Generalising Moments Analysis for Electroproduction

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Outline

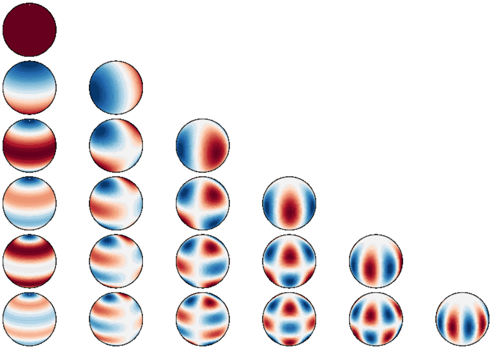

Figure: Example of some spherical harmonics. Red is +1 and blue is -1. Sourced from https://github.com/NVIDIA/torch-harmonics.

Motivation

- Understand resonance production in electron–proton collisions at next-generation facilities.
- Extend moments analysis beyond photoproduction to electroproduction.
- Analyse the structure of exotic hadrons not just there existence i.e. XYZ states
- Compare and connect:
  - Schilling & Wolf (SDME formalism for vector mesons) [1],
  - Moments analysis in photoproduction [2],
  - A unified electroproduction framework.

Goal

- Generalise the current electroproduction formalism so that moments analysis can be performed at current colliders and the future EIC.

The Electron Ion Collider (EIC)

- The EIC will be the next electron–proton collider and the first since HERA.
- Electron–proton collisions are much “cleaner” than proton–proton collisions.
- Provide much higher centre-of-mass / production energies than current LINAC experiments.
- So far, moments analysis has only been performed for photoproduction (e.g. GlueX).
- To perform moments at the EIC, the current electroproduction formalism must be generalised.

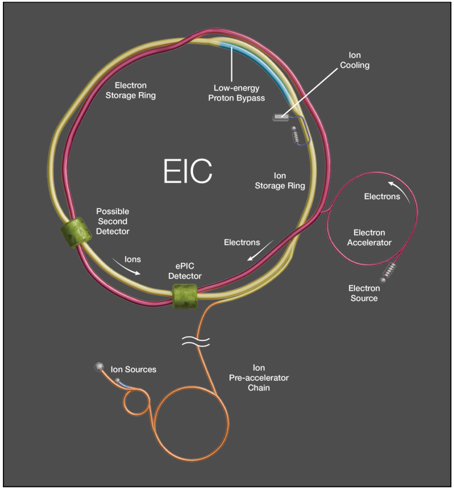

Figure: Schematic of the EIC. Taken from https://www.bnl.gov/eic/machine.php

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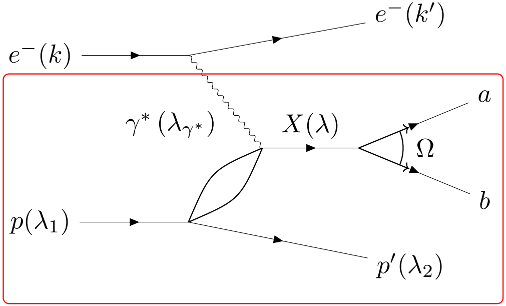
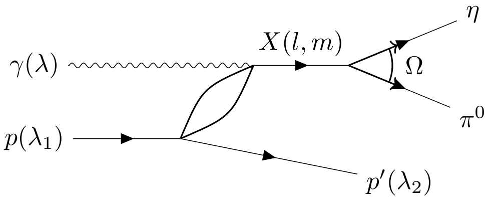
Basic Definitions

- **Electroproduction:** production of a resonance by a *virtual* photon emitted by a scattering electron.
- **Photoproduction:** production of a resonance by a *real* photon in a photon beam.





Moments Analysis Overview (I)

What are Moments?

- Akin to Fourier analysis but decomposing in terms of moments $H(LM)$ (Fourier coeffs) and spherical harmonics $Y_L^M(\Omega)$ (cosine and sine)
- Coefficients then identify and quantify the structure of the resonance

Why Moments?

- For a single resonance (e.g. spin-1 ρ meson), moments are equivalent to SDMEs.
- For multiple resonances with different spins, moments encode
  - interference between resonances,
  - relative magnitudes of partial waves.

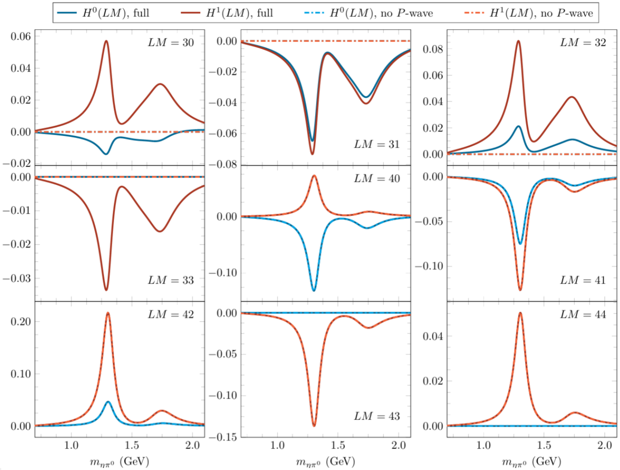


Figure: Example plot of S-, P- and D- wave contributions to the moments for a toy model of an $\eta\pi^0$ decay. Taken from V Matthieu et al. [2]

Moments Analysis Overview (II)

Moments Analysis

- From standard field theory we can write out our 2-body decay intensity as

$$I(\Omega,\Phi) = \frac{\mathrm{d}\sigma}{\mathrm{d}t\,\mathrm{d}m_{ab}\,\mathrm{d}\Omega\,\mathrm{d}\Phi} \tag{1}$$

$$= \kappa \sum_{\substack{\lambda,\lambda' \\ \lambda_1,\lambda_2}} A_{\lambda;\lambda_1\lambda_2}(\Omega)\,\rho^{\gamma^{(*)}}_{\lambda\lambda'}(\Phi)\,A^*_{\lambda';\lambda_1\lambda_2}(\Omega)\,. \tag{2}$$

- Can expand amplitudes out in a basis of spherical harmonics:

$$A_{\lambda;\lambda_1\lambda_2}(\Omega) = \sum_{\ell m} T^{\ell}_{\lambda m;\lambda_1\lambda_2}\; Y^m_\ell(\Omega) \tag{3}$$

- Intensity can also be expanded as a sum of different polarisation contributions

$$I(\Omega) = I^0(\Omega) + \mathbf{I}(\Omega)\mathbf{P}_{\gamma^{(*)}}(\Phi), \tag{4}$$

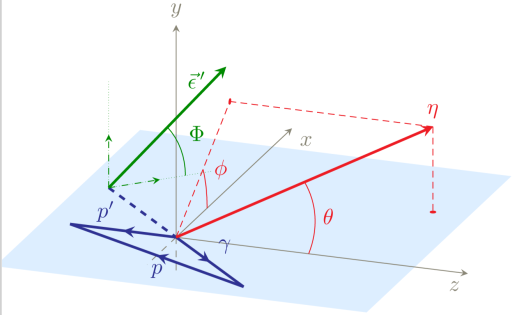


Figure: Diagram of the angles and vectors involved in the $\eta\pi^0$ photoproduction. Taken from [2]

Moments Analysis Overview (III)

Intensity Expansion

- For a simple 2-body decay, the intensities and moments can be written as

$$I^0(\Omega) = \sum_{L,M} H^0(LM)\, Y_L^M(\Omega), \quad \mathbf{I}(\Omega) = -\sum_{L,M} \mathbf{H}(LM)\, Y_L^M(\Omega), \tag{5}$$

$$H^0(LM) = \sum_{\substack{ll'\\mm'}} \left(\frac{2l'+1}{2l+1}\right) C^{l0}_{l'0L0}\, C^{lm}_{l'm'LM}\, \rho^{0,ll'}_{mm'} \text{ and } \mathbf{H}(LM) = \sum_{\substack{ll'\\mm'}} \left(\frac{2l'+1}{2l+1}\right) C^{l0}_{l'0L0}\, C^{lm}_{l'm'LM}\, \boldsymbol{\rho}^{ll'}_{mm'} \tag{6}$$

- Bold values are just vectors containing all $\alpha > 0$ terms
- This index α just corresponds to our different polarisation modes
- $\alpha > 0$ terms are negative purely by convention so that $H^1(00)$ is +ve for +ve naturality

Current Electroproduction Formalism (Schilling & Wolf)

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Setup

- Based on the work by Schilling and Wolf [1].
- Considers decay and electroproduction of a **vector meson**.
- Simpler case due to strict decay rules.

Observables

- Angular distribution expressed in terms of 28 observables:
  - 26 Spin Density Matrix Elements (SDMEs) $\rho^{\alpha}_{\lambda\lambda'}$.
  - 2 cross sections: σ_T (transverse) and σ_L (longitudinal).

Experimental Challenges

- Experimentally, we struggle to separate $\alpha = 0$ and $\alpha = 4$.
- Angular components are non-orthogonal.
- Requires a variation of beam energies to separate the components.

Current Electroproduction Formalism (Schilling & Wolf)

$$W^{\{0,4\}}(\theta,\phi) = \frac{3}{4\pi}\left[\frac{1}{2}(3\rho_{00}^{\{0,4\}} - 1)\cos^2\theta + \frac{1}{2}(1-\rho_{00}^{\{0,4\}}) \right.$$
$$\left. -\sqrt{2}\,\mathrm{Re}(\rho_{10}^{\{0,4\}})\sin(2\theta)\cos\phi - \rho_{1-1}^{\{0,4\}}\sin^2\theta\cos(2\phi)\right], \tag{7}$$

$$W^{\{1,5,8\}}(\theta,\phi) = \frac{3}{4\pi}\left[\rho_{11}^{\{1,5,8\}}\sin^2\theta + \rho_{00}^{\{1,5,8\}}\cos^2\theta \right.$$
$$\left. -\sqrt{2}\,\mathrm{Re}(\rho_{10}^{\{1,5,8\}})\sin(2\theta)\cos\phi - \rho_{1-1}^{\{1,5,8\}}\sin^2\theta\cos(2\phi)\right], \tag{8}$$

$$W^{\{2,3,6,7\}}(\theta,\phi) = \frac{3}{4\pi}\left[\sqrt{2}\,\mathrm{Im}(\rho_{10}^{\{2,3,6,7\}})\sin(2\theta)\sin\phi - \mathrm{Im}(\rho_{1-1}^{\{2,3,6,7\}})\sin^2\theta\sin(2\phi)\right]. \tag{9}$$

Why Move to Electroproduction

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Electroproduction vs Photoproduction

- Main difference is that we now have a virtual photons spin density matrix compared to a real one
- Gives us new cross section components going from just σ_T to σ_T, σ_L, σ_{TL} and σ_{LT}
- Virtual photon gives us Q^2 dependence $\Rightarrow$ different distance scales can be probed and structure of states uncovered
- The polarisation part of our virtual photon can be easily calculated from scattering kinematics so can easily tune experiments

Intensity In Terms of Polarisation Vector

- Can also write out our intensity in terms of the virtual-photon's polarisation vector to get

$$I(\Omega,\Phi) = (1+\epsilon+\delta)^{-1}\Big[I^0 - P_T I^1 \cos 2\Phi - P_T I^2 \sin 2\Phi + P_C P_0 I^3 + P_L I^4 + P_I I^5 \cos\Phi - P_I I^6 \sin\Phi + P_C I^7 (P_1 \cos\Phi + P_2 \sin\Phi) + P_C I^8 (P_1 \sin\Phi - P_2 \cos\Phi)\Big] \quad (10)$$

Practical Issues with SDMEs

Polarisation Variables

- Here P^i where $i = 0, 1, 2, 3$ are the virtual photon 4-polarisation vector components
- The other constants are the collected terms $P_T = \epsilon$, $P_C = \frac{2m}{Q}(1-\epsilon)$, $P_L = \epsilon + \delta$, $P_l = \sqrt{2\epsilon(1+\epsilon+2\delta)}$

Normalised Observables

- Common to quote SDMEs in terms of the 23 normalised observables r^{α}_{ik}.
  - $R = \sigma_L/\sigma_T$
  - δ is an electron mass correction
  - ε is a polarisation parameter
- Get photoproduction result back in the limit $R \to 0$
- This is standard in many current experiments and previous analyses (e.g. HERA [3], HERMES [4], COMPASS [5], etc).

$$r^{04}_{ik} = \frac{\rho^0_{ik} + (\varepsilon+\delta)R\,\rho^4_{ik}}{1+(\varepsilon+\delta)R}, \tag{11}$$

$$r^{\alpha}_{ik} = \begin{cases} \dfrac{\rho^{\alpha}_{ik}}{1+(\varepsilon+\delta)R}, & \alpha = 1\text{–}3, \\ \dfrac{\sqrt{R}\,\rho^{\alpha}_{ik}}{1+(\varepsilon+\delta)R}, & \alpha = 5\text{–}8. \end{cases} \tag{12}$$

Moments Analysis for Photoproduction (I)

Generalisation to Moments

- Generalisation of photoproduction in terms of moments by Vincent Mathieu et al.[2].
- Moments analysis applied to a toy model:

$$X \to \eta\pi^0.$$

Why $\eta^{(\prime)}\pi$?

- $\eta^{(\prime)}\pi$ decays are a *golden channel* for exotic mesons.
- Exotic behaviours appear in individual partial-wave resonances and their interferences.
- Allows spectroscopic identification of possible exotic states.

Note!

- As this is photoproduction we are only considering $\alpha = 0, 1, 2, 3$

Moments Analysis for Photoproduction (II)

From SDMEs to Partial Waves

- Replace SDMEs by their partial-wave expansion.
- Work in the **reflectivity basis**:
  - Basis change where $\pm$ reflectivity states correspond to $\pm$ naturality exchanges.
  - Decouples the system into two independent reflectivities/naturalities.
- Using parity relations, one can decouple target(nucleon) initial and final helicities:
  - $k = 0$: no spin-flip,
  - $k = 1$: single spin-flip.

$$[l]^{\epsilon}_{m;0} = {}^{(\epsilon)}T^{l}_{m;++} = T_{+1m;++} - \epsilon(-1)^{m}T_{-1-m;++} \tag{13}$$

$$[l]^{\epsilon}_{m;1} = {}^{(\epsilon)}T^{l}_{m;+-} = T_{+1m;+-} - \epsilon(-1)^{m}T_{-1-m;+-} \tag{14}$$

Interlude: Naturality

Definition

- Naturality (natural parity) is defined as

$$\eta = P(-1)^J,$$

where:

  - P is the parity,
  - J is the total angular momentum of the meson.

Examples

- **Natural** ($\eta = +1$) exchanges:
  - pomerons, ρs, vector mesons.
- **Unnatural** ($\eta = -1$) exchanges:
  - πs, pseudoscalar mesons.

Electroproduction SDMEs: Strategy

Aim

- Generalise the previously shown formalism.
- Express Schilling & Wolf SDMEs in terms of our reflectivity basis partial waves.

Procedure

- I followed the derivation from Schilling & Wolf.
  - Derive the virutal photon SDMEs from the leptonic part of the matrix element
  - Use these in the von Neumann equation to obtain the vector meson SDMEs.
  - Decompose the vector meson SDMEs in terms of our 9 α states.

Issue!

- Electroproduction introduces helicity-0 states that cannot be transformed by the previous reflectivity transform.

Electroproduction SDMEs: Formulae

Von Neumann Equation

$$\rho^{\alpha,ll'}_{mm'} = \sum_{\lambda,\lambda_1,\lambda_2} T^{l}_{\lambda m,\lambda_1\lambda_2} \rho^{\alpha}_{\lambda\lambda'} T^{l'*}_{\lambda' m',\lambda_1\lambda_2}, \tag{15}$$

Vector Meson SDMEs

$$\rho^{0,ll'}_{mm'} = \frac{\kappa}{2}\sum_{\lambda=\pm1} T^{l}_{\lambda m,\lambda_1\lambda_2} T^{l*}_{\lambda m,\lambda_1\lambda_2}$$

$$\rho^{1,ll'}_{mm'} = \frac{\kappa}{2}\sum_{\lambda=\pm1} T^{l}_{-\lambda m,\lambda_1\lambda_2} T^{l*}_{\lambda m,\lambda_1\lambda_2}$$

$$\rho^{2,ll'}_{mm'} = \frac{i\kappa}{2}\sum_{\lambda=\pm1} \lambda T^{l}_{-\lambda m,\lambda_1\lambda_2} T^{l*}_{\lambda m,\lambda_1\lambda_2}$$

$$\rho^{3,ll'}_{mm'} = \frac{\kappa}{2}\sum_{\lambda=\pm1} \lambda T^{l}_{\lambda m,\lambda_1\lambda_2} T^{l*}_{\lambda m,\lambda_1\lambda_2}$$

$$\rho^{4,ll'}_{mm'} = \kappa T^{l}_{0m,\lambda_1\lambda_2} T^{l*}_{0m,\lambda_1\lambda_2}$$

$$\rho^{5,ll'}_{mm'} = \frac{\kappa}{2\sqrt{2}}\sum_{\lambda=\pm1} \lambda\left(T^{l}_{0m,\lambda_1\lambda_2} T^{l*}_{\lambda m,\lambda_1\lambda_2} + T^{l}_{\lambda m,\lambda_1\lambda_2} T^{l*}_{0m,\lambda_1\lambda_2}\right)$$

$$\rho^{6,ll'}_{mm'} = \frac{i\kappa}{2\sqrt{2}}\sum_{\lambda=\pm1} \left(T^{l}_{0m,\lambda_1\lambda_2} T^{l*}_{\lambda m,\lambda_1\lambda_2} - T^{l}_{\lambda m,\lambda_1\lambda_2} T^{l*}_{0m,\lambda_1\lambda_2}\right)$$

$$\rho^{7,ll'}_{mm'} = \frac{\kappa}{2\sqrt{2}}\sum_{\lambda=\pm1} \left(T^{l}_{0m,\lambda_1\lambda_2} T^{l*}_{\lambda m,\lambda_1\lambda_2} + T^{l}_{\lambda m,\lambda_1\lambda_2} T^{l*}_{0m,\lambda_1\lambda_2}\right)$$

$$\rho^{8,ll'}_{mm'} = \frac{i\kappa}{2\sqrt{2}}\sum_{\lambda=\pm1} \lambda\left(T^{l}_{\lambda m,\lambda_1\lambda_2} T^{l*}_{0m,\lambda_1\lambda_2} - T^{l}_{0m,\lambda_1\lambda_2} T^{l*}_{\lambda m,\lambda_1\lambda_2}\right).$$

The Ansatz: New Reflectivity with T/L

Extended Reflectivity Basis

- Make an ansatz: define a new reflectivity basis as before, but introduce an additional index T/L

$$[l]^{\epsilon}_{Tm;k} = {}^{(\epsilon)}T_{Tm;++/+-} = T_{+1m;++/+-} - \epsilon(-1)^m T_{-1-m;++/+-} \quad (16)$$

$$[l]^{\epsilon}_{Lm;k} = {}^{(\epsilon)}T_{Lm;++/+-} = T_{0m;++/+-} - \epsilon(-1)^m T_{0-m;++/+-}. \quad (17)$$

- corresponding to transverse and longitudinal components of the partial wave.
- Each partial wave now has transverse and longitudinal pieces in the reflectivity basis.

k-basis reminder

- Here our k-basis again just refers to the fact we can reduce our nucelon/target spins from $\lambda_1\lambda_2$ =(++), (−−), (+−) and (−+) to k=0,1; as (++/−−) and (+−/−+) states are related through parity

Consistency Check

- To cast this into the k-basis, we first check the high-energy parity relation.
- Then we compare the number of degrees of freedom (d.o.f.) with the original system.

Proof

$$\hat{\mathcal{P}}\ {}^{(\epsilon)}T^{l}_{Lm;\lambda_1\lambda_2} = {}^{(\epsilon)}T_{L-m;-\lambda_1-\lambda_2} = \frac{1}{2}\left[T^{l}_{0-m;-\lambda_1-\lambda_2} - \epsilon(-1)^{m}T^{l}_{0m;-\lambda_1-\lambda_2}\right]$$

$$= \frac{1}{2}\left[(-1)^{m+\lambda_1-\lambda_2}T^{l}_{0m;\lambda_1\lambda_2} - \epsilon(-1)^{\lambda_1-\lambda_2}(-1)^{2m}T^{l}_{0-m;\lambda_1\lambda_2}\right]$$

$$\implies \text{under exchange m } \leftrightarrow -\text{m}$$

$${}^{(\epsilon)}T_{Lm;-\lambda_1-\lambda_2} = -\epsilon(-1)^{\lambda_1-\lambda_2}\,{}^{(\epsilon)}T_{Lm;\lambda_1\lambda_2}.$$

Casting L into k-basis

- Always have ${}^{(\epsilon)}T_{Lm;\lambda_1\lambda_2} = \pm\,{}^{(\epsilon)}T_{Lm;-\lambda_1-\lambda_2}$
- k-basis $[l]^{\epsilon}_{Lm;k}$ is valid also for the longitudinal part

Checking the Degrees of Freedom

Reflectivity Basis

- For our partial waves $[l]^{\epsilon}_{T\,m;k}$ and $[l]^{\epsilon}_{L\,m;k}$ the total d.o.f. is

$$\underbrace{2 \times 2(2l+1)}_{T} + \underbrace{2 \times \frac{2(2l+1)}{2}}_{L} = 6(2l+1), \tag{18}$$

  where the longitudinal part is symmetric under $-m \Leftrightarrow m$: $[l]^{\epsilon}_{L\,m;k} = -\epsilon\,(-1)^m\,[l]^{\epsilon}_{L\,m;k}$.
- For $m = 0$, only an **unnatural contribution remains**.

Standard Basis

- In the standard basis $T^{l}_{\lambda m;\lambda_1\lambda_2}$ (and considering parity):

$$\frac{3 \times 2^2(2l+1)}{2} = 6(2l+1), \tag{19}$$

Expanding the SDMEs

$$\rho^{0,ll'}_{mm'} = \kappa \sum_k [l]^{\epsilon}_{T-m;k}[l']^{\epsilon *}_{T-m';k} + (-1)^{m'-m}[l]^{\epsilon}_{Tm;k}[l']^{\epsilon *}_{Tm';k}$$

$$\rho^{1,ll'}_{mm'} = -\epsilon\kappa \sum_k (-1)^m [l]^{\epsilon}_{T-m;k}[l']^{\epsilon *}_{Tm';k} + (-1)^{m'}[l]^{\epsilon}_{Tm;k}[l']^{\epsilon *}_{T-m';k}$$

$$\rho^{2,ll'}_{mm'} = -i\epsilon\kappa \sum_k (-1)^m [l]^{\epsilon}_{T-m;k}[l']^{\epsilon *}_{Tm';k} - (-1)^{m'}[l]^{\epsilon}_{Tm;k}[l']^{\epsilon *}_{T-m';k}$$

$$\rho^{3,ll'}_{mm'} = \kappa \sum_k [l]^{\epsilon}_{T-m;k}[l']^{\epsilon *}_{T-m';k} - (-1)^{m'-m}[l]^{\epsilon}_{Tm;k}[l']^{\epsilon *}_{Tm';k}$$

$$\rho^{4,ll'}_{mm'} = 2\kappa [l]^{\epsilon}_{Lm}[l']^{\epsilon *}_{Lm'}$$

$$\rho^{5,ll'}_{mm'} = -\frac{\kappa}{\sqrt{2}}\epsilon \sum_k (-1)^m \left([l]^{\epsilon}_{Lm}[l']^{\epsilon *}_{Tm'} - [l]^{\epsilon}_{Tm}[l']^{\epsilon *}_{Lm'}\right) + (-1)^{m'}\left([l]^{\epsilon *}_{Lm}[l']^{\epsilon}_{Tm'} - [l]^{\epsilon *}_{Tm}[l']^{\epsilon}_{Lm'}\right)$$

$$\rho^{6,ll'}_{mm'} = \frac{i\kappa}{\sqrt{2}} \sum_k [l]^{\epsilon}_{Lm}[l']^{\epsilon *}_{Tm'} - [l]^{\epsilon}_{Tm}[l']^{\epsilon *}_{Lm'} - \epsilon\left((-1)^{m'}[l]^{\epsilon *}_{Lm}[l']^{\epsilon}_{Tm'} - (-1)^m [l]^{\epsilon *}_{Tm}[l']^{\epsilon}_{Lm'}\right)$$

$$\rho^{7,ll'}_{mm'} = -\frac{\kappa}{\sqrt{2}}\epsilon \sum_k (-1)^m \left([l]^{\epsilon}_{Lm}[l']^{\epsilon *}_{Tm'} + [l]^{\epsilon}_{Tm}[l']^{\epsilon *}_{Lm'}\right) + (-1)^{m'}\left([l]^{\epsilon *}_{Lm}[l']^{\epsilon}_{Tm'} + [l]^{\epsilon *}_{Tm}[l']^{\epsilon}_{Lm'}\right)$$

$$\rho^{8,ll'}_{mm'} = \frac{i\kappa}{\sqrt{2}} \sum_k [l]^{\epsilon}_{Lm}[l']^{\epsilon *}_{Tm'} - [l]^{\epsilon}_{Tm}[l']^{\epsilon *}_{Lm'} + \epsilon\left((-1)^{m'}[l]^{\epsilon *}_{Lm}[l']^{\epsilon}_{Tm'} - (-1)^m [l]^{\epsilon *}_{Tm}[l']^{\epsilon}_{Lm'}\right).$$

SDMEs in the New Reflectivity Basis

General Expression

- Expanding the SDMEs in terms of the new reflectivity basis yields a **completely general** formula.
- One can substitute any l (not restricted to vector mesons).
- Different choices of l produce different intensities and moment contributions.

Advantages

- Clean separation of transverse/longitudinal and natural/unnatural contributions.
- Direct bridge between SDMEs and partial-wave moments.

Moments and Intensities in Electroproduction

Definitions

- Definitions of intensities and moments were given previously in Eqs(5)(6).
- Difference between photo- and electroproduction are the inclusion of coefficients $\alpha = 4, 5, 6, 7, 8$.

Derivation Sketch

- Multiply SDMEs by Clebsch–Gordan coefficients.
- Multiply moments by corresponding angular parts (Wigner *D*-functions).
- Combine and simplify using parity and trace relations.
- Expand SDMEs in terms of partial waves in the new reflectivity basis.
- Express moments in terms of our partial waves
- For complicated amounts of possible *l*s use a symbolic solver (e.g. Mathematica) to calculate

Example for S- and P-Waves (Intensities)

$$I^0 = \frac{1}{4\pi}\left[3\left(\rho^{0,11}_{00} - \rho^{0,11}_{1-1} - \rho^{0,11}_{11}\right)\cos^2(\theta) + \left(3\rho^{0,01}_{00} + \rho^{0,10}_{00}\right)\cos(\theta) + \rho^{0,00}_{00} + 3\left(\rho^{0,11}_{1-1} + \rho^{0,11}_{11}\right)\right]$$

$$I^1 = \frac{1}{4\pi}\left[3\left(-\rho^{1,11}_{00} + \rho^{1,11}_{1-1} + \rho^{1,11}_{11}\right)\cos^2(\theta) - \left(3\rho^{1,01}_{00} + \rho^{1,10}_{00}\right)\cos(\theta) - \rho^{1,00}_{00} - 3\left(\rho^{1,11}_{1-1} + \rho^{1,11}_{11}\right)\right]$$

$$I^2 = \frac{1}{4\pi}\left[-3\left(\rho^{2,11}_{00} - \rho^{2,11}_{01} + 2\rho^{2,11}_{10}\right)\cos^2(\theta) - \left(3\rho^{2,01}_{00} + \rho^{2,10}_{00} + 2\rho^{2,10}_{10}\right)\cos(\theta) - \rho^{2,00}_{00} - 3\rho^{2,11}_{01}\right]$$

$$I^3 = \frac{1}{4\pi}\left[-3\left(\rho^{3,11}_{00} - \rho^{3,11}_{01} + 2\rho^{3,11}_{10}\right)\cos^2(\theta) - \left(3\rho^{3,01}_{00} + \rho^{3,10}_{00} + 2\rho^{3,10}_{10}\right)\cos(\theta) - \rho^{3,00}_{00} - 3\rho^{3,11}_{01}\right]$$

$$I^4 = \frac{1}{4\pi}\left[3\left(-\rho^{4,11}_{00} + \rho^{4,11}_{1-1} + \rho^{4,11}_{11}\right)\cos^2(\theta) - \left(3\rho^{4,01}_{00} + \rho^{4,10}_{00}\right)\cos(\theta) - \rho^{4,00}_{00} - 3\left(\rho^{4,11}_{1-1} + \rho^{4,11}_{11}\right)\right]$$

$$I^5 = \frac{1}{4\pi}\left[-3\left(\rho^{5,11}_{00} - \rho^{5,11}_{01} + 2\rho^{5,11}_{10}\right)\cos^2(\theta) - \left(3\rho^{5,01}_{00} + \rho^{5,10}_{00} + 2\rho^{5,10}_{10}\right)\cos(\theta) - \rho^{5,00}_{00} - 3\rho^{5,11}_{01}\right]$$

$$I^6 = \frac{1}{4\pi}\left[-3\left(\rho^{6,11}_{00} - \rho^{6,11}_{01} + 2\rho^{6,11}_{10}\right)\cos^2(\theta) - \left(3\rho^{6,01}_{00} + \rho^{6,10}_{00} + 2\rho^{6,10}_{10}\right)\cos(\theta) - \rho^{6,00}_{00} - 3\rho^{6,11}_{01}\right]$$

$$I^7 = \frac{1}{4\pi}\left[3\left(-\rho^{7,11}_{00} + \rho^{7,11}_{1-1} + \rho^{7,11}_{11}\right)\cos^2(\theta) - \left(3\rho^{7,01}_{00} + \rho^{7,10}_{00}\right)\cos(\theta) - \rho^{7,00}_{00} - 3\left(\rho^{7,11}_{1-1} + \rho^{7,11}_{11}\right)\right]$$

$$I^8 = \frac{1}{4\pi}\left[3\left(-\rho^{8,11}_{00} + \rho^{8,11}_{1-1} + \rho^{8,11}_{11}\right)\cos^2(\theta) - \left(3\rho^{8,01}_{00} + \rho^{8,10}_{00}\right)\cos(\theta) - \rho^{8,00}_{00} - 3\left(\rho^{8,11}_{1-1} + \rho^{8,11}_{11}\right)\right]$$

Example for S- and P-Waves (SDMEs)

$$\rho^{0,00}_{00} = 2\,|S_{T,0}|^2$$
$$\rho^{0,01}_{0-1} = S_{T,0}P_{T,-1}{}^* - S_{T,0}P_{T,1}{}^*$$
$$\rho^{0,01}_{00} = 2S_{T,0}P_{T,0}{}^*$$
$$\rho^{0,01}_{01} = S_{T,0}P_{T,1}{}^* - S_{T,0}P_{T,-1}{}^*$$
$$\rho^{0,10}_{-10} = P_{T,-1}S_{T,0}{}^* - P_{T,1}S_{T,0}{}^*$$
$$\rho^{0,10}_{00} = 2P_{T,0}S_{T,0}{}^*$$
$$\rho^{0,10}_{10} = P_{T,1}S_{T,0}{}^* - P_{T,-1}S_{T,0}{}^*$$
$$\rho^{0,11}_{-1-1} = |P_{T,-1}|^2 + |P_{T,1}|^2$$
$$\rho^{0,11}_{-10} = P_{T,-1}P_{T,0}{}^* - P_{T,1}P_{T,0}{}^*$$
$$\rho^{0,11}_{-11} = 2\Re\left(P_{T,1}P_{T,-1}{}^*\right)$$
$$\rho^{0,11}_{0-1} = P_{T,0}P_{T,-1}{}^* - P_{T,0}P_{T,1}{}^*$$
$$\rho^{0,11}_{00} = 2\,|P_{T,0}|^2$$
$$\rho^{0,11}_{01} = P_{T,0}P_{T,1}{}^* - P_{T,0}P_{T,-1}{}^*$$
$$\rho^{0,11}_{1-1} = 2\Re\left(P_{T,1}P_{T,-1}{}^*\right)$$
$$\rho^{0,11}_{10} = P_{T,1}P_{T,0}{}^* - P_{T,-1}P_{T,0}{}^*$$
$$\rho^{0,11}_{11} = |P_{T,-1}|^2 + |P_{T,1}|^2$$
$$\rho^{1,00}_{00} = -2\,|S_{T,0}|^2$$
$$\rho^{1,01}_{0-1} = S_{T,0}P_{T,1}{}^* - S_{T,0}P_{T,-1}{}^*$$
$$\rho^{1,01}_{00} = -2S_{T,0}P_{T,0}{}^*$$
$$\rho^{1,01}_{01} = S_{T,0}P_{T,-1}{}^* - S_{T,0}P_{T,1}{}^*$$

$$\rho^{1,11}_{-10} = P_{T,1}P_{T,0}{}^* - P_{T,-1}P_{T,0}{}^*$$
$$\rho^{1,11}_{-11} = |P_{T,-1}|^2 + |P_{T,1}|^2$$
$$\rho^{1,11}_{0-1} = P_{T,0}P_{T,1}{}^* - P_{T,0}P_{T,-1}{}^*$$
$$\rho^{1,11}_{00} = -2\,|P_{T,0}|^2$$
$$\rho^{1,11}_{01} = P_{T,0}P_{T,-1}{}^* - P_{T,0}P_{T,1}{}^*$$
$$\rho^{1,11}_{1-1} = |P_{T,-1}|^2 + |P_{T,1}|^2$$
$$\rho^{1,11}_{10} = P_{T,-1}P_{T,0}{}^* - P_{T,1}P_{T,0}{}^*$$
$$\rho^{1,11}_{11} = 2\Re\left(P_{T,1}P_{T,-1}{}^*\right)$$
$$\rho^{2,00}_{00} = 0$$
$$\rho^{2,01}_{0-1} = -i\left(S_{T,0}P_{T,-1}{}^* + S_{T,0}P_{T,1}{}^*\right)$$
$$\rho^{2,01}_{00} = 0$$
$$\rho^{2,01}_{01} = -i\left(S_{T,0}P_{T,-1}{}^* + S_{T,0}P_{T,1}{}^*\right)$$
$$\rho^{2,10}_{-10} = -i\left(P_{T,-1}\left(-S_{T,0}{}^*\right) - P_{T,1}S_{T,0}\right.$$
$$\rho^{2,10}_{00} = 0$$
$$\rho^{2,10}_{10} = -i\left(P_{T,-1}\left(-S_{T,0}{}^*\right) - P_{T,1}S_{T,0}\right.$$
$$\rho^{2,11}_{-1-1} = 2\Im\left(P_{T,-1}P_{T,1}{}^*\right)$$
$$\rho^{2,11}_{-10} = -i\left(P_{T,-1}\left(-P_{T,0}{}^*\right) - P_{T,1}P_{T,0}\right.$$
$$\rho^{2,11}_{-11} = -i\left(|P_{T,-1}|^2 - |P_{T,1}|^2\right)$$
$$\rho^{2,11}_{0-1} = -i\left(P_{T,0}P_{T,-1}{}^* + P_{T,0}P_{T,1}{}^*\right)$$
$$\rho^{2,11}_{00} = 0$$

$$\rho^{5,11}_{-11} = -\frac{1}{\sqrt{2}}\left(2i\Im\left(P_{T,1}P_{L,-1}{}^*\right) - P_{L,1}P_{T,-1}{}^* + P_{T,-1}P_{L,1}{}^*\right)$$
$$\rho^{5,11}_{0-1} = -\frac{1}{\sqrt{2}}\left(2\Re\left(P_{T,-1}P_{L,0}{}^*\right) + P_{L,-1}\left(-P_{T,0}{}^*\right) - P_{T,0}P_{L,-1}{}^*\right)$$
$$\rho^{5,11}_{00} = -\frac{1}{\sqrt{2}}\left(2P_{L,0}P_{T,0}{}^* - 2P_{T,0}P_{L,0}{}^*\right)$$
$$\rho^{5,11}_{01} = -\frac{1}{\sqrt{2}}\left(2\Re\left(P_{T,1}P_{L,0}{}^*\right) + P_{L,1}\left(-P_{T,0}{}^*\right) - P_{T,0}P_{L,1}{}^*\right)$$
$$\rho^{5,11}_{1-1} = -\frac{1}{\sqrt{2}}\left(2i\Im\left(P_{T,1}P_{L,-1}{}^*\right) - P_{L,1}P_{T,-1}{}^* + P_{T,-1}P_{L,1}{}^*\right)$$
$$\rho^{5,11}_{10} = -\frac{1}{\sqrt{2}}\left(2\Re\left(P_{T,1}P_{L,0}{}^*\right) + P_{L,1}\left(-P_{T,0}{}^*\right) - P_{T,0}P_{L,1}{}^*\right)$$
$$\rho^{5,11}_{11} = -\frac{1}{\sqrt{2}}\left(2P_{T,1}P_{L,1}{}^* - 2P_{L,1}P_{T,1}{}^*\right)$$
$$\rho^{6,00}_{00} = 0$$
$$\rho^{6,01}_{0-1} = \frac{i}{\sqrt{2}}\left(S_{L,-1}P_{T,0}{}^* + S_{L,0}P_{T,-1}{}^* - S_{T,-1}P_{L,0}{}^* - S_{T,0}P_{L,-1}{}^*\right)$$
$$\rho^{6,01}_{00} = 0$$
$$\rho^{6,01}_{01} = \frac{i}{\sqrt{2}}\left(S_{L,0}P_{T,1}{}^* + S_{L,1}P_{T,0}{}^* - S_{T,0}P_{L,1}{}^* - S_{T,1}P_{L,0}{}^*\right)$$
$$\rho^{6,10}_{-10} = \frac{i}{\sqrt{2}}\left(P_{L,-1}S_{T,0}{}^* - P_{L,0}S_{T,-1}{}^* - P_{T,-1}S_{L,0}{}^* + P_{T,0}S_{L,-1}{}^*\right)$$
$$\rho^{6,10}_{00} = 0$$

Future Work

University of Glasgow

Validation and Toy Models

- Test the formalism using dedicated toy models and input some form-factor Q^2 dependence.
- Compare with existing photoproduction results in real/quasi-real photon limit.

Applications to Data

- Investigate current reaction data at polarised electron-beam experiments (e.g. JLab / CLAS).
- At the EIC, exploit access to a polarised proton beam by extending the treatment to polarised targets (nucleons),
- Extend from 2-body to 3-body decays for richer spectroscopy.

Summary & Questions

- Reviewed the Schilling & Wolf SDME formalism for electroproduction.
- Connected moments analysis in photoproduction to a generalised electroproduction framework.
- Introduced an extended reflectivity basis with transverse/longitudinal components and checked its consistency.
- Laid groundwork to apply moments analysis to electroproduction to the EIC and previous exlectroproduction experiments.

Any Questions?

References I

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[5] G. D. Alexeev et al. "Spin density matrix elements in exclusive ρ^0 meson muoproduction". In: *Eur. Phys. J. C* 83.10 (2023), p. 924. DOI: 10.1140/epjc/s10052-023-11359-4. arXiv: 2210.16932 [hep-ex].

Backup Slides

- Use these for additional derivation details, intermediate steps, or checks.
- Add specific algebra for:
  - parity relations for Transverse reflectivity,
  - k-basis example, can describe coordinates in terms of (1,0),(-1,0) (0,-1) and (0,-1) but ofcourse these 4 are equivalently just (1,0) and (0,1) or (-1,0) and (0,-1) as they are -ves of each other
  - Screenshot of mathematica and outputs?,
  - Maybe have matrix and decomposition?

SDME Basis Matrices and Coefficient Vector

Full Spin Density Matrix

$$\begin{pmatrix} 1-\sqrt{1-\varepsilon^2}\,P\cos\alpha_2 & e^{-i\Phi}\left(\sqrt{\varepsilon(1+\varepsilon+2\delta)}-\sqrt{\varepsilon(1-\varepsilon)}\,P\cos\alpha_2\right) & -\varepsilon e^{-2i\Phi} \\ e^{i\Phi}\left(\sqrt{\varepsilon(1+\varepsilon+2\delta)}-\sqrt{\varepsilon(1-\varepsilon)}\,P\cos\alpha_2\right) & 2(\varepsilon+\delta) & -e^{-i\Phi}\left(\sqrt{\varepsilon(1+\varepsilon+2\delta)}+\sqrt{\varepsilon(1-\varepsilon)}\,P\cos\alpha_2\right) \\ -\varepsilon e^{2i\Phi} & -e^{i\Phi}\left(\sqrt{\varepsilon(1+\varepsilon+2\delta)}+\sqrt{\varepsilon(1-\varepsilon)}\,P\cos\alpha_2\right) & 1+\sqrt{1-\varepsilon^2}\,P\cos\alpha_2 \end{pmatrix}$$

Why can we decompose this?

- As our spin density matrix is a hermitian matrix it can always be decomposed into N^2 basis matrices (N is the dimension of the matrix)
- General complex matrices form a vector space over $\mathbb{R}$ of dimension $2N^2$
- Hermitian matrices are a subspace of this with half the d.o.f so they form a vector space of N^2

SDME Basis Matrices and Coefficient Vector

γ^* Vector of Coefficients

$$\Pi = \{\, 1,\ -\varepsilon\cos(2\Phi),\ -\varepsilon\sin(2\Phi),\ \tfrac{2m}{Q}(1-\varepsilon)P_0,\ \varepsilon+\delta,\ \sqrt{2\varepsilon(1+\varepsilon+2\delta)}\cos\Phi,\ \sqrt{2\varepsilon(1+\varepsilon+2\delta)}\sin\Phi,\ \tfrac{2m}{Q}(1-\varepsilon)(P_1\cos\Phi+P_2\sin\Phi),\ \tfrac{2m}{Q}(1-\varepsilon)(P_1\sin\Phi-P_2\cos\Phi)\,\}$$

Vector of Vector Meson SDME Coefficients

$$\Pi = \frac{1}{1+(\varepsilon+\delta)R}\{\, 1,\ -\varepsilon\cos(2\Phi),\ -\varepsilon\sin(2\Phi),\ \tfrac{2m}{Q}(1-\varepsilon)P_0,\ \varepsilon+\delta,\ \sqrt{2\varepsilon(1+\varepsilon+2\delta)}\cos\Phi,\ \sqrt{2\varepsilon(1+\varepsilon+2\delta)}\sin\Phi,\ \tfrac{2m}{Q}(1-\varepsilon)(P_1\cos\Phi+P_2\sin\Phi),\ \tfrac{2m}{Q}(1-\varepsilon)(P_1\sin\Phi-P_2\cos\Phi)\,\}$$

SDME Basis Matrices and Coefficient Vector

Basis Matrices

$$\Sigma^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Sigma^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \Sigma^2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$\Sigma^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \Sigma^4 = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Sigma^5 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$\Sigma^6 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad \Sigma^7 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Sigma^8 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

Mathematica Script and Outputs Demo

k-Basis Example

- Why can we convert our nucleon lambdas into this spin-flip/non-spin-flip basis?
  - From our parity relation we have (ignoring other indices) $T_{++} = \pm T_{--}$ and $T_{+-} = \pm T_{-+}$
  - Redundant to have all four summed over if $(--)$ is needed we just take $\pm$ of $(++)$
- $+$ factor is trivial but why can we reduce our basis even when they are related by the factor -1?

Example:

- Can describe a standard 2-D cartesian basis in terms of $(1, 0), (0, 1), (-1, 0), (0, -1)$
- With this we can describe every point on a 2-D plane
- Clearly redundant as $(-1, 0) = -(1, 0)$
- So we just express our basis in terms of the linearly independent $(1, 0)$ and $(0, 1)$
- Same principle but applied to a more abstract case

Transverse Parity Proof

Proof

$$
\begin{aligned}
\hat{\mathcal{P}}\ {}^{(\epsilon)}T^{l}{}_{Tm;\lambda_1\lambda_2} = {}^{(\epsilon)}T_{T-m;-\lambda_1-\lambda_2} &= \frac{1}{2}\left[T^{l}_{1-m;-\lambda_1-\lambda_2} - \epsilon(-1)^{m}T^{l}_{-1m;-\lambda_1-\lambda_2}\right] \\
&= \frac{1}{2}\left[-(-1)^{m+\lambda_1-\lambda_2}T^{l}_{-1m;\lambda_1\lambda_2} + \epsilon(-1)^{\lambda_1-\lambda_2}(-1)^{2m}T^{l}_{1-m;\lambda_1\lambda_2}\right] \\
&\implies \text{under exchange m } \leftrightarrow -\text{m} \\
{}^{(\epsilon)}T_{Tm;-\lambda_1-\lambda_2} &= \epsilon(-1)^{\lambda_1-\lambda_2}\,{}^{(\epsilon)}T_{Tm;\lambda_1\lambda_2}.
\end{aligned}
$$