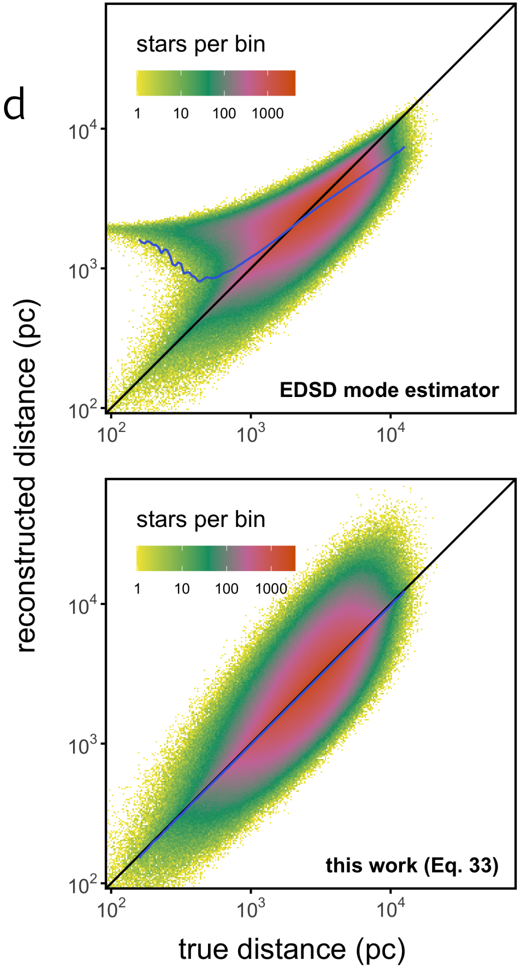
But feel free to use other quantiles for the confidence intervals if you prefer...

Prior comparison – simulations

5M sources, parallax error is quadratic sum of 10% of parallax and 0.1 mas error floor.

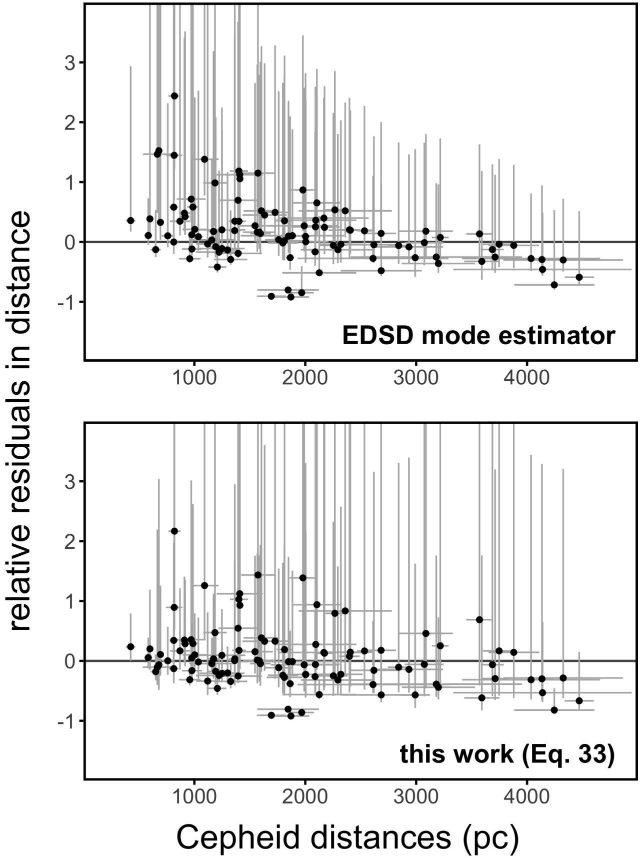


5M sources, uniformly distributed fractional error on interval [0.01, 1], no error floor.



Prior comparison – real data

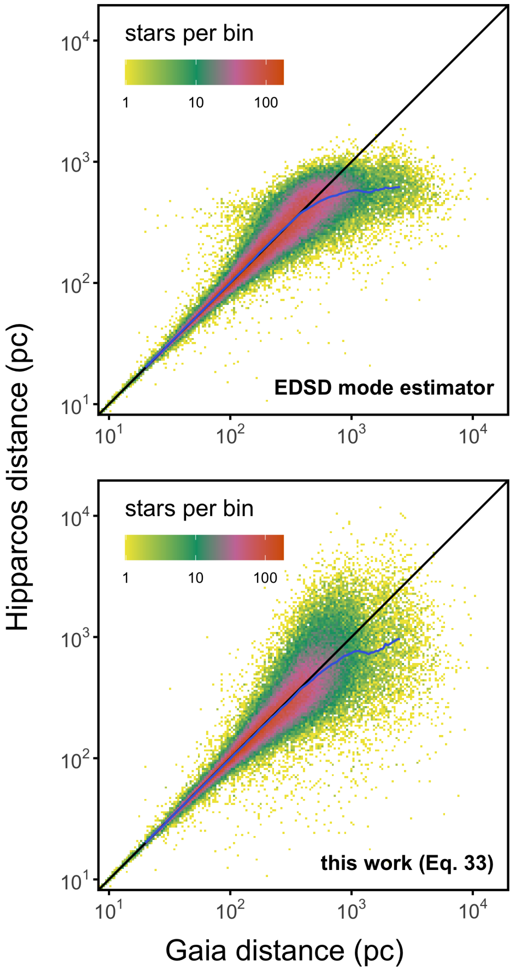
Cepheid example from
Astraatmadja & Bailer-Jones (2016)



