Glueballs within constituent approaches

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Nuclear and subnuclear physics unit
University of Mons

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Concerning the dynamic of the system, it is (often) ruled by a phenomenological QCD-inspired Hamiltonian.

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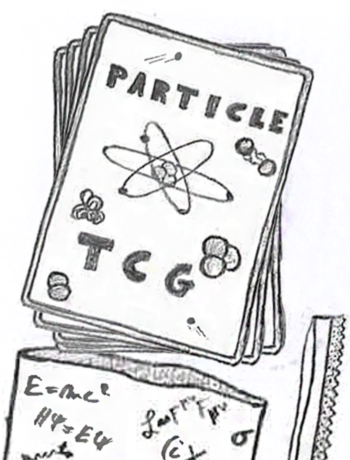

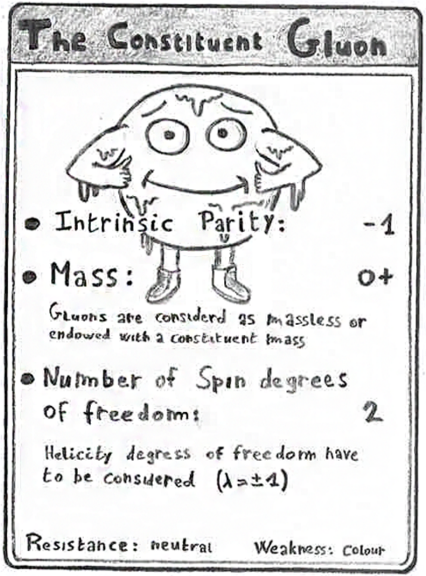

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From one-body helicity states... [5]

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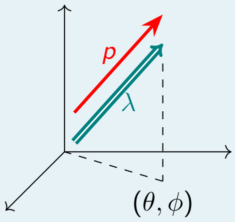

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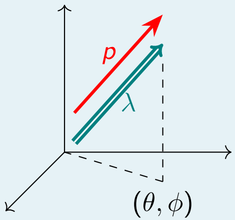

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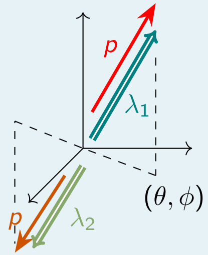

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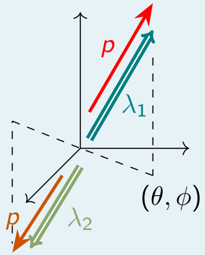

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Application to Two-gluon Glueballs

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Decomposition of a two-gluon glueball in the helicity basis :

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It includes utra-relativistic kinematics, linear confinement ($\sigma = 0.185\,\text{GeV}^2$) and Coulombic short-range interaction ($\alpha_s = 0.450$),

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Application to Glueballs
Glueball spectrum

Let's take a look at the results

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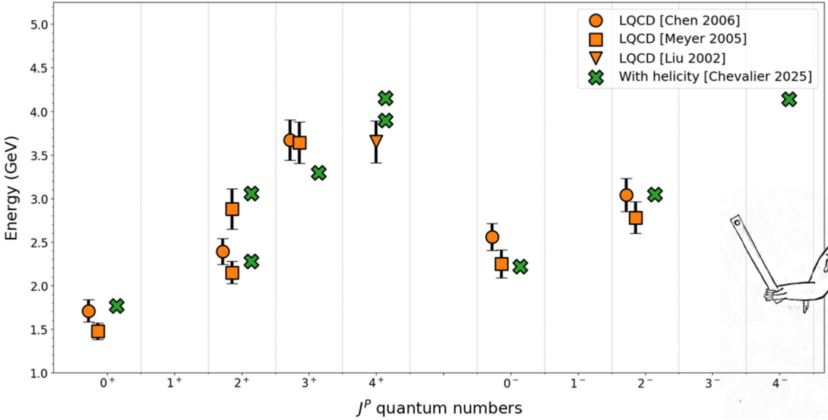

