Probing $\eta\pi$ Production with Finite Energy Sum Rules Work in Progress

NADINE HAMMOUD

Institute of Cosmos Sciences, University of Barcelona

With
Vincent Mathieu, Adam P. Szczepaniak

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Exotic Spectroscopy

Mesons: $q\bar{q}$

$P = (-1)^{L+1}$

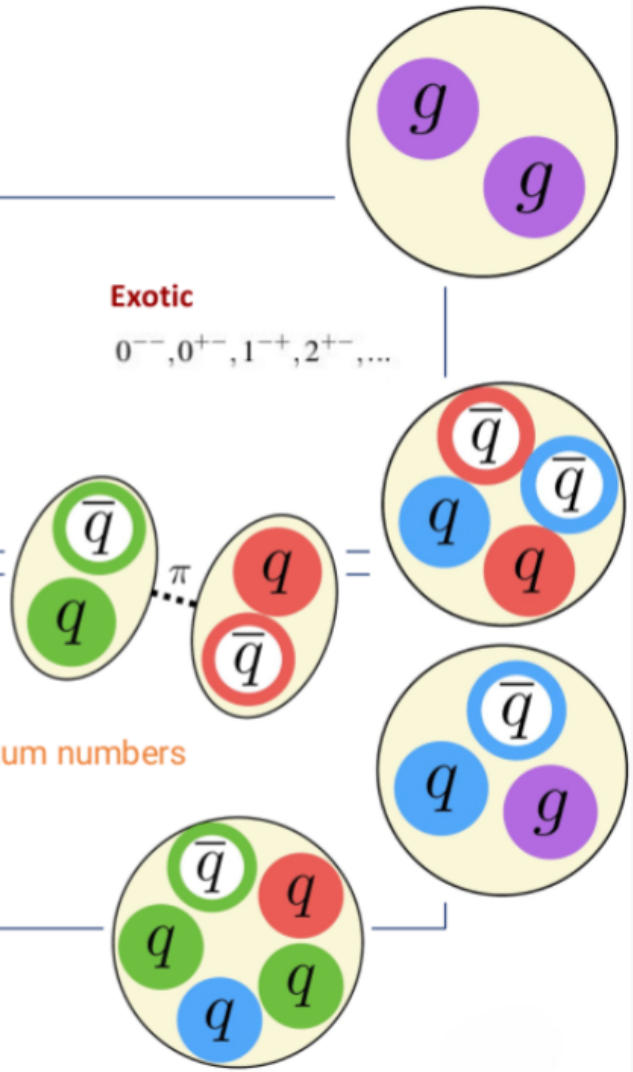
$C = (-1)^{L+S}$

$J^{PC} = 0^{-+}, 0^{++}, 1^{--}, 1^{+-}, 1^{++}, 2^{--}, 2^{-+}, 2^{++}, \ldots$

Exotic

$0^{--}, 0^{+-}, 1^{-+}, 2^{+-}, \ldots$

glueballs or hybrid mesons or multi-quark states or molecules



Baryons: qqq

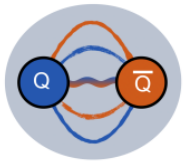
Observation is difficult:

- 'exotics' hide in plain sight since they have the same quantum numbers
- Large masses

Only structure to distinguish them

Lightest Exotic Meson

Exotic mesons

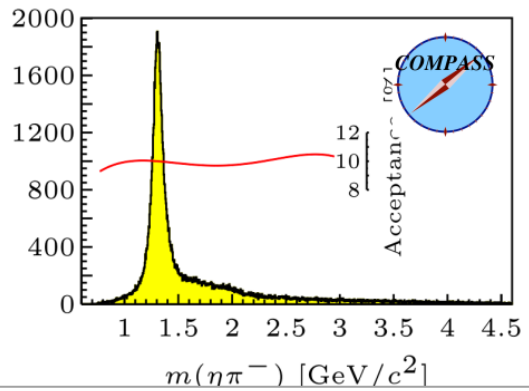


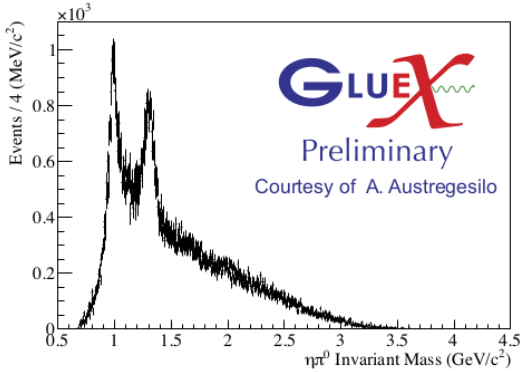
The lightest is π_1 $\quad J^{PC} = 1^{-+}$

$$\begin{array}{cccc} 0^{--} & 0^{-+} & 0^{+-} & 0^{++} \\ 1^{--} & \boxed{1^{-+}} & 1^{+-} & 1^{++} \\ 2^{--} & 2^{-+} & 2^{+-} & 2^{++} \\ 3^{--} & 3^{-+} & 3^{+-} & 3^{++} \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

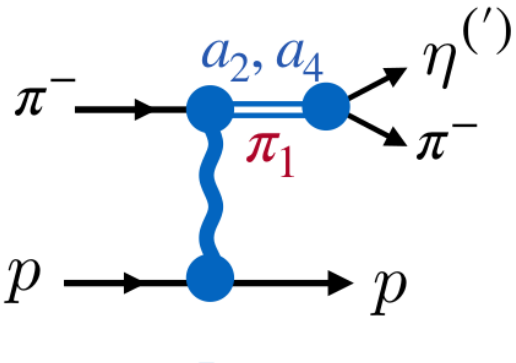
Decay mode $\quad \pi_1 \to \pi\eta$

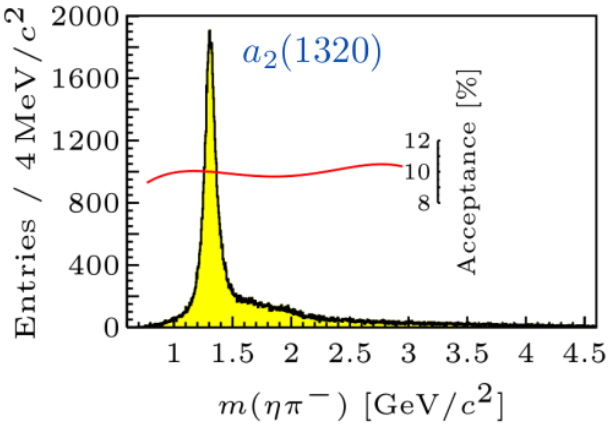
$$1^{-+} = \left(0^{-+} \otimes 0^{-+}\right)_{P\text{-wave}}$$

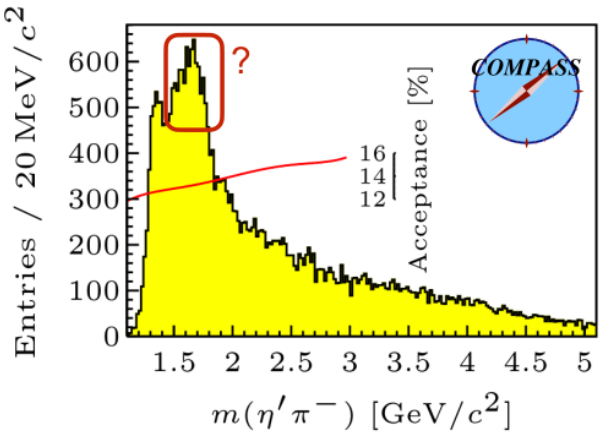


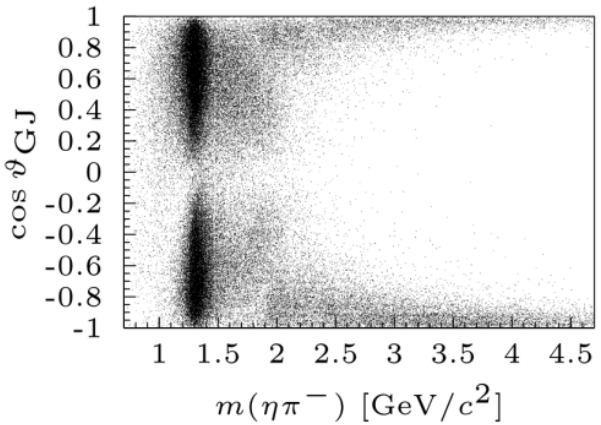


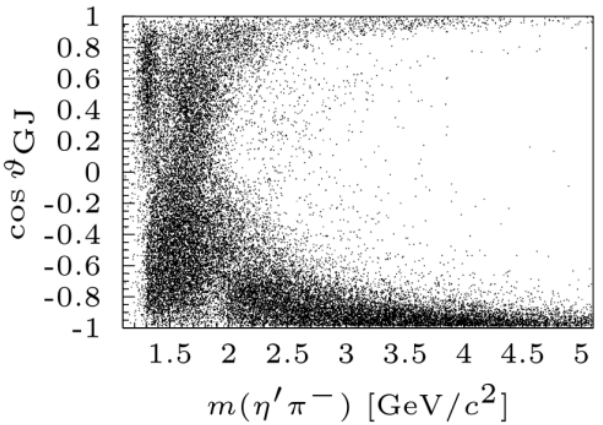
$\eta\pi$ at COMPASS COMPASS PLB740 (2015)





