# EW SMEFT – Exercise sheet 2 – Solutions

## The $W$ mass in SMEFT

1. (a)

$$m_W^2 = \frac{g^2}{v_T^2}4 = \frac{\hat{g}^2\hat{v}_T^2}{4}\left[1 + 2\frac{\delta g}{g} + 2\frac{\delta v_T}{v_T}\right] \tag{1}$$

   (b)

$$\begin{aligned} m_W^2 &= \frac{\hat{g}^2\hat{v}_T^2}{4}\left[1 - \frac{c_\theta^2}{c_{2\theta}}\Delta m_Z^2 + \frac{s_\theta^2}{c_{2\theta}}(\Delta\alpha - \Delta G_F)\right] \quad (2)\\ &= \frac{\hat{g}^2\hat{v}_T^2}{4}\left[1 - \frac{1}{c_{2\theta}}\left(s_{2\theta}\bar{C}_{HWB} + \frac{c_\theta^2}{2}\bar{C}_{HD} + s_\theta^2(2\bar{C}_{Hl}^{(3)} - \bar{C}'_{ll})\right)\right] \quad (3)\end{aligned}$$

   (c)

$$m_W^2 = \frac{\hat{g}^2\hat{v}_T^2}{4}\left[1 - \frac{\Delta G_F}{2} + \frac{\Delta G_F}{2}\right] = \frac{\hat{g}^2\hat{v}_T^2}{4} \tag{4}$$

2. (a) From the definition:

$$\begin{aligned} \cos^2\theta &= \frac{g^2}{g^2 + (g')^2} - \frac{s_{4\theta}}{4}\bar{C}_{HWB} \quad (5)\\ &= \frac{\hat{g}^2}{\hat{g}^2 + (\hat{g}')^2}\left[1 + 2s_\theta^2\left(\frac{\delta g}{g} - \frac{\delta g'}{g'}\right)\right] - \frac{s_{4\theta}}{4}\bar{C}_{HWB} \quad (6)\end{aligned}$$

   (b) $\{\alpha, G_F, m_Z\}$ scheme:

$$\begin{aligned} \cos^2\theta &= \frac{\hat{g}^2}{\hat{g}^2 + (\hat{g}')^2}\left[1 + \frac{s_\theta^2}{c_{2\theta}}(\Delta\alpha - \Delta G_F - \Delta m_Z^2) - \frac{s_{4\theta}}{4c_\theta^2}\bar{C}_{HWB}\right] \quad (7)\\ &= \frac{\hat{g}^2}{\hat{g}^2 + (\hat{g}')^2}\left[1 + \frac{s_\theta^2}{c_{2\theta}}\left(-\frac{1}{2}\bar{c}_{HD} - 2\bar{C}_{Hl}^{(3)} + \bar{C}'_{ll} - \frac{2}{s_{2\theta}}\bar{C}_{HWB}\right)\right] \quad (8)\end{aligned}$$

   $\{m_W, G_F, m_Z\}$ scheme:

$$\begin{aligned} \cos^2\theta &= \frac{\hat{g}^2}{\hat{g}^2 + (\hat{g}')^2}\left[1 + \Delta m_Z^2 - \frac{s_{4\theta}}{4c_\theta^2}\bar{C}_{HWB}\right] \quad (9)\\ &= \frac{\hat{g}^2}{\hat{g}^2 + (\hat{g}')^2}\left[1 + \frac{\bar{C}_{HD}}{2} + t_\theta\bar{C}_{HWB}\right] \quad (10)\end{aligned}$$

(c) The formula is easy to verify using the expressions in terms of Δ's.
As an intuitive interpretation, it can be derived differentiating the SM relation

$$m_W^2 = c_\theta^2 m_Z^2 \,. \tag{11}$$

and noting that $\Delta m_Z^2$ enters with a minus sign because it's the contribution to the input quantity, and not a predicted shift. The term in $\bar{C}_{HWB}$ is practically subtracting the genuine correction to $\theta$ from $\delta c_\theta^2/c_\theta^2$. This makes sense because the $c_\theta$ we have in the SM relation actually stands for the ratio $g^2/(g^2+(g')^2)$, and is unrelated to the mass diagonalization in the neutral gauge sector. Since the $\bar{C}_{HWB}$ dependence comes from the latter, it has to cancel in this equation.

(d)

$$\rho = \frac{1}{g^2}\frac{g^2}{c_\theta^2}\left(1+\frac{t_\theta}{2}\bar{C}_{HWB}\right)^2 \frac{g^2 v_T^2}{4}\frac{4}{(g^2+(g')^2)v_T^2(1+\Delta m_Z^2)} \tag{12}$$

$$= 1 - \Delta m_Z^2 + 2\frac{gg'}{g^2+(g')^2}\bar{C}_{HWB} \tag{13}$$

$$= 1 - \frac{\bar{C}_{HD}}{2} \tag{14}$$

(e)

$$\rho = \frac{\hat{g}^2+(\hat{g}')^2}{\hat{g}^2}\left[1+\frac{2(\hat{g}')^2}{\hat{g}^2+(\hat{g}')^2}\left(-\frac{\delta g}{g}+\frac{\delta g'}{g'}\right)+\frac{2\hat{g}\hat{g}'}{\hat{g}^2+(\hat{g}')^2}\bar{C}_{HWB}\right]\frac{\hat{g}^2(1+\delta m_W^2/m_W^2)}{\hat{g}^2+(\hat{g}')^2}\times$$

$$\times\left[1+2c_\theta^2\frac{\delta g}{g}+s_\theta^2 2\frac{\delta g'}{g'}+2\frac{\delta v_T}{v_T}+\Delta m_Z^2\right]^{-1} \tag{15}$$

$$= 1 - 2\left(\frac{\delta g}{g}+\frac{\delta v_T}{v_T}\right) - \Delta m_Z^2 + s_{2\theta}\bar{C}_{HWB} + \frac{\delta m_W^2}{m_W^2} \tag{16}$$

(f) $\{\alpha, G_F, m_Z\}$ scheme:

$$\rho = 1 + \frac{s_\theta^2}{c_{2\theta}}\left[\Delta m_Z^2 + \Delta G_F - \Delta\alpha\right] + s_{2\theta}\bar{C}_{HWB} - \frac{c_\theta^2}{c_{2\theta}}\Delta m_Z^2 + \frac{s_\theta^2}{c_{2\theta}}(\Delta\alpha - \Delta G_F) \tag{17}$$

$$= 1 - \Delta m_Z^2 + s_{2\theta}\bar{C}_{HWB} = 1 - \frac{\bar{C}_{HD}}{2} \tag{18}$$

$\{m_W, G_F, m_Z\}$ scheme:

$$\rho = 1 - \Delta m_Z^2 + s_{2\theta}\bar{C}_{HWB} + 0 \tag{19}$$

$$= 1 - \frac{\bar{C}_{HD}}{2} \tag{20}$$

(g)

$$(\rho - 1) = \frac{\delta m_W^2}{m_W^2} - \frac{\delta c_\theta^2}{c_\theta^2} + t_\theta\bar{C}_{HWB} = s_{2\theta}\bar{C}_{HWB} - \Delta m_Z^2 \tag{21}$$

This holds in any input scheme: in fact we can get it directly from (16) and (6).

## The W decay width in SMEFT

3.

$$|A(W^- \to e^- \nu_e)|^2 = \frac{1}{3} g^2 m_W^2 \left[ 1 + 2\frac{\delta g}{g} + \bar{2} C_{Hl}^{(3)} \right] \tag{22}$$

4.

$$|A(W^- \to \bar{u} d)|^2 = g^2 m_W^2 \left[ 1 + 2\frac{\delta g}{g} + \bar{2} C_{Hq}^{(3)} \right] \tag{23}$$

where this time the 3 from the average over polarizations goes away with the quark color factor.

5.

$$\Gamma(W^- \to e^- \nu_e) = \frac{|A|^2}{16\pi m_W} = \frac{g^2 \hat{m}_W}{48\pi} \left[ 1 + 2\frac{\delta g}{g} + \bar{2} C_{Hl}^{(3)} + \frac{\delta m_W}{m_W} \right] \tag{24}$$

$$\Gamma(W^- \to \bar{u} d) = \frac{|A|^2}{16\pi m_W} = \frac{g^2 \hat{m}_W}{16\pi} \left[ 1 + 2\frac{\delta g}{g} + \bar{2} C_{Hq}^{(3)} + \frac{\delta m_W}{m_W} \right] \tag{25}$$

where we have noted that $|A|^2$ was computed in terms of the Lagrangian parameter $m_W$, and we have replaced it here with the $\hat{m}_W(1 + \delta m_W / m_W)$.

6.

$$\Gamma_W = 3\Gamma(W^- \to e^- \nu_e) + 2\Gamma(W^- \to \bar{u} d) \tag{26}$$

$$= \frac{3 g^2 \hat{m}_W}{16\pi} \left[ 1 + \frac{1}{3}\left( 2\frac{\delta g}{g} + \bar{2} C_{Hl}^{(3)} + \frac{\delta m_W}{m_W} \right) + \frac{2}{3}\left( 2\frac{\delta g}{g} + \bar{2} C_{Hq}^{(3)} + \frac{\delta m_W}{m_W} \right) \right] \tag{27}$$

$$= \frac{3 g^2 \hat{m}_W}{16\pi} \left[ 1 + 2\frac{\delta g}{g} + \frac{\delta m_W}{m_W} + \frac{2}{3}(\bar{C}_{Hl}^{(3)} + 2\bar{C}_{Hq}^{(3)}) \right] \tag{28}$$

$\{\alpha, G_F, m_Z\}$ scheme:

$$\frac{\delta \Gamma_W}{\Gamma_W} = \frac{3}{2 c_{2\theta}} \left[ s_\theta^2 \Delta\alpha - c_\theta^2 (\Delta m_Z^2 + \Delta G_F) \right] + \frac{1}{2}\Delta G_F + \frac{2}{3}\left( \bar{C}_{Hl}^{(3)} + 2\bar{C}_{Hq}^{(3)} \right) \tag{29}$$

$$= -\frac{3 t_{2\theta}}{2} \bar{C}_{HWB} - \frac{3}{4}\frac{c_\theta^2}{c_{2\theta}} \bar{C}_{HD} + \left( \frac{3 c_\theta^2}{c_{2\theta}} - 1 \right) \frac{\bar{C}'_{ll}}{2} - \left( \frac{3 c_\theta^2}{c_{2\theta}} - \frac{5}{3} \right) \bar{C}_{Hl}^{(3)} + \frac{4}{3} \bar{C}_{Hq}^{(3)} \tag{30}$$

$\{m_W, G_F, m_Z\}$ scheme:

$$\frac{\delta \Gamma_W}{\Gamma_W} = -\Delta G_F + \frac{2}{3}\left( \bar{C}_{Hl}^{(2)} + 2\bar{C}_{Hq}^{(3)} \right) \tag{31}$$

$$= \bar{C}'_{ll} + \frac{4}{3}\left( \bar{C}_{Hq}^{(3)} - \bar{C}_{Hl}^{(3)} \right) \tag{32}$$

## Jacobian formulation of input shifts

7. The Jacobian and the input Δ's are

$$J_{mW} = \frac{gv_T^2}{2}\begin{pmatrix}1 & & \frac{g}{v_T} \\ 1 & \frac{g'}{g} & \frac{g^2+(g')^2}{gv_T} \\ & & -\frac{2\sqrt{2}}{gv_T^5}\end{pmatrix} \tag{33}$$

$$\Delta\vec{O}_{mW} = \begin{pmatrix}0 & \frac{v_T^2(g^2+(g')^2}{4}\Delta m_Z^2, & \frac{\Delta G_F}{\sqrt{2}v_T^2}\end{pmatrix}^T \tag{34}$$

Inverting the Jacobian:

$$J_{mW}^{-1} = \frac{2}{gv_T^2}\begin{pmatrix}1 & & \frac{g^2v_T^4}{2\sqrt{2}} \\ -\frac{g}{g'} & \frac{g}{g'} & \frac{gg'v_T^4}{2\sqrt{2}} \\ & & -\frac{gv_T^5}{2\sqrt{2}}\end{pmatrix} \tag{35}$$

hence

$$\vec{G} = \hat{\vec{G}} + \begin{pmatrix}-\frac{1}{2}g\Delta G_F \\ -\frac{(g^2+(g')^2)}{2g'}\Delta m_Z^2 - \frac{g'}{2}\Delta G_F \\ \frac{v_T}{2}\Delta G_F\end{pmatrix} = \begin{pmatrix}\hat{g}\,[1-\Delta G_F/2] \\ \hat{g}'[1-\Delta m_Z^2/(2s_\theta^2)-\Delta G_F/2] \\ \hat{v}_T[1+\Delta G_F/2]\end{pmatrix} \tag{36}$$

8.

$$\frac{\partial\vec{O}_\alpha}{\partial\vec{O}_{mW}} = \begin{pmatrix}\frac{\partial\alpha}{\partial m_W^2} & \frac{\partial\alpha}{\partial m_Z^2} & \frac{\partial\alpha}{\partial G_F} \\ & 1 & \\ & & 1\end{pmatrix} = \begin{pmatrix}-\frac{c_{2\theta}}{\pi v_T^2} & \frac{c_\theta^4}{\pi v_T^2} & \frac{g^2v_T^2s_\theta^2}{2\sqrt{2}\pi} \\ & 1 & \\ & & 1\end{pmatrix} \tag{37}$$

then

$$J_\alpha = \frac{\partial\vec{O}_\alpha}{\partial\vec{O}_{mW}}J_{mW} = \frac{1}{2}\begin{pmatrix}\frac{g(g')^4}{(g^2+(g')^2)^2\pi} & \frac{g^4(g')}{(g^2+(g')^2)^2\pi} & \\ gv^2 & g'v^2 & (g^2+(g')^2)v \\ & & -\frac{2/\sqrt{2}}{v^3}\end{pmatrix} \tag{38}$$

The inverse Jacobian is

$$J_\alpha^{-1} = \frac{1}{c_{2\theta}}\begin{pmatrix}-\frac{2\pi}{g} & \frac{2c_\theta^4}{gv_T^2} & \frac{gv_T^2c_\theta^2}{\sqrt{2}} \\ \frac{2\pi}{gt_\theta} & -\frac{s_{2\theta}s_\theta^2}{gv_T^2} & -\frac{gv_T^2s_\theta^2t_\theta}{\sqrt{2}} \\ 0 & 0 & -\frac{v_T^2c_{2\theta}}{\sqrt{2}}\end{pmatrix} \tag{39}$$

and with the observables corrections

$$\Delta\vec{O}_\alpha = \begin{pmatrix}\frac{g^2(g')^2}{4\pi(g^2+(g')^2)}\Delta\alpha & \frac{g^2+(g')^2v_T^2}{4}\Delta m_Z^2 & \frac{1}{\sqrt{2}v_T^3}\Delta G_F\end{pmatrix}^T \tag{40}$$

one finally obtains

$$\vec{G} = \hat{\vec{G}} + \begin{pmatrix}g(s_\theta^2\Delta\alpha - c_\theta^2(\Delta m_Z^2+\Delta G_F)/(2c_{2\theta}) \\ g'(s_\theta^2(\Delta m_Z^2+\Delta G_F) - c_\theta^2\Delta\alpha)/(2c_{2\theta}) \\ v_T\Delta G_F/2\end{pmatrix} \tag{41}$$

which is the same result as in the appendix.

9. Let's use the Jacobian to define a new input scheme: $\vec{O}_{\text{new}} = \{m_W^2, m_Z^2, \alpha\}$.
   Start from the $\vec{O}_{mW}$ set, and replace $G_F$ with $\alpha$.

   (a) The SM solutions are

$$\hat{g} = 2\sqrt{\frac{\pi\alpha}{1 - m_W^2/m_Z^2}} \qquad \hat{g}' = \frac{2m_Z\sqrt{\pi\alpha}}{m_W} \qquad \hat{v} = m_W\sqrt{\frac{1 - m_W^2/m_Z^2}{\pi\alpha}} \tag{42}$$

   (b)

$$J_{\text{new}} = \begin{pmatrix} 1 & & \\ & 1 & \\ -\frac{c_{2\theta}}{\pi v_T^2} & \frac{c_\theta^4}{\pi v_T^2} & \frac{g^2 v_T^2 s_\theta^2}{2\sqrt{2}\pi} \end{pmatrix} J_{mW} = \frac{g v_T^2}{2} \begin{pmatrix} 1 & 0 & \frac{g}{v_T} \\ 1 & t_\theta & \frac{g}{v_T c_\theta^2} \\ \frac{s_\theta^4}{\pi v_T^2} & \frac{s_{2\theta} c_\theta^2}{2\pi v_T^2} & 0 \end{pmatrix} \tag{43}$$

   Therefore

$$J_{\text{new}}^{-1} = \frac{2}{g v_T^2} \begin{pmatrix} \frac{1}{t_\theta^2} & -\frac{c_\theta^4}{s_\theta^2} & \frac{\pi v_T^2}{s_\theta^2} \\ -t_\theta & \frac{s_{2\theta}}{2} & \frac{2\pi v_T^2}{s_{2\theta}} \\ -\frac{c_{2\theta}}{s_\theta^2}\frac{v_T}{g} & \frac{c_\theta^4}{s_\theta^2}\frac{v_T}{g} & -\frac{\pi v_T^3}{g s_\theta^2} \end{pmatrix} \tag{44}$$

   (c)

$$\vec{G} = \hat{\vec{G}} + \frac{1}{2} \begin{pmatrix} \hat{g}(\Delta m_Z^2/t_\theta^2 - \Delta\alpha) \\ -\hat{g}'(\Delta m_Z^2 + \Delta\alpha) \\ \hat{v}_T(\Delta\alpha - \Delta m_Z^2/t_\theta^2) \end{pmatrix} \tag{45}$$

   Hence, in this scheme

$$\frac{\delta g}{g} = \frac{1}{2}\left(\frac{\Delta m_Z^2}{t_\theta^2} - \Delta\alpha\right) = \frac{1}{t_\theta}\bar{C}_{HWB} + \frac{1}{4t_\theta^2}\bar{C}_{HD} \tag{46}$$

$$\frac{\delta g'}{g'} = -\frac{1}{2}(\Delta m_Z^2 + \Delta\alpha) = -\frac{\bar{C}_{HD}}{4} \tag{47}$$

$$\frac{\delta v_T}{v_T} = \frac{1}{2}\left(\Delta\alpha - \frac{\Delta m_Z^2}{t_\theta^2}\right) = -\frac{1}{t_\theta}\bar{C}_{HWB} - \frac{1}{4t_\theta^2}\bar{C}_{HD} \tag{48}$$