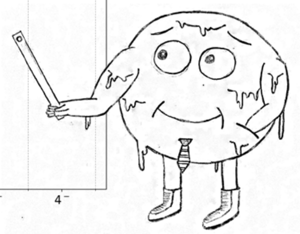

Figure: Comparison of two-gluon glueball spectra

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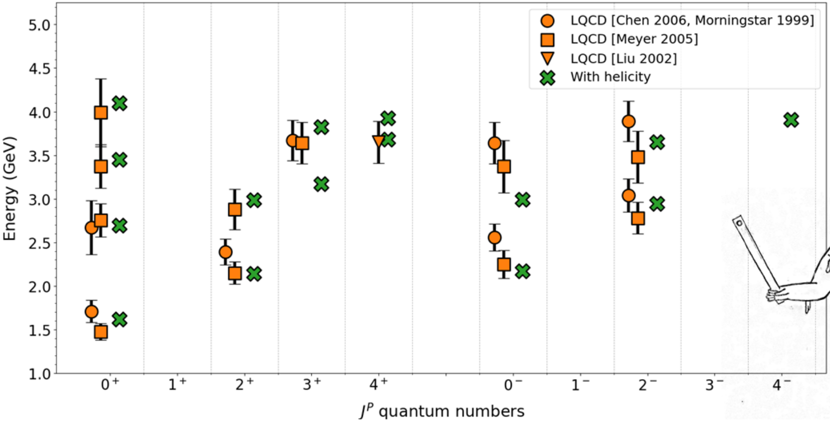

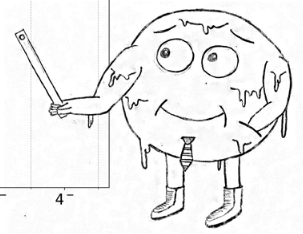

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Application to Glueballs

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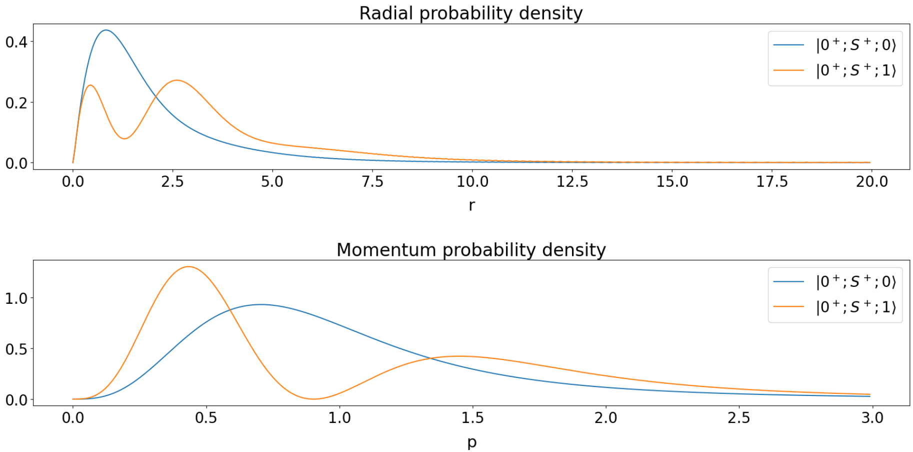
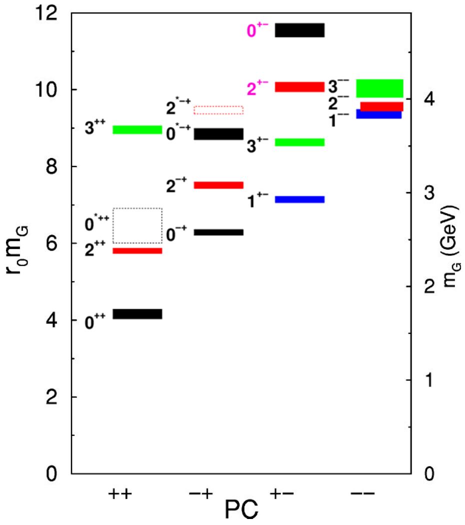


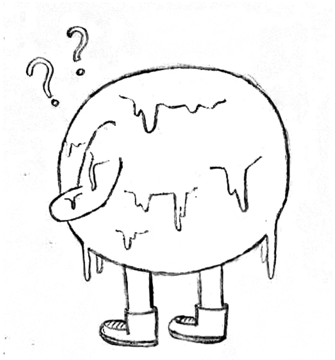
Figure: Two-gluon glueball probability densities for the two lowest 0^+ states.

Outlooks and Generalisations

Three-gluon Glueballs

Motivation





Glueball spectrum from lattice QCD calculations
(from Morningstar, Peardon (1999) Phys.Rev.D, **60**, 034509).

Quick look at results

Three-gluon Glueball

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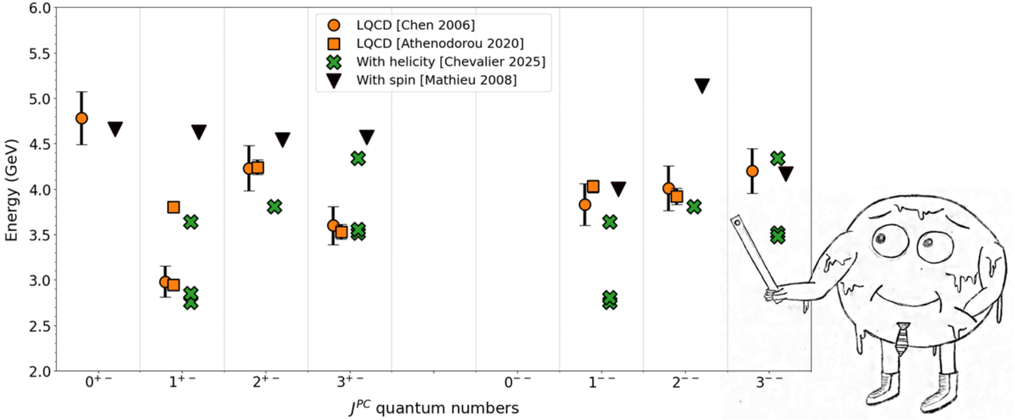

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Outlooks

Other mentions

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  - glueball masses follow the Casimir scaling,
  - gluons generates adjoint strings.

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Outlooks

Other mentions

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- ... compute observables of interest, such as decay rates for glueballs.

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Conclusion

Take home message

"The glueball spectrum is easily reproduced in constituent approaches, as long as helicity degrees of freedom are considered"

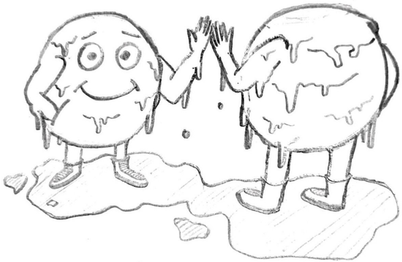


Figure: Rare picture of two gluons meeting each other to make a glueball.

A Few References about Glueballs

A non-exhaustive list

Experimentally-oriented review :

- Crede, Meyer (2009) Prog.Part.Nucl.Phys., **63**, 74

Intermediary review :

- Llanes-Estrada (2021) Eur.Phys.J.Spec.Top, **230**, 1575
- Vadacchino (2023) *arXiv:2305.04869*

Theoretically-oriented reviews :

- Mathieu, Kochelev, Vento (2009) Int.J.Mod.Phys.E, **18**, 1

Constituent approaches : (among many other studies)

- Mathieu, Buisseret, Semay (2008) Phys.Rev.D, **77**, 114022
- Szczepaniak, Swanson (2003) Phys.Lett.B, **577**, 61
- Chevalier, Mathieu (2025) Phys.Rev.D, **112**, 014015

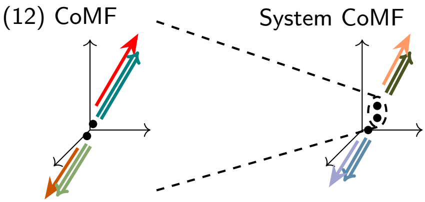
Slide-ppendix : the Two-gluon Glueball Spectrum

State	M_{model}	LQCD	LQCD
$\|\Psi; S_+, 0^+\rangle$	1.641	1.710	1.475
$\|\Psi; S_-, 0^-\rangle$	2.184	2.560	2.250
$\|\Psi; D_+, 2^+\rangle$	2.214	2.390	2.150
$\|\Psi; S_+, 2^+\rangle$	2.944	N.A.	2.880
$\|\Psi; S_-, 2^-\rangle$	2.939	3.040	2.780
$\|\Psi; D_-, 3^+\rangle$	3.159	3.670	3.385
$\|\Psi; D_+, 4^+\rangle$	3.668	N.A.	3.640
$\|\Psi; S_+, 4^+\rangle$	3.871	N.A.	N.A.
$\|\Psi; S_-, 4^-\rangle$	3.871	N.A.	N.A.

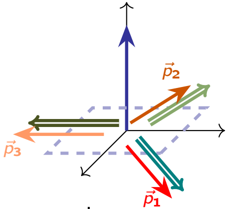
Table: Comparison of two-gluon glueball spectra. Masses from our constituent model, M_{model}, are compared to lattice QCD results (column 1 from [Chen 2006], column 2 form [Meyer 2005]). Energies are provided in GeV.

Need of complete sets of helicity states for three-body systems. Two different sets: Wick's states [9] & Berman's states [10].

Wick performs **two successive two-body couplings**.



Berman starts with **tensor products of three one-body helicity states**...



...and integrate the system over its orientations.

[9] Wick (1962) Annals Phys., **18**, 65

[10] Berman, Jacob (1965) Phys.Rev., **139 B**, 1023

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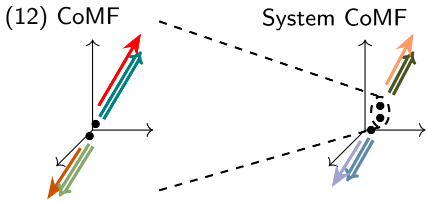

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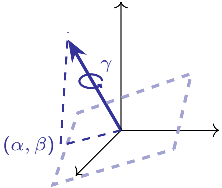

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The same methodology is used than for two-body systems.

- Symmetrised helicity states are integrated with a helicity momentum wave function,
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Symmetry implemented with Berman's definition $\Rightarrow$ Passage in Wick's one $\Rightarrow$ Matrix element computed on Wick's states

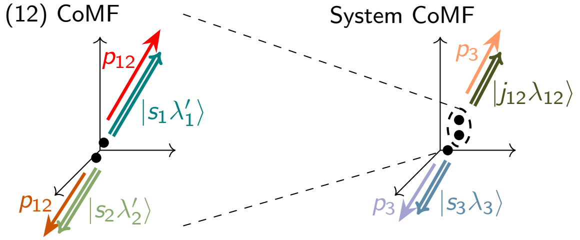
Slide-ppendix : the Three-body Helicity Formalism

Explicit definition of Wick's three-body helicity states

Property: Relation to two-body J-helicity states

$$|p\theta\phi; j_{12}\lambda_{12}s_3\lambda_3; p_{12}s_1\lambda_1' s_2\lambda_2'\rangle \propto |m_{12}; p\theta\phi; j_{12}\lambda_{12}; s_1\lambda_1' s_2\lambda_2'\rangle \otimes |m_3; p(\pi+\phi)(\pi-\theta); s_3\lambda_3\rangle\,,$$

$$|p_3; JM; j_{12}\lambda_{12}s_3\lambda_3; p_{12}s_1\lambda_1' s_2\lambda_2'\rangle \propto \int \mathrm{d}\cos\theta\,\mathrm{d}\phi\, D^{J*}_{M\,\lambda_{12}-\lambda_3}(\phi,\theta,0)\,|p\theta\phi; j_{12}\lambda_{12}s_3\lambda_3; p_{12}s_1\lambda_1' s_2\lambda_2'\rangle\,.$$



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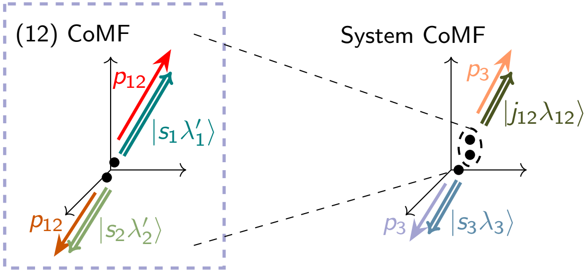

Parity and symmetry of Berman's states

Concerning Berman's state parity and symmetry, one can show that

$$\Pi\,|JM\mu; w_1 w_2 w_3; \lambda_1\lambda_2\lambda_3\rangle = \eta_1\eta_2\eta_3(-1)^{-s_1-s_2-s_3-\mu}\,|JM\mu; w_1 w_2 w_3; -\lambda_1-\lambda_2-\lambda_3\rangle\,,$$
$$\mathbb{P}_{12}\,|JM\mu; w_1 w_2 w_3; \lambda_1\lambda_2\lambda_3\rangle = (-1)^{J+\mu+\lambda_1+\lambda_2-\lambda_3}\,|JM-\mu; w_2 w_1 w_3; \lambda_2\lambda_1\lambda_3\rangle\,,$$
$$\mathbb{P}_{13}\,|JM\mu; w_1 w_2 w_3; \lambda_1\lambda_2\lambda_3\rangle = (-1)^{J-\mu-\lambda_1-\lambda_2-\lambda_3}e^{-i\varphi_{13}\mu}\,|JM-\mu; w_3 w_2 w_1; \lambda_3\lambda_2\lambda_1\rangle\,,$$
$$\mathbb{P}_{23}\,|JM\mu; w_1 w_2 w_3; \lambda_1\lambda_2\lambda_3\rangle = (-1)^{\lambda_1-\lambda_2-\lambda_3+J+\mu}e^{i\varphi_{23}\mu}\,|JM-\mu; w_1 w_3 w_2; \lambda_1\lambda_3\lambda_2\rangle\,.$$

where

$$\varphi_{ij} = \frac{p_k^2 - p_i^2 - p_j^2}{2p_i p_j}\,, \qquad p_i^2 = w_i^2 - m_i^2\,.$$

As a result, parity and symmetry mixes different helicities as well as different μ values.

From Berman's definition to Wick's one

For three massless particles, one can show that

$$|JM\mu; w_1 w_2 w_3; \lambda_1\lambda_2\lambda_3\rangle \propto \sum_{j_{12}=|\lambda_1-\lambda_2|}^{\infty} \sum_{\lambda_{12}=-j_{12}}^{j_{12}} e^{i\pi\lambda_{12}/2}\sqrt{\frac{2j_{12}+1}{2}}\, d^{j_{12}}_{\lambda_{12}\,\lambda_1-\lambda_2}(u)\, d^{J}_{\mu\,\lambda_{12}-\lambda_3}(\pi/2)\,|p_3;\ JM;\ j_{12}\lambda_{12}s_3\lambda_3;\ p_{12}s_1\lambda_1 s_2\lambda_2\rangle$$

where

$$\begin{cases} u = \arccos\left(\left(w_1 - w_2\right)/w_3\right), \\ p_{12} = \sqrt{(w_1+w_2)^2 - w_3^2}/2, \\ p_3 = w_3. \end{cases}$$

.

Hamiltonian for three-gluon calculations

We simply generalise the two-gluon Hamiltonian,

$$H = w_1 + w_2 + w_3 + V(r_{12}) + V(r_{13}) + V(r_{23}) \quad \text{with} \quad V(r) = \sigma r - \frac{3\alpha_s}{2r} + C.$$

A physical spectrum can be obtained by fixing the above parameters. Two sets of parameters can be used to obtain similar spectra,

- economical values : $\sigma = 0.086\,\text{GeV}^2$, $\alpha_s = 0.450$ and $C = 0\,\text{GeV}$,
- more physical values : $\sigma = 0.150\,\text{GeV}^2$, $\alpha_s = 0.450$ and $C = -0.375\,\text{GeV}$.

Slide-ppendix : the Physical Three-gluon Glueball Spectrum

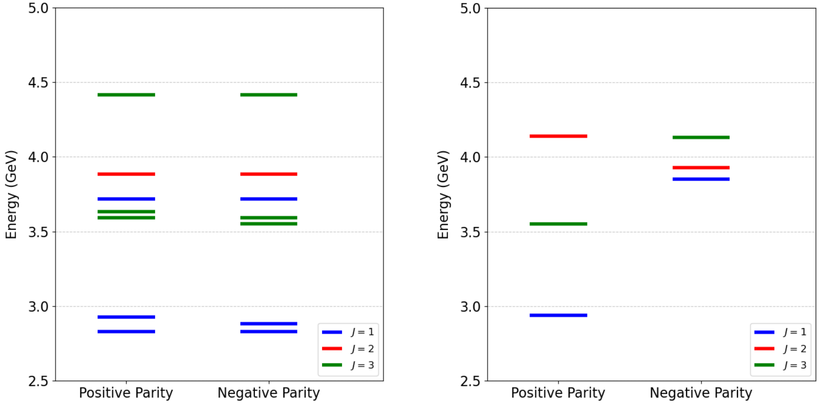

Figure: Comparison of glueball spectra obtained using the helicity formalism with physical parameters [12] (left) and quenched lattice QCD [13] (right). In both cases, spectra are provided in GeV.

[12] Chevalier, Mathieu (2025) Phys.Rev.D, **112**, 014015

[13] Chen et al. (2006) Phys.Rev.D, **73**, 014516