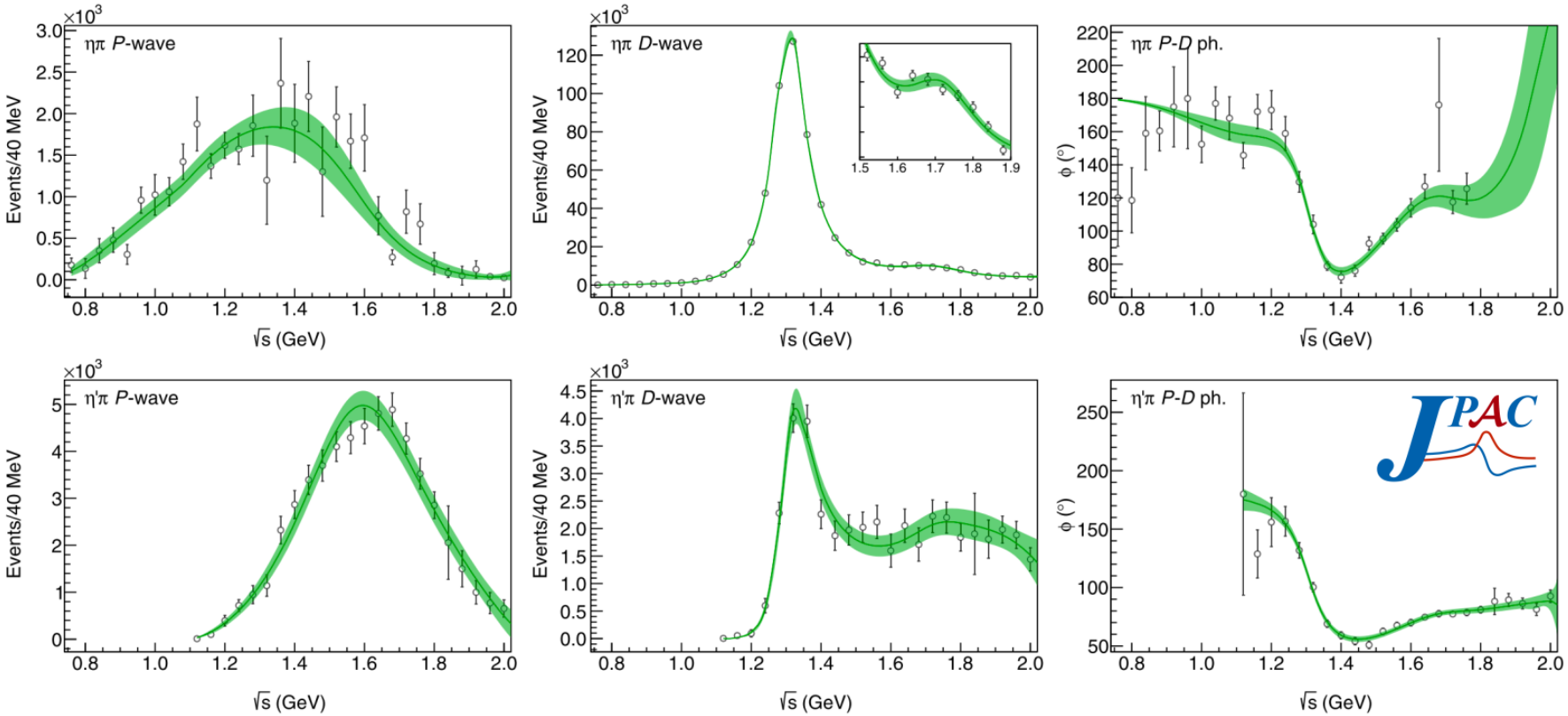






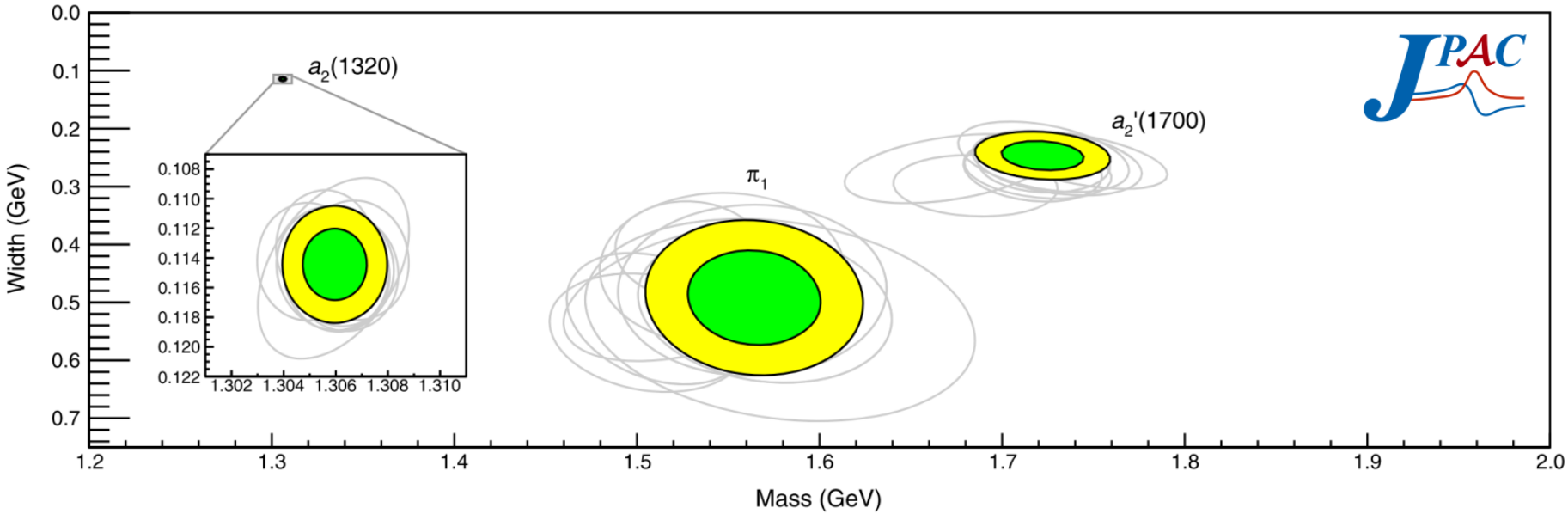
Low Energy Fit of P and D waves Rodas et al PRL122 (2019)

$\pi_1(1400)$ vs $\pi_1(1600)$

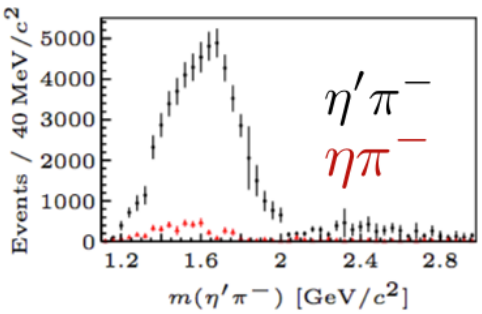


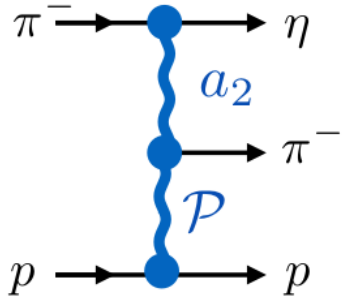
Low Energy Fit of P and D waves Rodas et al PRL122 (2019)

Poles	Mass (MeV)	Width (MeV)
$a_2(1320)$	$1306.0 \pm 0.8 \pm 1.3$	$114.4 \pm 1.6 \pm 0.0$
$a_2'(1700)$	$1722 \pm 15 \pm 67$	$247 \pm 17 \pm 63$
π_1	$1564 \pm 24 \pm 86$	$492 \pm 54 \pm 102$



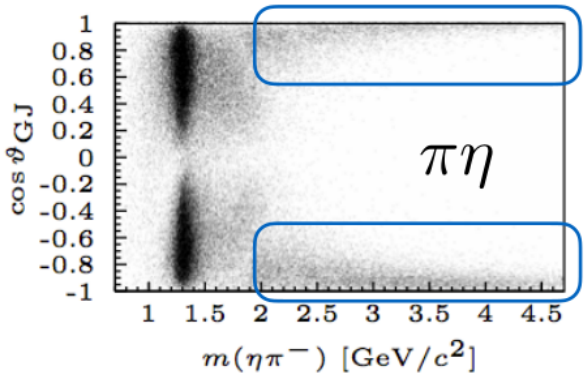
$\eta\pi$ at COMPASS COMPASS PLB740 (2015)

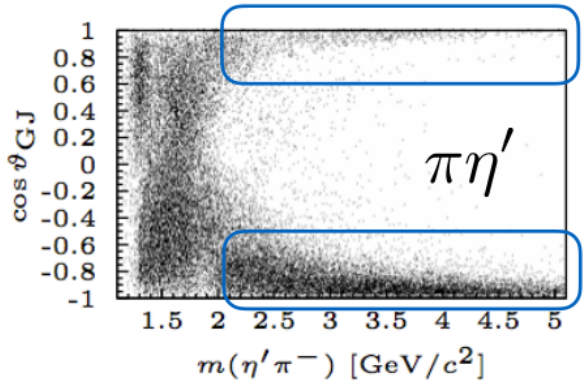




$\cos\theta_{GF} \sim 1 \rightarrow \eta$ forward

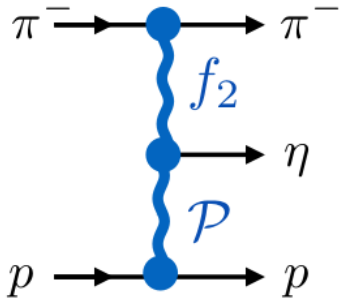
$\cos\theta_{GF} \sim -1 \rightarrow \eta$ backward

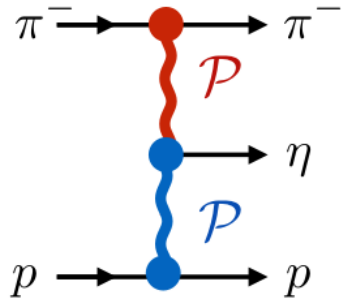




Exotic meson related to
Forward-backward asymmetry

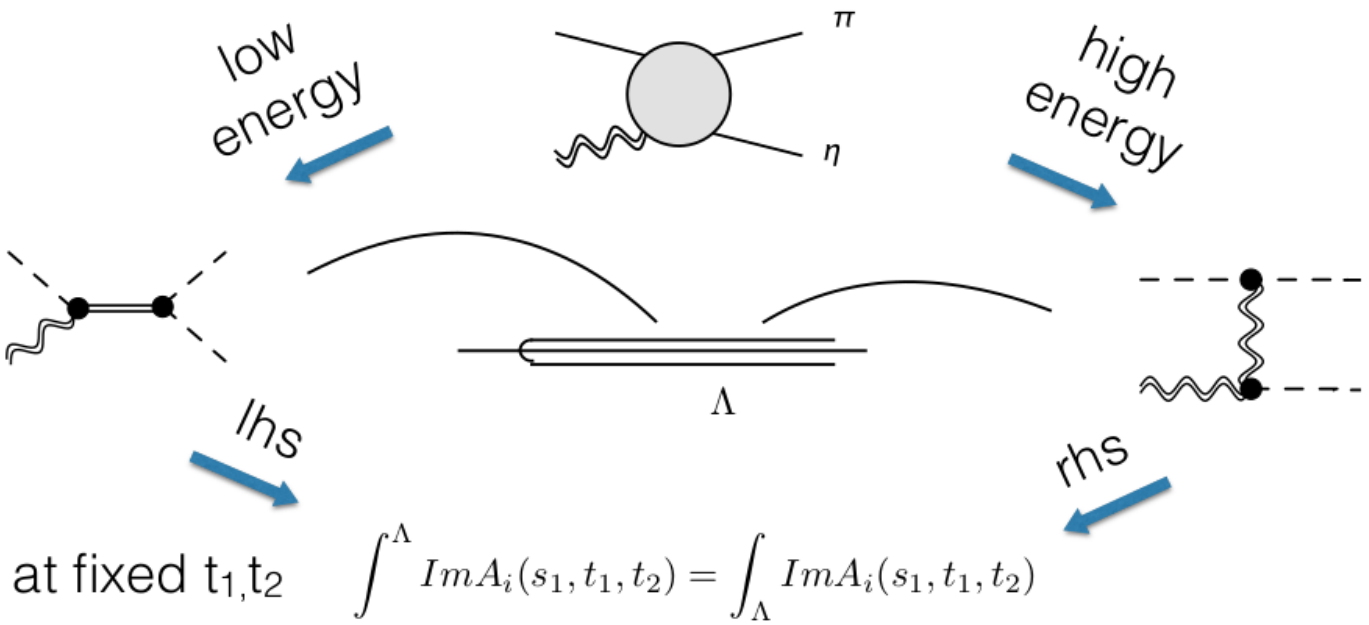
Asymmetry related to
even-odd waves interferences





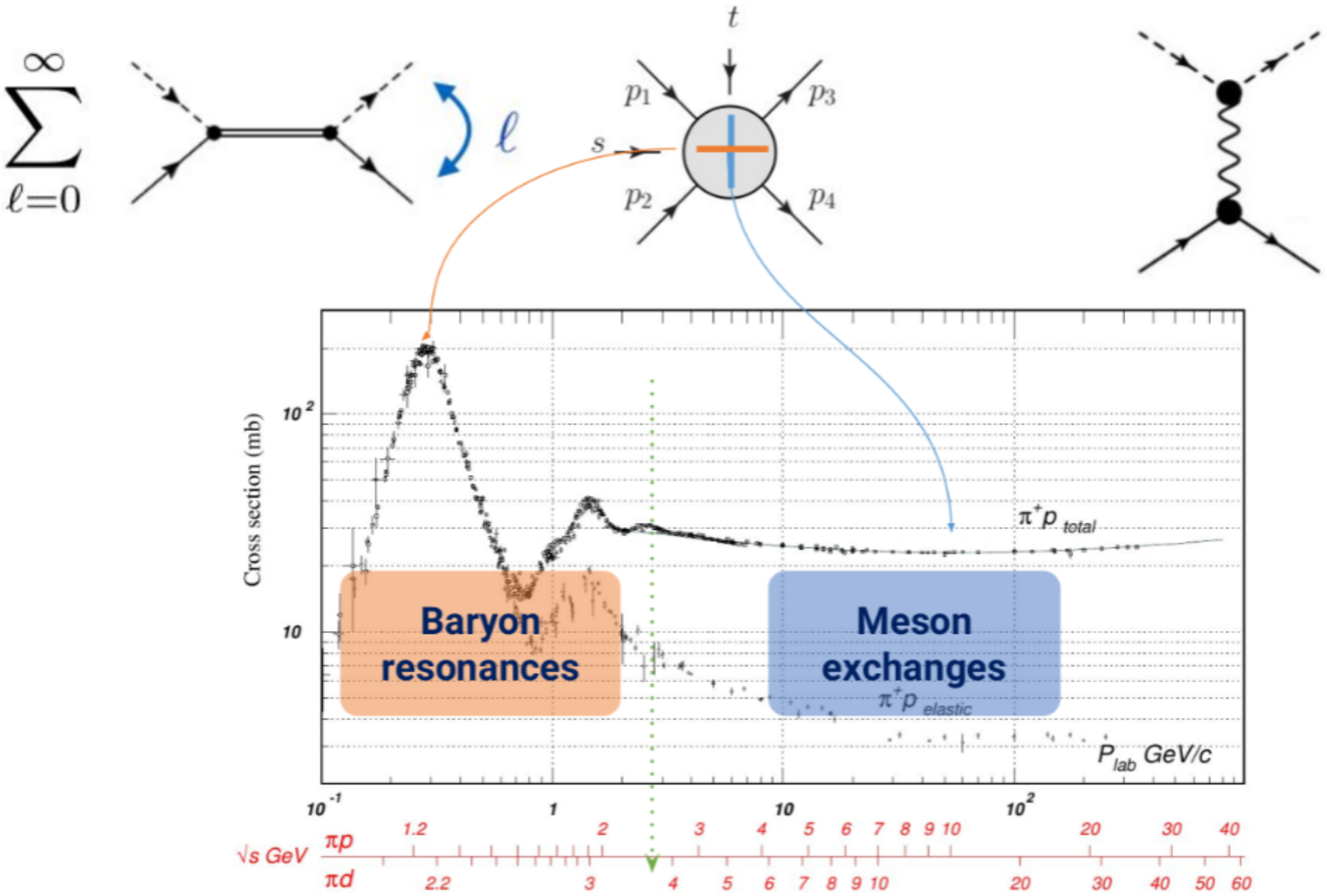
Finite Energy Sum rules

$$A(s_1, t_1, t_2, s_2/s) = \sum_i A_i(s_1, t_1, t_2) \left(\frac{s_2}{s}\right)^2$$



- Derived using Cauchy's theorem: $\oint_C A(s,t)ds = 0$:
- Connect low-energy and high-energy dynamics.
- Predict high-energy observables from low-energy data.
- Constrain low-energy models using high-energy results.

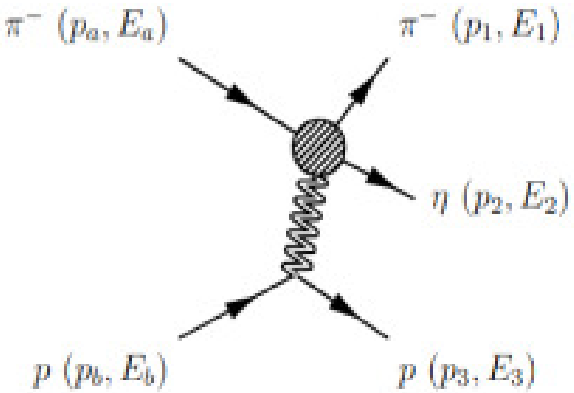
Finite Energy Sum rules

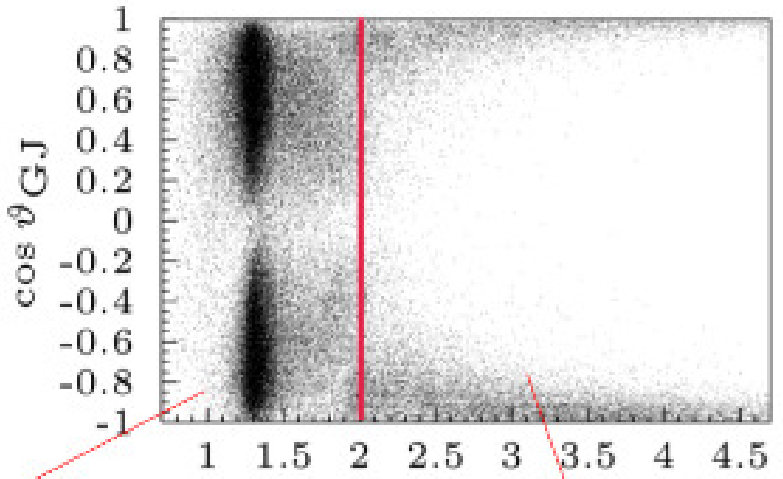


Summary: Why Study $\pi p \to \pi\eta p$?

- High-quality data available from the COMPASS experiment.
- Theoretical Motivation:
  - Test the analytic structure and consistency of scattering amplitudes.
  - Apply Finite Energy Sum Rules (FESR) for the **first time** to a $2 \to 3$ process; a novel theoretical challenge.
- Phenomenological Goals:
  - Connect resonance-region dynamics to high-energy behavior through FESR.
  - Constrain resonance parameters using high-energy data potentially shed light on exotic candidates like the π_1.
- Applications:
  - Improve modeling of vertex functions in double-Regge exchange frameworks.
  - Reduce uncertainty in the extraction of the π_1 pole position (see PRL 122, 042002 (2019)).

Analysis of $\pi p \to \pi \eta p$

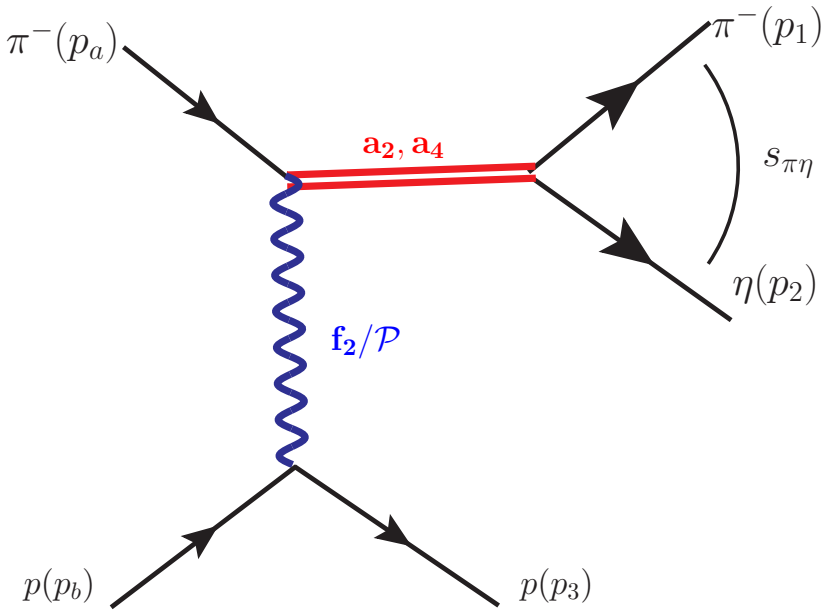




Low Energy Limit: Resonance production, Single Regge exchange

High Energy Limit : Double Regge exchange

SRL Amplitude $s \to \infty$, fixed $s_{\pi\eta}$



At low energies, the amplitude is primarily dominated by resonance production and can be expressed as a sum of partial wave amplitudes, which can be extracted from experimental data:

$$A^{PW}(s_{\eta\pi}, \Omega) = \sum_{l=1}^{6} a_{l,1}(s_{\eta\pi}) \sin(\Phi)\, Y_l^1(\theta, 0) + a_{2,2}(s_{\eta\pi}) \sin(2\Phi)\, Y_2^2(\theta, 0)$$

SRL Amplitude

To extract the partial wave amplitudes from data, we use the intensities and relative phases provided in the COMPASS analysis (Phys. Lett., B740:303-311, 2015). The partial waves are written as

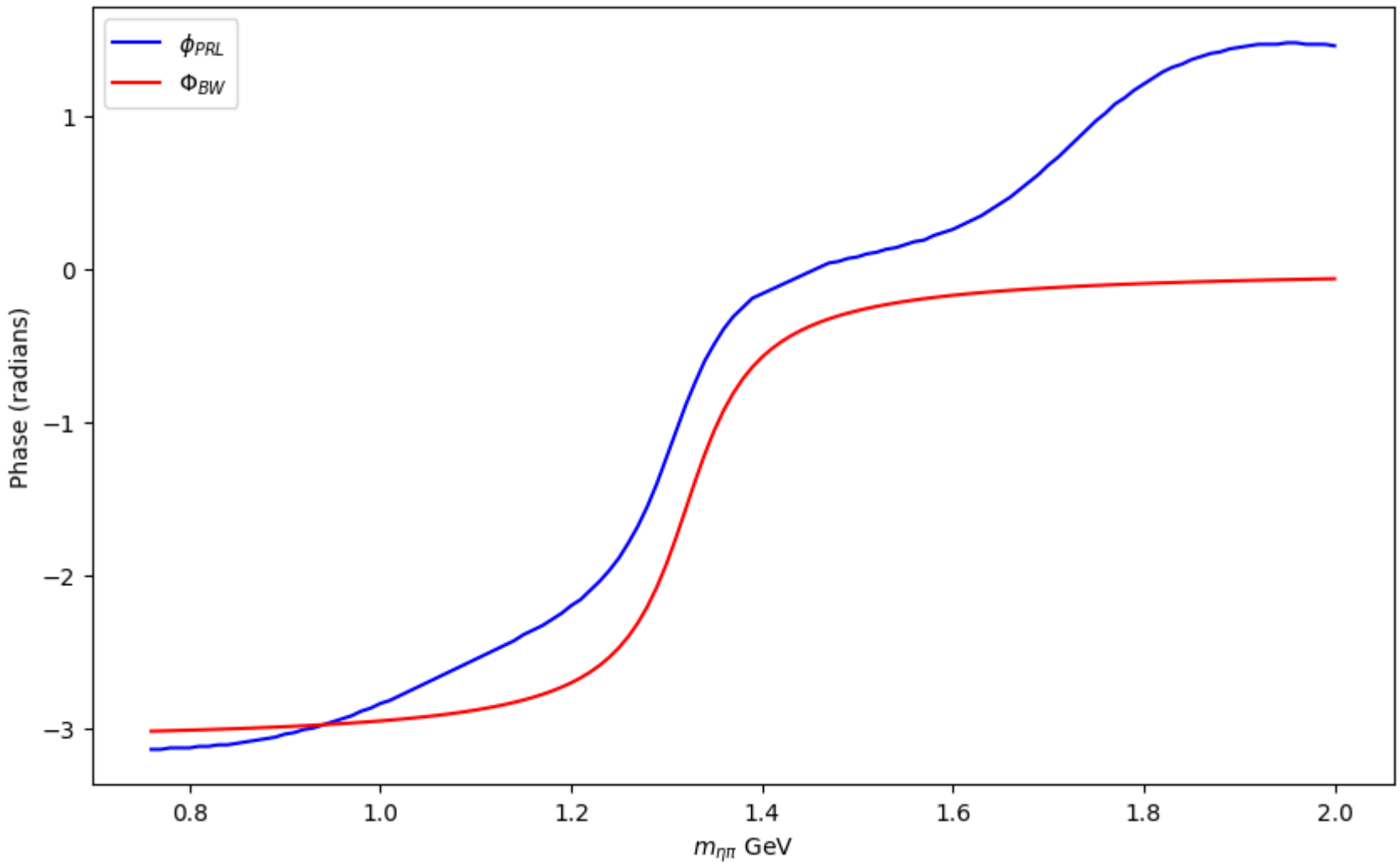
$$a_{l,M}(s_{\eta\pi}) = \sqrt{\frac{I_{l,M}(s_{\eta\pi})}{k}}\, e^{i(\delta_{l,M}+\phi_{2,1})}$$

where $I_{l,M}$ and $\delta_{l,M}$ are the experimental intensity and phases relative to the $l = 2,\ M = 1$.

$$k(s_{\eta\pi}) = \frac{\lambda^{1/2}(s_{\eta\pi}, m_\eta^2, m_\pi^2)}{2\sqrt{s_{\eta\pi}}}$$

is the breakup momentum factor.

SRL Amplitude



The $l = 2, M = 1$ wave phase calculated from the Breit Wigner of a_2 meson vs. the one calculated from the theoretical results of PRL model PRL 122, 042002 (2019)

SRL Amplitude

To obtain an analytic amplitude, we remove the dynamical singularities by dividing out the kinematic factor:

$$K = 4\sqrt{s_{\eta\pi}}\,|p_2|\,|q|\,|p_a|\sin\epsilon\sin\theta\sin\phi$$

In the single Regge limit, $s, s_{\eta p} \to \infty$ with fixed $\kappa = s_{\eta p}/s$, the kinematic factor simplifies to:

$$4\sqrt{s_{\eta\pi}}\,|q|\,|p_a|\sin\epsilon \simeq 2s\sqrt{-u_p}, \qquad K \simeq 2s\sqrt{-u_p}\cdot|p_2|\sin\theta\sin\phi$$

The reduced partial wave amplitude, obtained after dividing out K, reads:

$$\widehat{A}^{PW}(s_{\eta\pi},\Omega) = \frac{A^{PW}}{K} = \frac{1}{2s\sqrt{-u_p}}\left[\sum_{\ell=1}^{6}\frac{a_{\ell,1}}{|p_2|}\frac{Y_\ell^1(\theta,0)}{\sin\theta} + \frac{1}{2}\sqrt{\frac{15}{2\pi}}\frac{a_{2,2}}{|p_2|}\sin\theta\cos\phi\right]$$

DRL Amplitude $s, s_{\pi\eta}, s_{\eta p} \to \infty$

The generic amplitude for the double Regge exchange can be expressed as in arXiv:2104.10646v2 [hep-ph] 19 Jul 2021 :

$$T(\alpha_1, \alpha_2; s_{\pi\eta}, s_{\eta p}) = K\,\Gamma(1-\alpha_1)\,\Gamma(1-\alpha_2)\,\frac{(\alpha' s_{\pi\eta})^{\alpha_1}(\alpha' s_{\eta p})^{\alpha_2}}{\alpha' s}$$
$$\left[\frac{\xi_1\,\xi_{21}}{\kappa^{\alpha_1}}\,V(\alpha_1, \alpha_2, \kappa) + \frac{\xi_2\,\xi_{12}}{\kappa^{\alpha_2}}\,V(\alpha_2, \alpha_1, \kappa)\right]$$

where

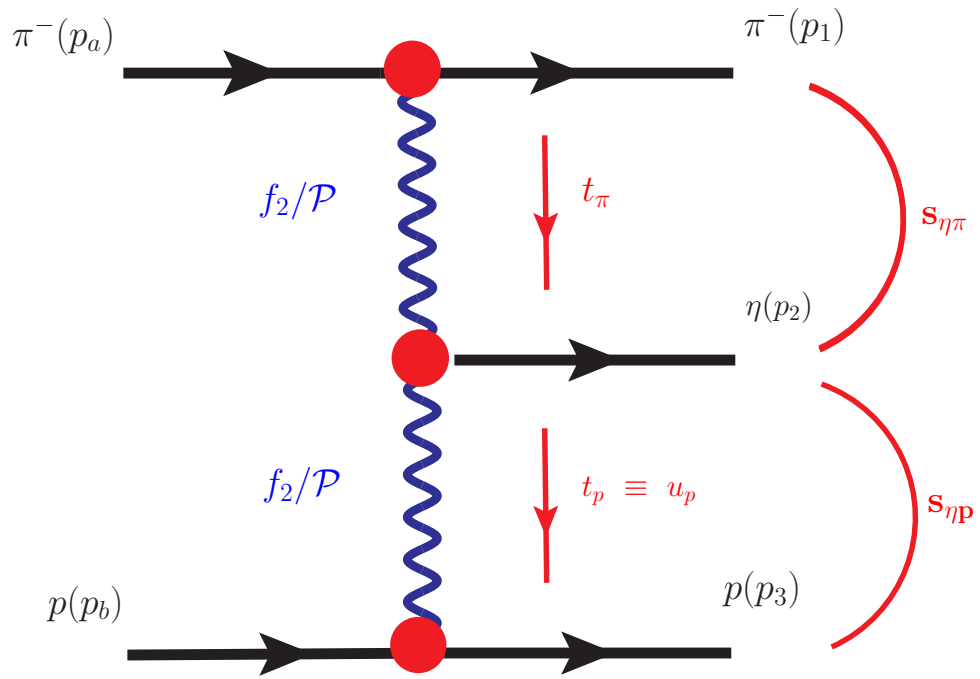
$$V(\alpha_1, \alpha_2, \kappa) = \frac{\Gamma(\alpha_1 - \alpha_2)}{\Gamma(1-\alpha_2)}\,{}_1F_1\,(1-\alpha_1, 1-\alpha_1+\alpha_2, -\kappa)$$

$$\kappa = \frac{\alpha' s_{\pi\eta} s_{\eta p}}{s}$$

$$\xi_n = \frac{1 + e^{-i\pi\alpha_n}}{2}$$

$$\xi_{nm} = \frac{1 + e^{-i\pi(\alpha_n - \alpha_m)}}{2}$$

DRL Amplitude- Fast pion



The total amplitude $A(s_{\eta\pi}, \Omega)$ is the sum of four possible double-Regge amplitudes

$$A(s_{\eta\pi}, \Omega) = c_{f_2\mathbb{P}}\, T_{f_2\mathbb{P}} + c_{f_2 f_2}\, T_{f_2 f_2} + c_{\mathbb{P}\mathbb{P}}\, T_{\mathbb{P}\mathbb{P}} + c_{\mathbb{P} f_2}\, T_{\mathbb{P} f_2}$$

Where c were fitted to the data in arXiv:2104.10646v2 [hep-ph] 19 Jul 2021, with

$$\alpha_{f_2}(t) = 0.47 + 0.89\, t$$
$$\alpha_{\mathbb{P}}(t) = 1.08 + 0.25\, t$$

We define the total amplitude $\hat{A}$ for the **fast pion** case to be:

$$\widehat{A}^{DRL}(s_{\eta\pi}, \Omega) = \frac{A(s_{\eta\pi}, \Omega)}{K} = \frac{\sum_{i=1}^{4} c_i T^i}{K} = \sum_{i=1}^{4} c_i \widehat{T}^i$$

FESR in ν-LHS

Define the crossing-symmetric variables:

$$\nu = s_{\eta\pi} - \frac{C}{2}, \qquad C = 2m_\pi^2 + m_\eta^2 + u_p - t_\pi, \qquad \sigma = \tfrac{1}{2}(s - u_1)$$

The SRL amplitude:

$$\begin{aligned} A(\sigma,\nu) &= (\sigma^\alpha + (-\sigma)^\alpha)\Big[F(\sigma,\nu,u_p,t_\pi,\tfrac{s_{\eta p}}{\sigma}) + F(-\sigma,-\nu,u_p,t_\pi,-\tfrac{s_{\eta p}}{\sigma})\Big] \\ &= (\sigma^\alpha + (-\sigma)^\alpha)\sum_{n=0}^{\infty}\left[c_n(\nu) + (-1)^n c_n(-\nu)\right]\Big(\tfrac{s_{\eta p}}{\sigma}\Big)^n. \end{aligned}$$

Thus the LHS of the FESR becomes:

$$LHS = \sum_{n=0}^{\infty}\Big(\tfrac{s_{\eta p}}{\sigma}\Big)^n(1+(-1)^n)\int_0^\Lambda d\nu\, c_n(\nu) = 2\sum_{m=0}^{\infty}\Big(\tfrac{s_{\eta p}}{\sigma}\Big)^{2m}\int_0^\Lambda d\nu\, c_{2m}(\nu).$$

FESR in ν-RHS

The DRL amplitude can be written as:

$$\widehat{T}(\sigma,\nu,\xi) = \frac{1}{4}\Big[(\alpha'\sigma)^{2\alpha_2-1} + (-\alpha'\sigma)^{2\alpha_2-1}\Big]\sum_{j=0}^{\infty} c_j\Big[(\alpha'\nu)^{\alpha_1-\alpha_2} + (-\alpha'\nu)^{\alpha_1-\alpha_2}\Big]\Big(-\alpha'\nu\,\frac{s_{\eta p}}{\sigma}\Big)^j$$

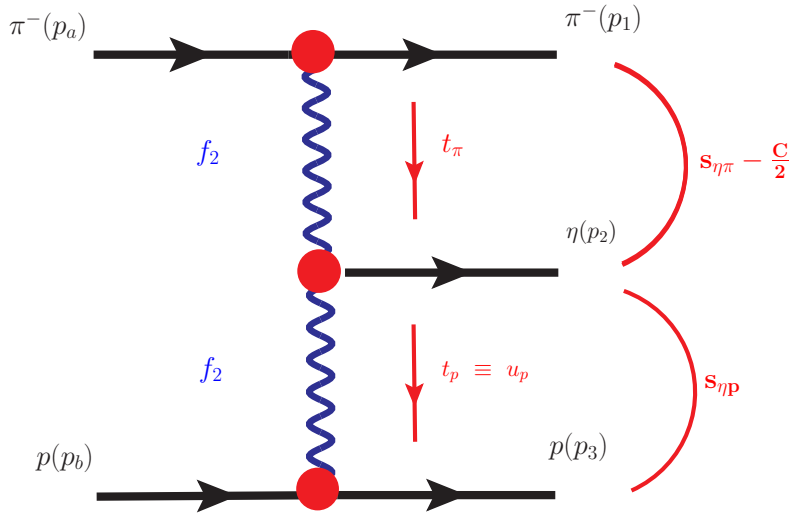
$$= \frac{1}{4}(\alpha')^{\alpha_1-1}\Big[(\sigma)^{2\alpha_2-1} + (-\sigma)^{2\alpha_2-1}\Big]\Big[F(\sigma,\nu,\tfrac{s_{\eta p}}{\sigma}) + F(-\sigma,-\nu,-\tfrac{s_{\eta p}}{\sigma})\Big]$$

Taking $\nu = \Lambda e^{i\theta}$ and computing the closed contour integral for both functions we reach:

$$RHS = -2i\sum_i c_i\left[|\xi_2|\,\sigma^{\alpha_2-1}\,(\alpha')^{\alpha_1-1}\right]\sum_{j=even}^{\infty}\frac{\Lambda^{\alpha_1-\alpha_2+1}}{j!}\,\frac{\Gamma(1-\alpha_2+j)}{\Gamma(2-\alpha_2+\alpha_1+j)}\left(-\alpha'\Lambda\frac{s_{\eta p}}{\sigma}\right)^j.$$

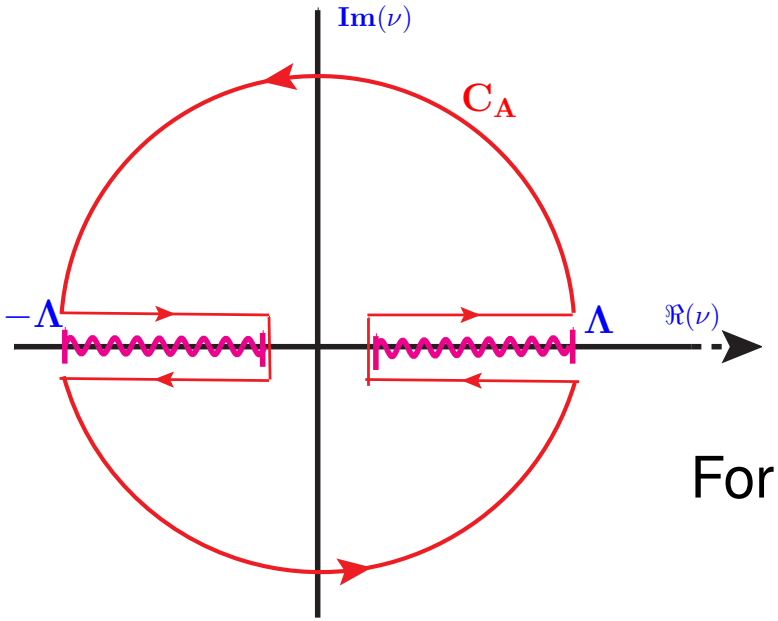
FESR

We choose the minimal model with a single bottom Regge exchange, i.e. the f_2, and fix $u_p = -0.2$ GeV^2. The FESR in ν is applied for different cutoff values.



Channel	Parameter	MIN	MIN+$f/\mathbb{P}$	MIN+$\mathbb{P}/\mathbb{P}$
$\eta\pi$	$c_{f_2\mathbb{P}}$	—	−0.20	—
	$c_{f_2f_2}$	−11.82	−8.99	−10.86
	$c_{\mathbb{P}\mathbb{P}}$	—	—	0.0073

Table: Fitted parameters for the DRL amplitude from arXiv:2104.10646v2 [hep-ph] (2021)



$$C = 2m_\pi^2 + m_\eta^2 + u_p - t_\pi$$

$$\nu = s_{\eta\pi} - \frac{C}{2}$$

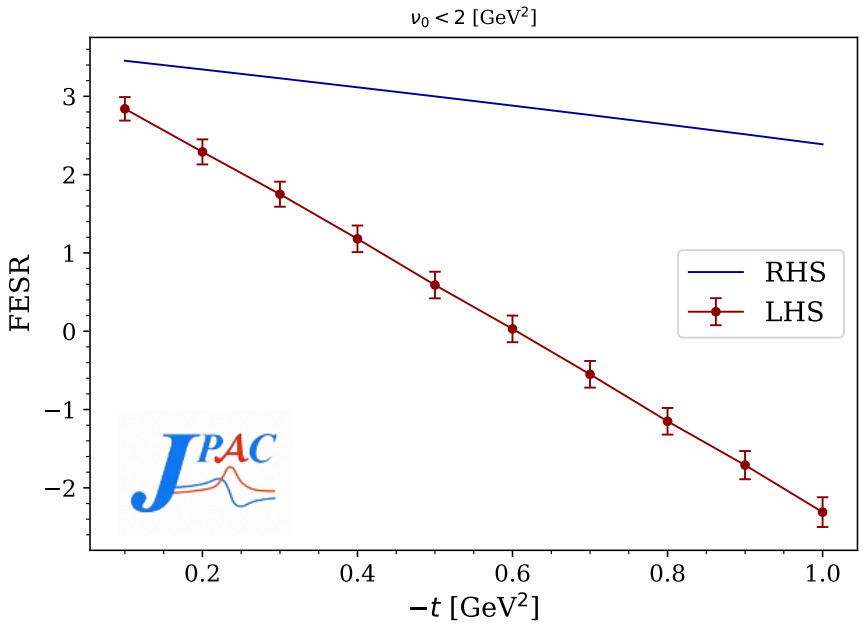
For $j = 0$ the FESR will read:

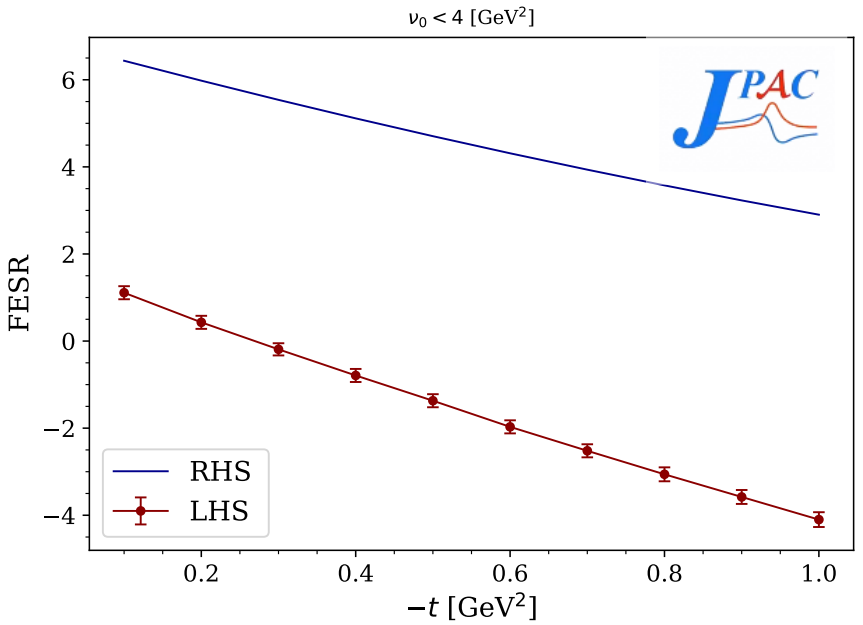
$$LHS = 4\int_0^\Lambda d\nu \frac{1}{2p_\eta \sigma \sqrt{-u_p}}\Bigg[\sum_{\ell=1}^{6} Ima_{\ell,1}(\nu)\frac{Y_\ell^1(\theta,0)}{\sin\theta} + \frac{1}{2}\sqrt{\frac{15}{2\pi}}Ima_{2,2}(\nu)\frac{m_\eta^2 + m_p^2 + 2E_2E_3 - 2p_1p_3\cos\epsilon\cos\theta}{2p_1p_3\sin\epsilon}\Bigg]$$

$$RHS = -2i\, c_{f_2f_2}\, \pi|\xi_2|\, \sigma^{\alpha_2-1}(\alpha')^{\alpha_1-1}\frac{\Gamma(1-\alpha_2)}{\Gamma(2-\alpha_2+\alpha_1)}\Lambda^{\alpha_1-\alpha_2+1}$$

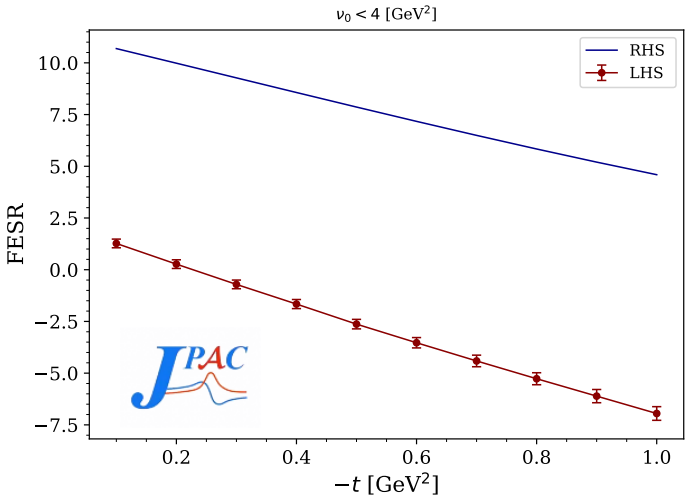
FESR-Preliminary Results

$u_p = -0.2\ \text{GeV}^2$

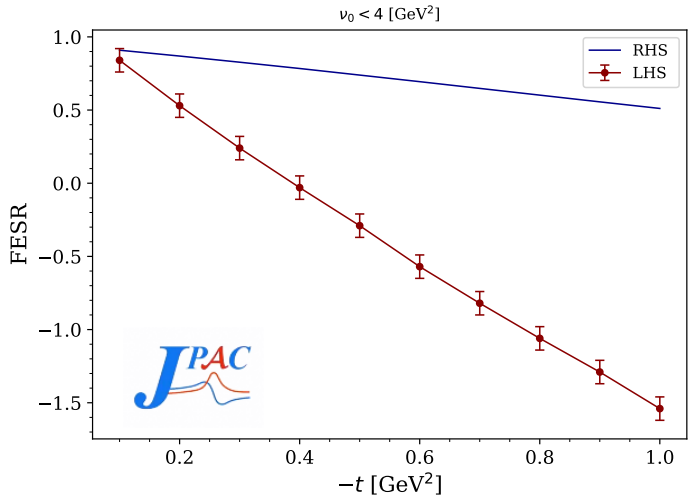




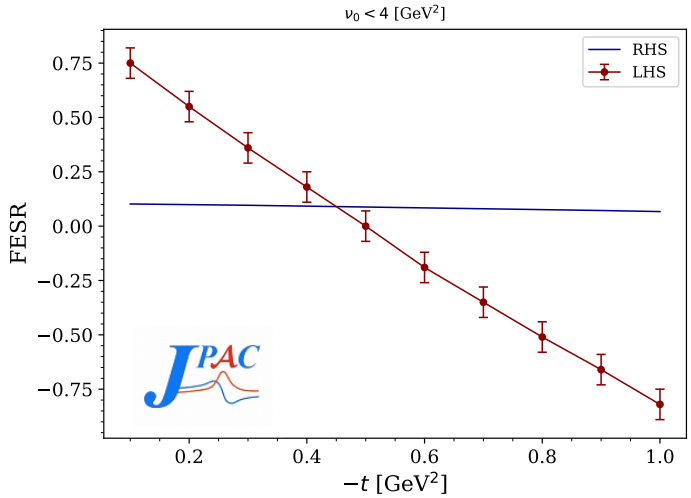
FESR-Preliminary Results



(a) $u_p = -0.1$



(b) $u_p = -0.6$



(c) $u_p = -1$

Summary

- Established the relevance of Finite Energy Sum Rules (FESR) for probing the dynamics of the reaction $\pi^- p \to \pi^- \eta p$.
- Presented the first theoretical derivation of both the low-energy (LHS) and high-energy (RHS) contributions of FESR for a $2 \to 3$ process in the fast-pion limit.
- Implemented the zeroth moment ($j = 0$) across multiple cutoffs and partial-wave configurations; a persistent LHS-RHS mismatch is observed, with the discrepancy reaching up to a factor of ~ 4.
- Examined the sensitivity of the FESR to variations in the parameter u_p.

Outlook

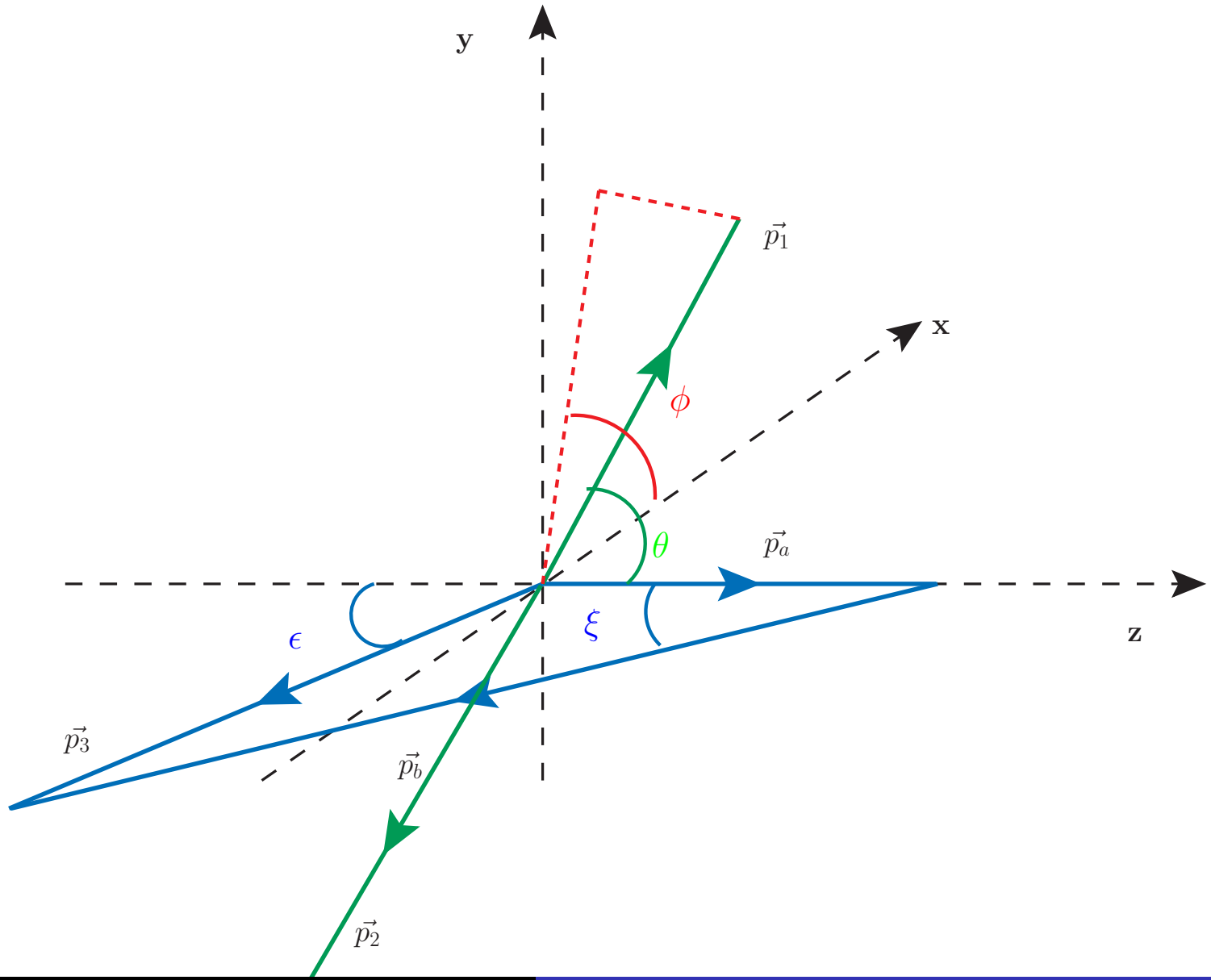
- Identify and understand the underlying source of the mismatch between the LHS and RHS of the FESR.
- Extend the analysis to the $\eta'\pi$ channel.
- Investigate the forward process (**Fast-η**) to study potential forward-backward asymmetries.
- Incorporate and reassess the analysis using the updated COMPASS partial-wave fit.


Thank
you

Back-Up Slides

Kinematics

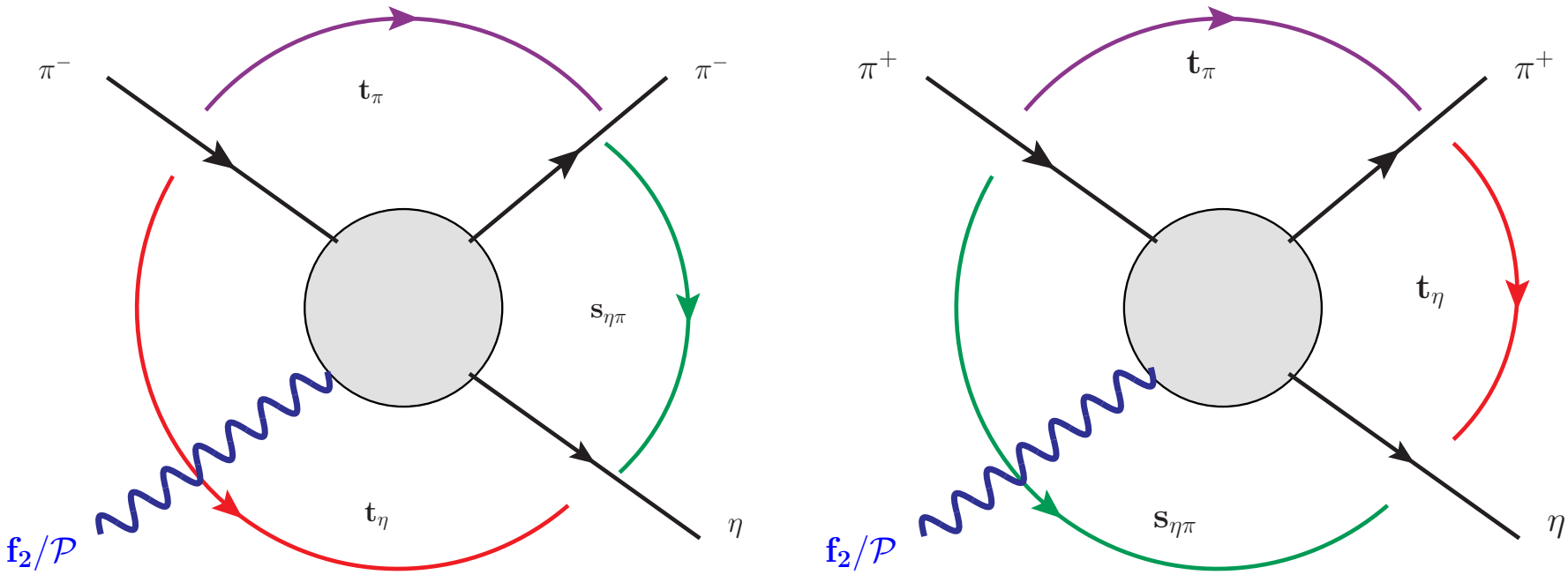
12-GJ-frame:



$$
\begin{aligned}
\boldsymbol{p}_a &= |\boldsymbol{p}_a|(0,0,1) \\
\boldsymbol{p}_b &= |\boldsymbol{p}_b|(-\sin\xi, 0, -\cos\xi) \\
\boldsymbol{p}_3 &= |\boldsymbol{p}_3|(-\sin\epsilon, 0, -\cos\epsilon) \\
\boldsymbol{p}_1 &= |\boldsymbol{p}_1|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \\
\boldsymbol{p}_2 &= -|\boldsymbol{p}_1|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)
\end{aligned}
$$

$$
s = (p_a + p_b)^2\,, \qquad t_i = (p_a - p_i)^2\,, \quad u_i = (p_b - p_i)^2\,, \qquad s_{ij} = (p_i + p_j)^2\,.
$$

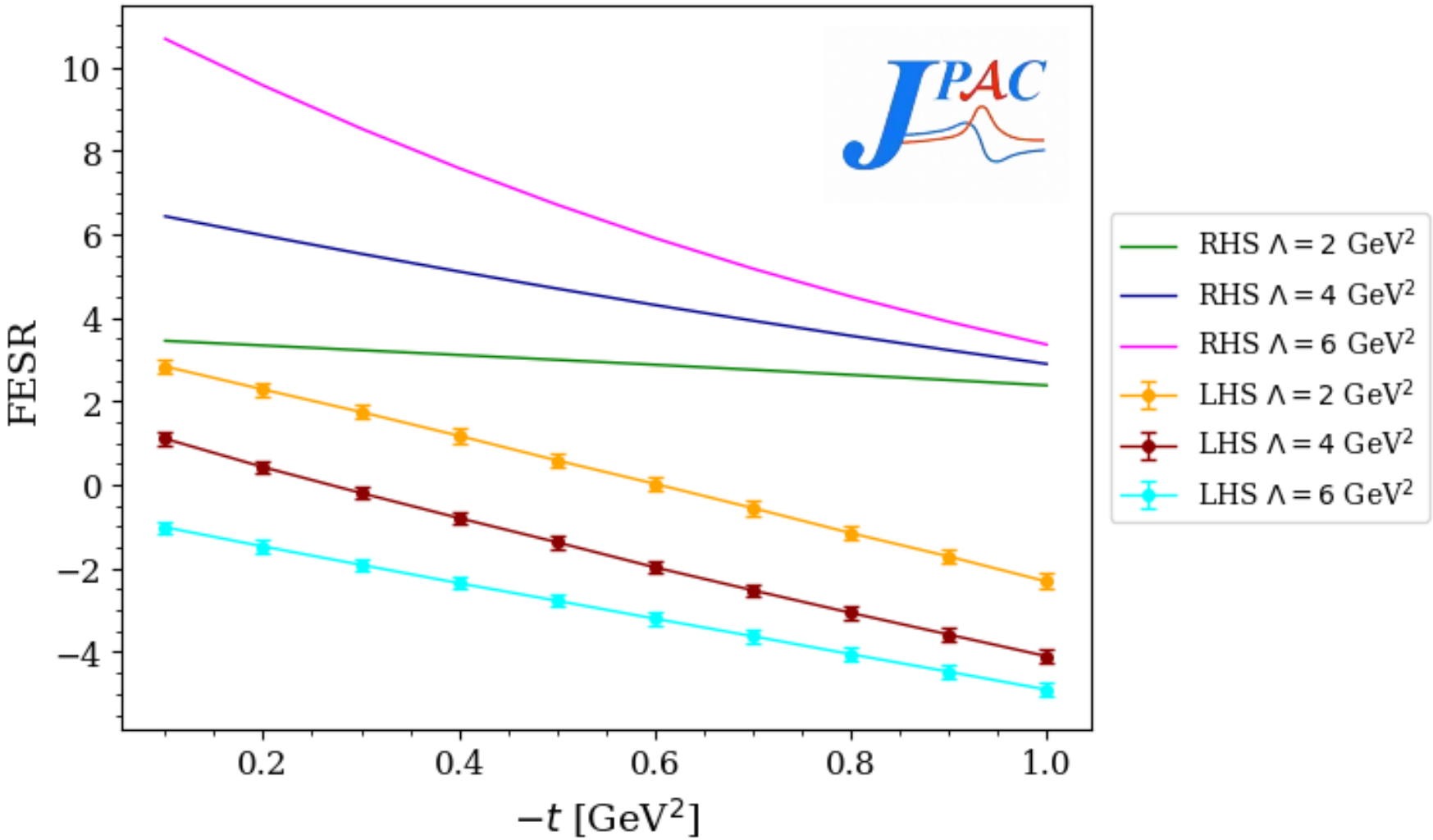
Crossed-Channels



(a) "s-channel" (b) "u-channel"

Figure: Representation of the crossed channels for $\pi p \to \pi\eta p$

FESR-Preliminary Results



FESR-Preliminary Results

$\Lambda = 4\ \text{GeV}^2$