Generalized Fermi acceleration

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Outline:

1. **A generalized view of Fermi acceleration**
   [M.L. 19,   arXiv:1903.05917]

2. **Application to particle acceleration in relativistic turbulence**
   [PhD work of Camilia Demidem, Demidem+19a, b, in prep]

General principles of particle acceleration

Standard lore:

→ Lorentz force: $\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = q\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right)$

→ recall: $\boldsymbol{E} \cdot \boldsymbol{B}$ and $\boldsymbol{E}^2 - \boldsymbol{B}^2$ Lorentz scalars

Case 1: $\boldsymbol{E} \cdot \boldsymbol{B} = 0 \text{ and } \boldsymbol{E}^2 - \boldsymbol{B}^2 < 0$

→ generic because it corresponds to ideal MHD assumptions…

→ ∃ a frame in which $\mathbf{E}_{|p}$ vanishes: the plasma rest frame for ideal MHD

→ examples: Fermi-type scenarios (turbulence, shear, shocks)

Case 2: $\boldsymbol{E} \cdot \boldsymbol{B} \neq 0 \text{ or } \boldsymbol{E}^2 - \boldsymbol{B}^2 > 0$

→ acceleration can proceed unbounded along **E** (or at least $\mathbf{E}_{\parallel}$)…

→ examples: reconnection, gaps

Fermi acceleration scenarios

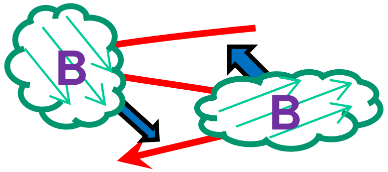
Ideal MHD:

→ ***E*** field is 'motional', i.e. if plasma moves at velocity $\boldsymbol{\beta}_{\rm p}$: $\mathbf{E} = -\boldsymbol{\beta}_{\rm p} \times \mathbf{B}$

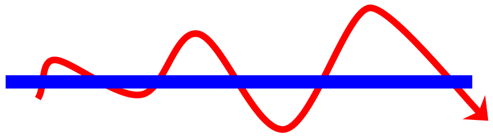
→ **need scattering to push particles across *B***

⇒ t_{acc} scales with the scattering time t_{scatt} (time needed to enter random walk)

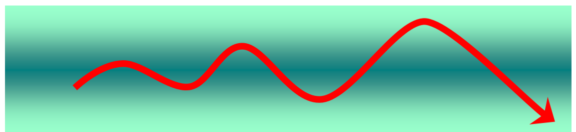
→ examples: - turbulent Fermi acceleration



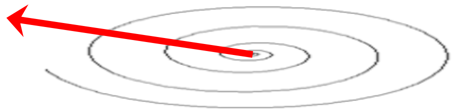
- Fermi acceleration at shock waves



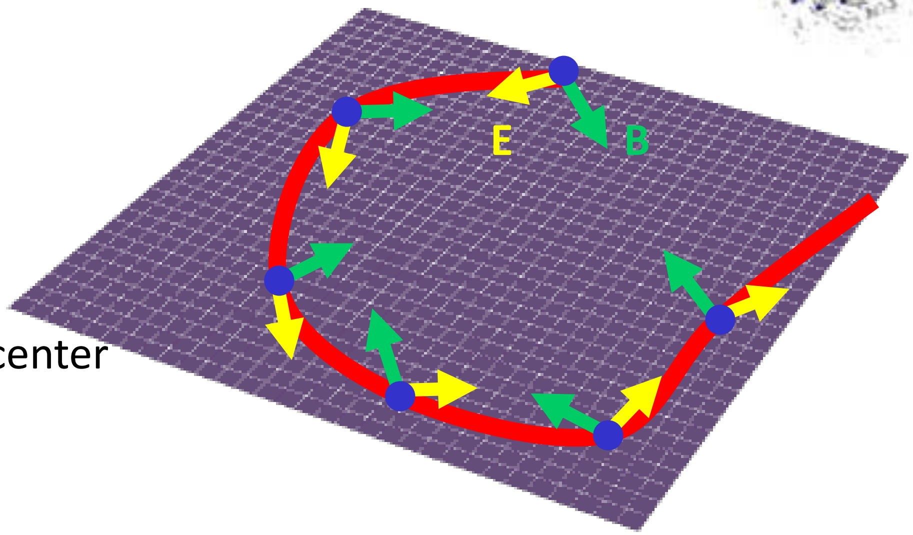
- acceleration in sheared velocity fields



- magnetized rotators



Fermi acceleration



<u>General problem</u>**:** integrate trajectories of particles in a non-uniform **(E,B)** configuration…

→ various schemes depending on the situation…

- explicit integration of trajectories with guiding center approximations in known E,B…

- in Fermi 1, go to local rest frame downstream, then upstream, then downstream…

- quasilinear theory: assume particles propagate on straight orbits and accumulate energy increments…

- in complex flows, derive transport equation and identify energy gain terms…

Fermi acceleration… generalized

General problem: integrate trajectories of particle
in a non-uniform **(E,B)** configuration…

→ various schemes depending on the situation…

- explicit integration of trajectories with guiding c

Fermi 49

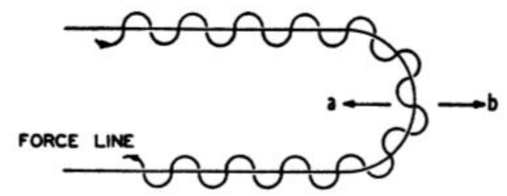
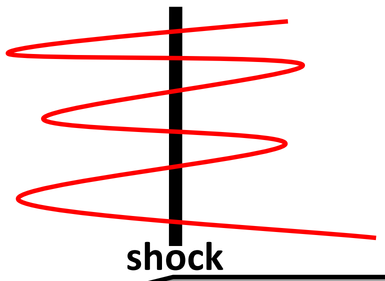
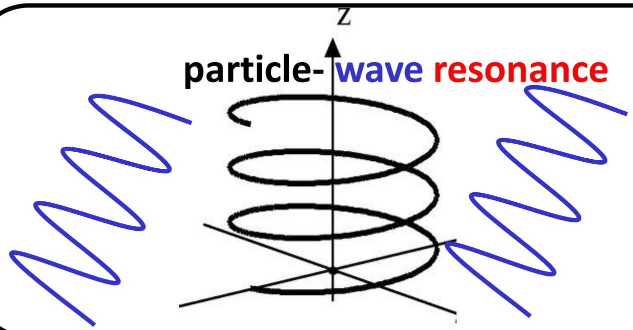


FIG. 1. Type *B* reflection of a cosmic-ray particle.





downstream, th

- quasilinear theory: assu
energy increments…

2.1. *The Transport Equation*

From Webb (1989), the diffusive energetic particle transport equation for the mean scattering frame distribution $f(x^\alpha, p')$ may be written as:

$$L(f) = \nabla_\alpha(cu^\alpha f + q^\alpha) + \frac{1}{p'^2}\frac{\partial}{\partial p'}\left[-\frac{1}{3}p'^3 c\nabla_\beta u^\beta f - p'(p'^0)^2 \dot{u}_\alpha q^\alpha - (\Gamma p'^4 \tau_c + p'^2 D_{pp})\frac{\partial f}{\partial p'}\right] = 0\,, \qquad (2.1)$$

- in complex flows, derive transport equation and identify energy gain terms…

Present scheme: at each point along the particle trajectory, define: $\boldsymbol{\beta}_\mathbf{u} = \dfrac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}$

… and deboost by this velocity to go to the reference frame in which **E** vanishes to compute the force (elastic scattering on B!), model trajectory as random walk…

… here a local transform, hence a problem of general relativity …

Moving from a global to a local comoving inertial frame

→ GR allows to consider a curved space-time…

→ define:

(1) coordinate basis in lab frame $\mathcal{R}_\mathrm{L}$ with line element:

$$\mathrm{d}s^2 = g_{\mu\nu}(\mathrm{x})\, \mathrm{d}x^\mu \mathrm{d}x^\nu$$

(2) a locally inertial frame $\overline{\mathcal{R}}_\mathrm{L}$ at each space-time point, set up by the vierbein:

$$e_\mathrm{L}{}^{\overline{a}}{}_\mu(\mathrm{x}) : \quad \eta_{\overline{a}\overline{b}}\, e_\mathrm{L}{}^{\overline{a}}{}_\mu e_\mathrm{L}{}^{\overline{b}}{}_\nu = g_{\mu\nu}$$

4-velocity in local frame $\quad u^{\overline{a}} = e_\mathrm{L}{}^{\overline{a}}{}_\mu\, u^\mu \quad$ 4-velocity in global frame

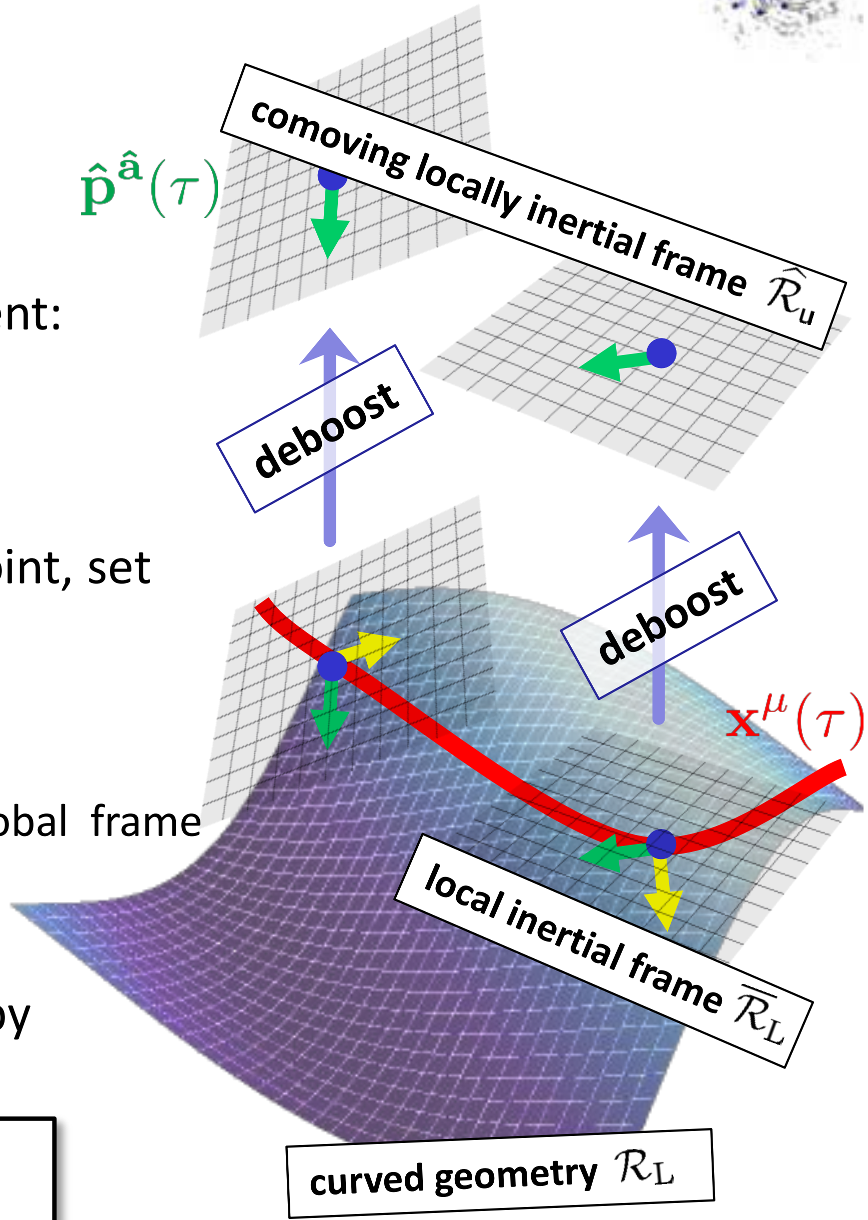
(3) a comoving locally inertial frame $\widehat{\mathcal{R}}_\mathrm{u}$, obtained by (Lorentz) deboosting from the former by $u^{\overline{a}}$:

$$e^{\hat{a}}{}_\mu(\mathrm{x}) : \quad \eta_{\hat{a}\hat{b}}\, e^{\hat{a}}{}_\mu e^{\hat{b}}{}_\nu = g_{\mu\nu}$$

$$u^{\hat{a}} = \{1, 0, 0, 0\} = e^{\hat{a}}{}_\mu\, u^\mu$$

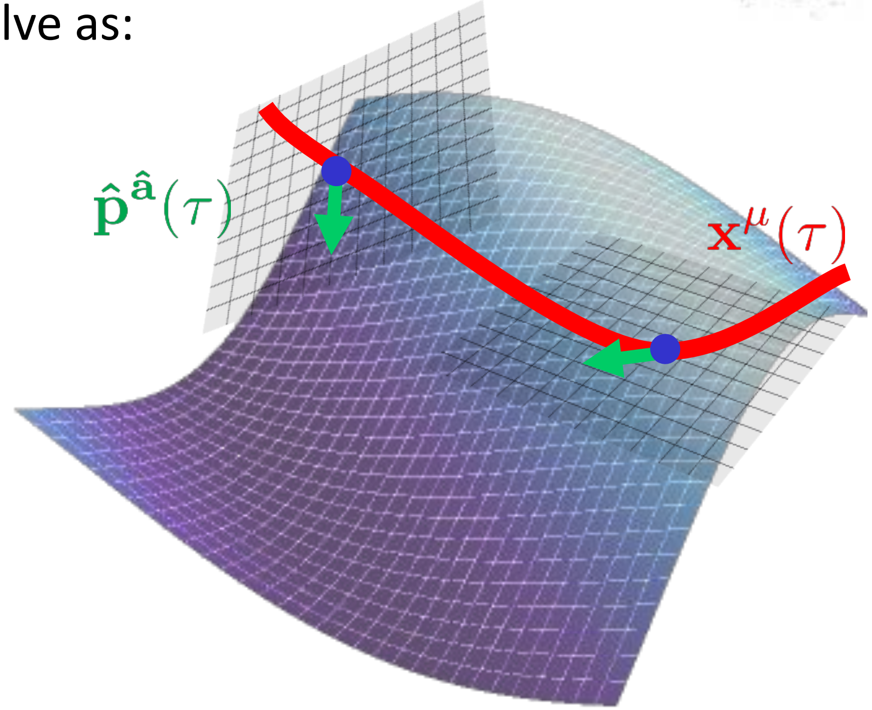
$$p^{\hat{a}} = e^{\hat{a}}{}_\mu\, p^\mu$$

momentum $p^{\hat{a}}$ lives in the comoving locally inertial frame



Generalized Fermi acceleration: particle kinetics

→ in the comoving locally inertial frame, momenta evolve as:



$$\frac{\mathrm{d}\hat{p}^{\hat{a}}}{\mathrm{d}\tau} = \frac{q}{m}\,\widehat{F}^{\hat{a}}{}_{\hat{b}}\,\hat{p}^{\hat{b}} - \widehat{\Gamma}^{\hat{a}}_{\hat{b}\hat{c}}\,\frac{\hat{p}^{\hat{b}}\hat{p}^{\hat{c}}}{m}$$

Lorentz force | inertial correction

→ in particular:

$$\frac{\mathrm{d}\hat{p}^{\hat{0}}}{\mathrm{d}\tau} = e^{\beta}{}_{\hat{b}}\,e^{\gamma}{}_{\hat{c}}\,\nabla_{\gamma}e^{\hat{0}}{}_{\beta}\,\frac{\hat{p}^{\hat{b}}\hat{p}^{\hat{c}}}{m}$$

… and the vierbein ