EW SMEFT – Exercise sheet 2

The W mass in SMEFT

1. Compute the correction to m_W^2 in the Warsaw basis, at tree level and up to Λ^{-4} corrections.

   (a) Remember that, at LO, the W pole mass in the SMEFT Lagrangian is just $m_W^2 = g^2 v_T^2/4$, and leave the result written in terms of $\delta g/g$ etc.

   (b) Using the shift formulas in the appendix, specialize the result to the input parameters set $\{\alpha, G_F, m_Z\}$ and write the correction to m_W in terms of the Wilson coefficients.

   (c) Do the same for the input parameters set $\{m_W, G_F, m_Z\}$. Verify that the correction to m_W vanishes in this case, as expected for any input quantities,

2. Now we will relate the m_W correction to the ρ parameter.

   (a) Compute the relative correction to $\cos^2\theta$ (i.e. $\delta c_\theta^2/c_\theta^2$) in the Warsaw basis, again at LO and up to Λ^{-4} corrections.
   Remember that the angle is defined by

   $$\theta = \arctan\left[\frac{g'}{g} + \frac{1}{2}\frac{gg'}{g^2+(g')^2}\bar{C}_{HWB}\right] \tag{1}$$

   As above, leave the result in the inputs-independent form, as a function of $\delta g/g$ etc.

   (b) Write $\delta c_\theta^2/c_\theta^2$ in terms of Wilson coefficients, specializing to the $\{\alpha, G_F, m_Z\}$ and $\{m_W, G_F, m_Z\}$ input schemes.

   (c) By comparing with the result of the previous exercise, verify that the following relation holds in both schemes

   $$\frac{\delta m_W^2}{m_W^2} = -\Delta m_Z^2 + \frac{\delta c_\theta^2}{c_\theta^2} + \frac{s_{4\theta}}{4c_\theta^2}\bar{c}_{HWB} \tag{2}$$

   Can you give an intuitive interpretation of this formula?

   (d) The ρ parameter can be defined *à la Veltman*, from the ratio of Z (neutral) and W (charged) currents, i.e.:

   $$\rho \equiv \frac{g_Z^2}{g^2}\frac{m_W^2}{m_Z^2} \tag{3}$$

   Where m_Z, m_W are the pole masses and g_Z is defined such that, in unitary gauge, the covariant derivative for a chiral fermionic field ψ contains the term

   $$D_\mu\psi = -ig_Z Z_\mu\left(T_3 - Qs_\theta^2\right)\psi + \dots \tag{4}$$

   being $T_3 = \pm 1/2$ and Q the isospin and electric charge of ψ respectively.

Compute ρ to order Λ^{-2} in the Warsaw basis.
Use the Lagrangian expression *before* defining the input parameters and write the result in terms of Wilson coefficients.
Hint: remember from lecture 1 that a generic covariant derivative contains the term

$$D_\mu \psi = -i\frac{g}{c_\theta} Z_\mu \left(T_3 - Q s_\theta^2\right)\left[1 + \frac{t_\theta}{2}\bar{C}_{HWB}\right]\psi + \dots \tag{5}$$

having defined the angle θ as in (1).

(e) Now repeat the calculation with the Lagrangian defined *after* defining inputs: write ρ as a function of hat quantities and $\delta g/g$ etc.

(f) Write ρ in terms of Wilson coefficients, specializing to the $\{\alpha, G_F, m_Z\}$ and $\{m_W, G_F, m_Z\}$ input schemes. Verify that, in both cases, you get the same result and that this also coincides with the result found at point (a).

(g) Deduce the relation between $\delta m_W^2/m_W^2$, $\delta c_\theta^2/c_\theta^2$ and $(\rho - 1)$.

The W decay width in SMEFT

3. In this exercise we will compute the SMEFT correction to the total decay width of the W boson. We will ignore all fermion masses and mixings.

   (a) Compute the *squared* amplitude for a decay $W^- \to e^-\nu_e$, averaged over the polarizations of the W boson, considering only one lepton flavor and expanding to linear order in the SMEFT. The relevant Feynman rule is

$$W_\mu^- \bar{e}\nu \qquad -\frac{ig}{\sqrt{2}}(\gamma^\mu P_L)\left[1 + \frac{\delta g}{g} + \bar{C}_{Hl}^{(3)}\right] \tag{6}$$

   (b) Repeat for $W^- \to \bar{u}d$. The Feynman rule in this case is

$$W_\mu^- \bar{d}u \qquad -\frac{ig}{\sqrt{2}}(\gamma^\mu P_L)\left[1 + \frac{\delta g}{g} + \bar{C}_{Hq}^{(3)}\right] \tag{7}$$

   (c) Compute the decay widths $\Gamma(W^- \to e^-\nu_e)$ and $\Gamma(W^- \to \bar{u}d)$.
   Remember that

$$\Gamma = \frac{|A|^2}{16\pi m_W} \tag{8}$$

   (d) Compute the total decay width Γ_W and express it as $\Gamma_W^{SM}\left[1 + \delta\Gamma_W/\Gamma_W\right]$.
   Then specialize the result to the $\{\alpha, G_F, m_Z\}$ and $\{m_W, G_F, m_Z\}$ input schemes.

Jacobian formulation of input shifts

⚠ For this exercise you will need Mathematica to invert matrices.
One can ask whether there is a simple way to translate between different EW input schemes. In fact, when working to $\mathcal{O}(\Lambda^{-2})$, the translation can be done quite easily using a Jacobian description of the whole inputs procedure:

Let's define the vector $\vec{G} = (g, g', v_T)$ of the 3 Lagrangian parameters in the EW sector of the SM. In order to fix their numerical values, we need to relate them to 3 observables, that we don't specify yet. We will call $\vec{O}$ the vector formed by them.
Each of these observables can be computed in the SMEFT using the canonically normalized Lagrangian. The prediction for observable O_n has the form

$$O_n(\vec{G}, C_i) = O_n^{SM}(\vec{G}) + \Delta O_n(\vec{G}, C_i) \tag{9}$$

where the first term is the SM prediction and the second is the $\mathcal{O}(\Lambda^{-2})$ correction.
What we do when fixing the input scheme is solving the system $\vec{O}(\vec{G}, C_i) = \hat{\vec{O}}$ for $\vec{G}$, where $\hat{\vec{O}}$ are the measured values for $\vec{O}$. The solution can be written

$$\vec{G} = \hat{\vec{G}} - J^{-1}\Delta\vec{O} \tag{10}$$

where $\hat{\vec{G}}$ is the SM solution and J is a Jacobian matrix defined by

$$J_{nk} = \frac{\partial O_n^{SM}}{\partial G_k} \tag{11}$$

4. Choose the input observables $\vec{O}_{mW} = \{m_W^2, m_Z^2, G_F\}$ and re-derive the shifts presented in class (and given in the appendix) using the formula (10).

   Remember the starting point:

$$m_W^2 = \frac{g^2 v_T^2}{4} \qquad m_Z^2 = \frac{(g^2 + (g')^2)v_T^2}{4}(1 + \Delta m_Z^2) \qquad G_F = \frac{1}{\sqrt{2}v_T^2}(1 + \Delta G_F) \tag{12}$$

   You don't need to open the Δ's at this stage.

5. Using Eq. (10) you can see that the solution for the scheme with $\vec{O}_a = \{\alpha, m_Z^2, G_F\}$ will be given by

$$\vec{G} = \hat{\vec{G}}_a - (J_\alpha^{-1})\Delta\vec{O}_\alpha \tag{13}$$

   where $\hat{\vec{G}}_a$ is the new SM solution (which is trivial to find). More interestingly, deriving by parts:

$$J_\alpha = \frac{\partial \vec{O}_\alpha^{SM}}{\partial \vec{G}} = \frac{\partial \vec{O}_\alpha^{SM}}{\partial \vec{O}_{mW}^{SM}} \frac{\partial \vec{O}_{mW}^{SM}}{\partial \vec{G}} = \frac{\partial \vec{O}_\alpha^{SM}}{\partial \vec{O}_{mW}^{SM}} J_{mW} \tag{14}$$

   where trivially

$$\frac{\partial \vec{O}_\alpha^{SM}}{\partial \vec{O}_{mW}^{SM}} = \begin{pmatrix} \frac{\partial\alpha}{\partial m_W^2} & \frac{\partial\alpha}{\partial m_Z^2} & \frac{\partial\alpha}{\partial G_F} \\ & 1 & \\ & & 1 \end{pmatrix} \tag{15}$$

   with all observable predictions computed in the SM. So the new Jacobian J_α in α scheme can be computed very easily, once the Jacobian J_{mW} in m_W scheme is known.

   Do this computation and verify that eq. (13) gives the result presented in the lecture for the α scheme.

   Hint: for the Jacobian, you will need to express α as a function of the $\mathcal{O}_{mW}$ observables. To do this, take $\alpha = \frac{1}{4\pi}\frac{(gg')^2}{g^2+(g')^2}$ and replace g, g' with the SM solutions in m_W scheme.

6. Let's use the Jacobian to define a new input scheme: $\vec{O}_{\text{new}} = \{m_W^2, m_Z^2, \alpha\}$.

   Start from the $\vec{O}_{mW}$ set, and replace G_F with α.

   (a) Compute the SM solutions $\vec{\bar{G}}$ in the new scheme.

   (b) Compute the jacobian as $J_{\text{new}} = \frac{\partial \vec{O}^{SM}_{\text{new}}}{\partial \vec{O}^{SM}_{mW}} J_{mW}$.

   (c) Put everything together in (10) to find the result for the parameters and their shfits.

A Input shift expressions

In the $\{\alpha, m_Z, G_F\}$ scheme:

$$\frac{\delta g}{g} = \frac{1}{2c_{2\theta}}\left[-c_\theta^2\left(\Delta m_Z^2 + \Delta G_F\right) + s_\theta^2 \Delta\alpha\right] \tag{16}$$

$$\frac{\delta g'}{g'} = \frac{1}{2c_{2\theta}}\left[s_\theta^2\left(\Delta m_Z^2 + \Delta G_F\right) - c_\theta^2 \Delta\alpha\right] \tag{17}$$

$$\frac{\delta v_T}{v_T} = \frac{\Delta G_F}{2} \tag{18}$$

In the $\{m_W, m_Z, G_F\}$ scheme:

$$\frac{\delta g}{g} = -\frac{\Delta G_F}{2} \tag{19}$$

$$\frac{\delta g'}{g'} = -\frac{1}{2}\left[\Delta G_F + \frac{\Delta m_Z^2}{s_\theta^2}\right] \tag{20}$$

$$\frac{\delta v_T}{v_T} = \frac{\Delta G_F}{2} \tag{21}$$

And the Δ's are

$$\Delta G_F = 2\bar{C}_{Hl}^{(3)} - \bar{C}'_{ll} \tag{22}$$

$$\Delta m_Z^2 = \frac{2gg'}{g^2 + (g')^2}\bar{C}_{HWB} + \frac{\bar{C}_{HD}}{2} \tag{23}$$

$$\Delta\alpha = -\frac{2gg'}{g^2 + (g')^2}\bar{C}_{HWB} \tag{24}$$