mean +/- std. dev.:
0.16 +/- 0.54

mean +/- std. dev.:
0.04 +/- 0.51

Hipparcos – Gaia comparison

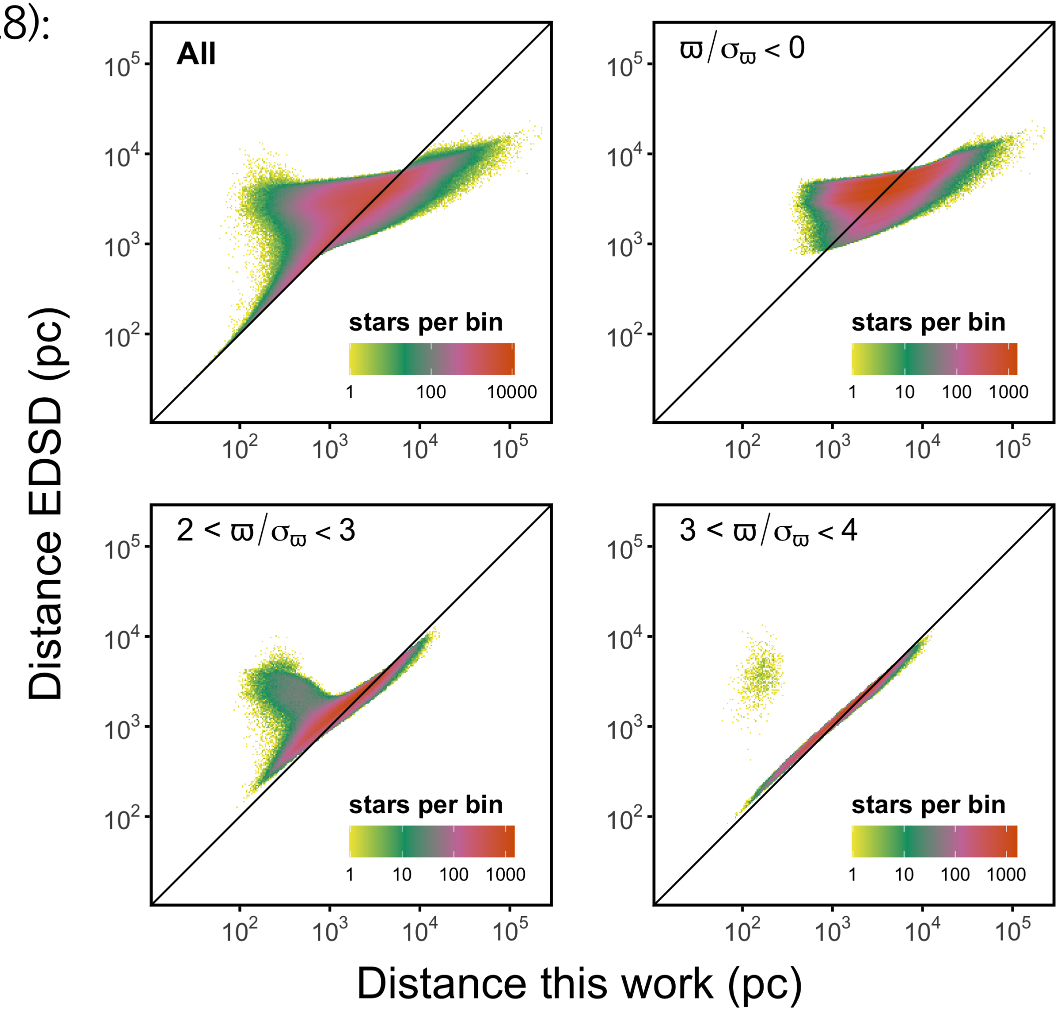


Prior comparison – real data

5M random sources from Bailer-Jones et al. (2018):

EDSD mode estimator vs. this work

▷ Different scale lengths are used by Bailer-Jones (2018), *(335 – 2600 pc, median 1473 pc)* producing the spread in the features.



Magnitude estimator

From the measured flux $\phi \pm \sigma$ estimate the magnitude as:

$$m = -2.5 \cdot \log_{10}\left(\phi + \sigma \cdot C\left(\frac{\phi}{\sigma}, \frac{1}{2}\right)\right) + zp$$

The lower and upper error intervals (consistent with std. dev of a normally distributed distance) are:

$$\sigma^{-} = m + 2.5 \cdot \log_{10}\left(\phi + \sigma \cdot C\left(\frac{\phi}{\sigma}, \Phi(-1)\right)\right) + zp$$

$$\sigma^{+} = -2.5 \cdot \log_{10}\left(\phi + \sigma \cdot C\left(\frac{\phi}{\sigma}, \Phi(1)\right)\right) + zp - m$$

Colour and absolute magnitude

Combine pdfs for two magnitudes to pdf of the colour,
pdfs for magnitude and distance to pdf of absolute magnitude
via convolution:

$$s_i = \frac{\mu_i}{\sigma_i}\,,\; i = 1,2$$

$$a = \left(\frac{10^{0.4\,zp_1}}{\sigma_1}\right)^2 + \left(\frac{10^{0.4\,(zp_2+z)}}{\sigma_2}\right)^2$$

$$b = \frac{s_1}{\sigma_1}\,10^{0.4\,zp_1} + \frac{s_2}{\sigma_2}\,10^{0.4\,(zp_2+z)}$$

pdf of colour distribution:

$$f_z(z) = \frac{2}{\pi}\,\frac{\ln(10)}{2.5}\,\frac{1}{\sigma_1\,\sigma_2}\,\frac{1}{\left[1+\mathrm{erf}\left(\frac{s_1}{\sqrt{2}}\right)\right]\left[1+\mathrm{erf}\left(\frac{s_2}{\sqrt{2}}\right)\right]}\,10^{0.4\,(zp_1+zp_2+z)}\,\frac{1}{a}\,\mathrm{e}^{\frac{b^2}{2a}-\frac{s_1^2+s_2^2}{2}}\left(\mathrm{e}^{-\frac{b^2}{2a}} + \sqrt{\frac{\pi}{2\,a}}\,b\left[1+\mathrm{erf}\left(\frac{b}{\sqrt{2\,a}}\right)\right]\right)$$

$$s_\phi = \frac{\mu_\phi}{\sigma_\phi}\,,\; s_\varpi = \frac{\mu_\varpi}{\sigma_\varpi}$$

$$a' = \frac{\left(10^{0.4\,zp}\right)^2}{2\,\sigma_\phi^2}$$

$$b' = \frac{\left(10^{0.2\,M-1}\right)^2}{2\,\sigma_\varpi^2} - \frac{s_\phi}{\sigma_\phi}\,10^{0.4\,zp}$$

$$c' = \frac{s_\varpi}{\sigma_\varpi}\,10^{0.2\,M-1}$$

pdf of absolute magnitude distribution:

$$f_M(M) = \frac{2}{\pi}\,\frac{\ln(10)}{2.5}\,\frac{1}{\sigma_\phi\,\sigma_\varpi}\,\frac{1}{1+\mathrm{erf}\left(\frac{s_\phi}{\sqrt{2}}\right)}\,\frac{1}{1+\mathrm{erf}\left(\frac{s_\varpi}{\sqrt{2}}\right)}\,\mathrm{e}^{-\frac{1}{2}\left(s_\phi^2+s_\varpi^2\right)}\,10^{0.4\,zp+0.2\,M-1}\int_0^\infty x^2\,\mathrm{e}^{-a'\,x^4-b'\,x^2+c'\,x}\,\mathrm{d}x$$

Summary

- Where possible, work with parallaxes or fluxes

- $$D = \frac{1}{\varpi + \sigma_\varpi C\left(\frac{\varpi}{\sigma_\varpi}, \frac{1}{2}\right)}$$

  $$m = -2.5 \cdot \log_{10}\left(\phi + \sigma \cdot C\left(\frac{\phi}{\sigma}, \frac{1}{2}\right)\right) + zp$$

  $$C(x, p) = \sqrt{2} \cdot \mathrm{erf}^{-1}\left(1 - p \cdot \left[1 + \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right]\right)$$

  are good "all purpose" estimators for distance and magnitudes
  (there might be better ones for particular problems, but these can always be used)

- Priors do no wonders, if there's no more signal, the distribution will go to a limit distribution which will only depend on the noise. Be aware.

The End.

Results for magnitudes from fluxes

Probability density function:

$$f_m(m) = \sqrt{\frac{2}{\pi}} \frac{\ln(10)}{2.5} \frac{1}{\sigma} \frac{1}{1 + \mathrm{erf}\left(\frac{\mu}{\sqrt{2}\sigma}\right)} x\, \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad x = 10^{0.4\,(zp-m)}$$

Cumulative distribution function:

$$F_m(m) = \frac{1}{1 + \mathrm{erf}\left(\frac{\mu}{\sqrt{2}\,\sigma}\right)} \left[1 - \mathrm{erf}\left(\frac{1}{\sqrt{2}\,\sigma} \left(10^{0.4\,(zp-m)} - \mu\right)\right)\right]$$

Quantile function:

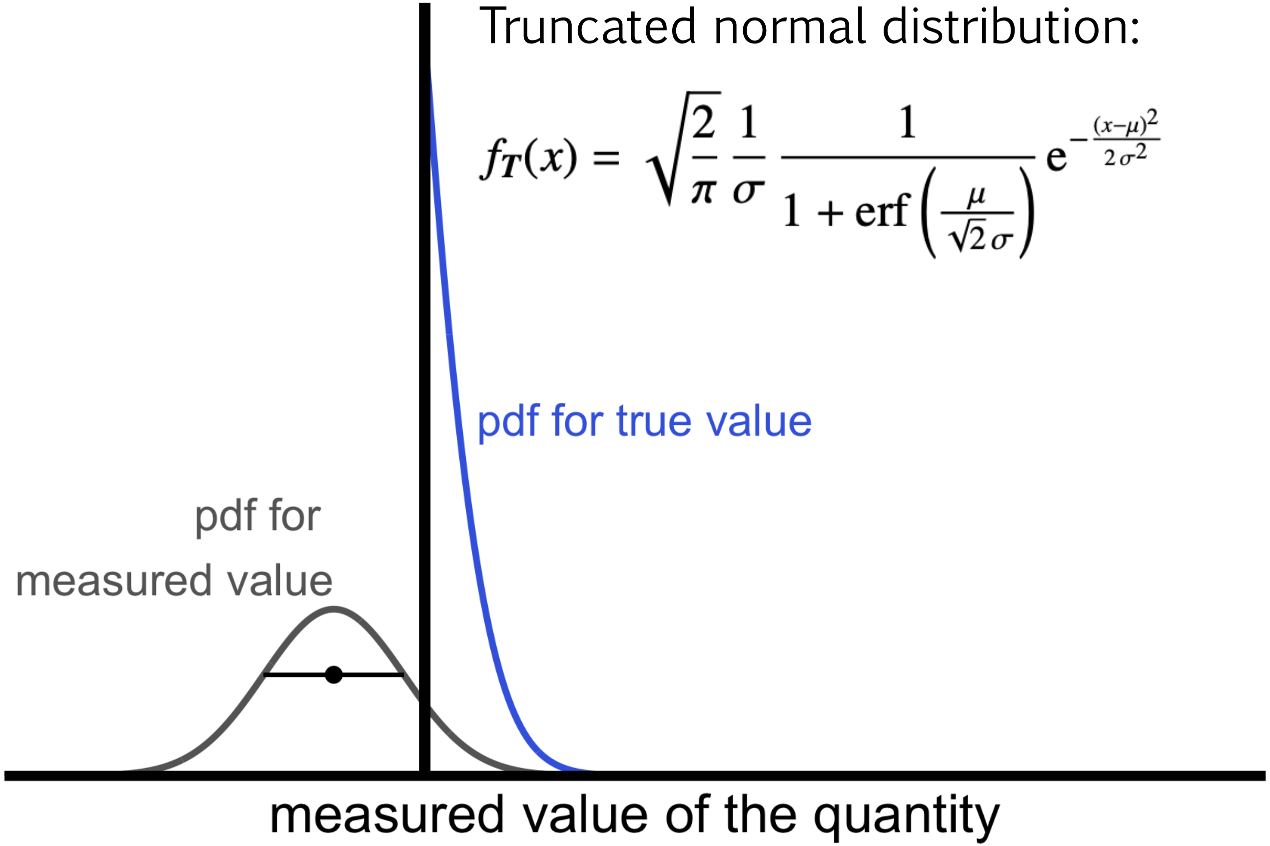
$$m(F_m) = -2.5 \cdot \log_{10}\left(\mu + \sigma \cdot C\left(\frac{\mu}{\sigma}, F_{\mathbf{m}}\right)\right) + zp$$

Measurements and probability

Truncated normal distribution:

$$f_T(x) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\sigma}\,\frac{1}{1+\mathrm{erf}\left(\frac{\mu}{\sqrt{2}\,\sigma}\right)}\,\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-t^2}\,\mathrm{d}t$$



Homogeneous space density

Example pdfs for
the distance of a
star with ϖ 10 mas
and different S/N

Using
*“exponential decay
space density prior”*

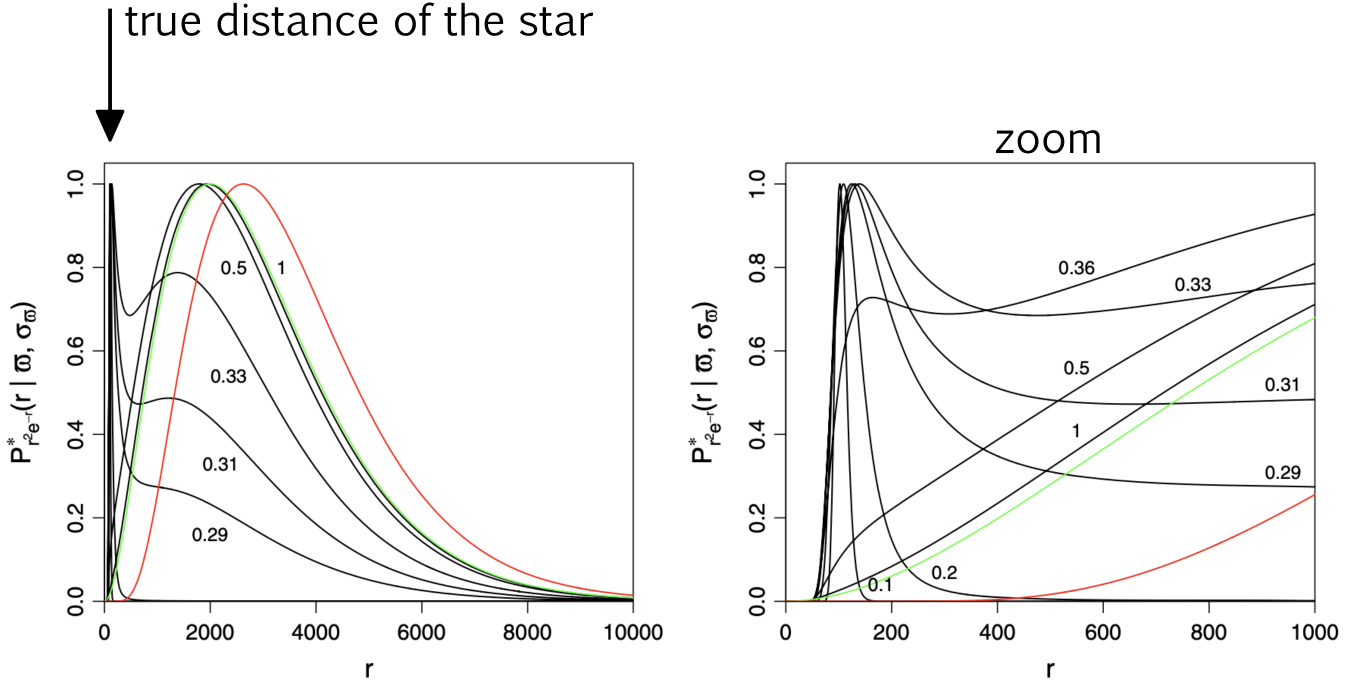


Fig. 12 from Bailer-Jones, PASP, 127, 994 (2015)

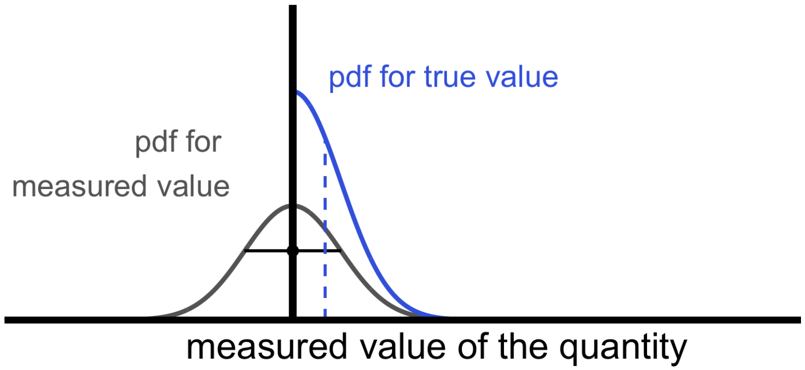
- ▷ pdfs extend way too far out
- ▷ mean and median are not suitable as estimators
- ▷ Use (inner) mode as estimator → *actually not using the prior!*

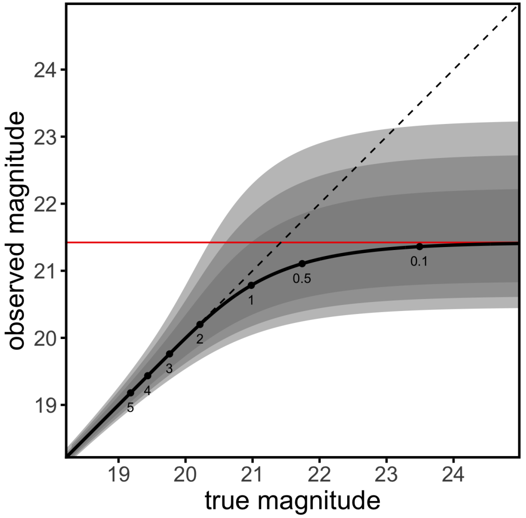
Limit distributions

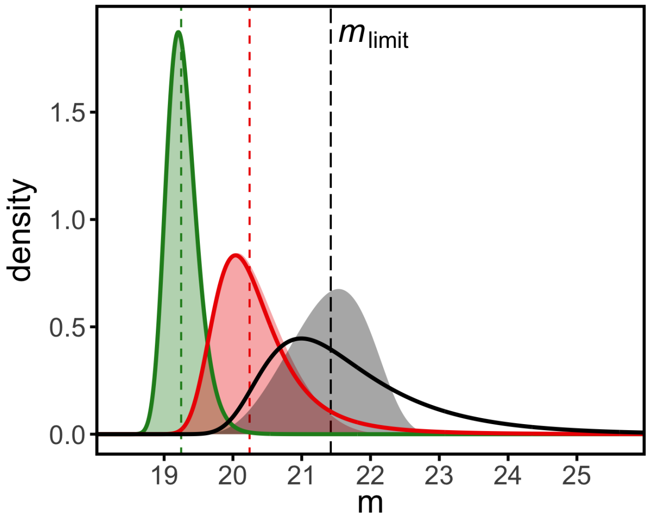
The distributions for distance and magnitude converge to limit distributions for vanishing signals.

The medians are:

$$m_{\text{limit}} = -2.5 \cdot \log_{10}\left(\sigma\ \sqrt{2}\,\text{erf}^{-1}\left(\frac{1}{2}\right)\right) + zp \qquad D_{\text{limit}} = \frac{1}{\sigma\ \sqrt{2}\,\text{erf}^{-1}\left(\frac{1}{2}\right)}$$







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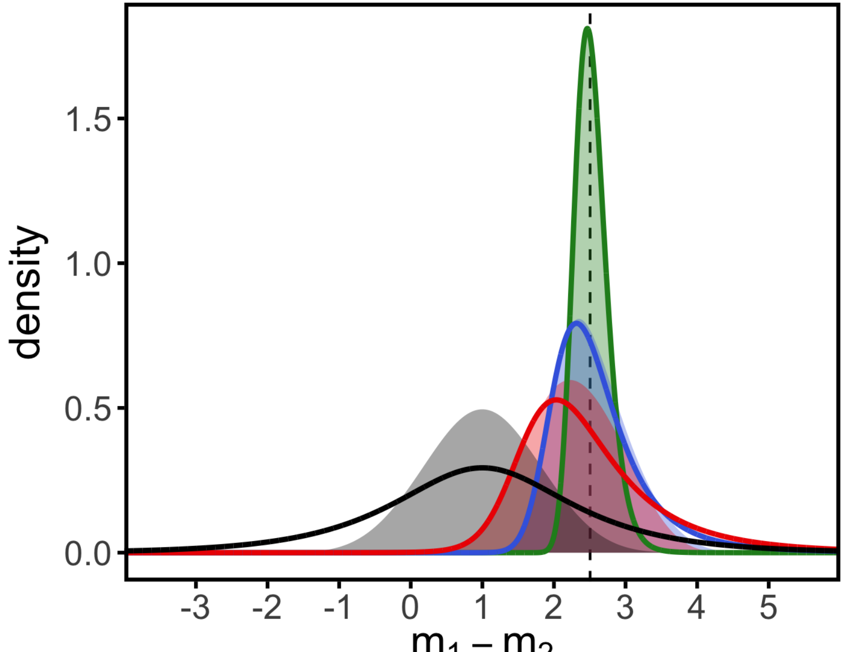
Examples for colour distributions:



Examples for absolute magnitude distributions: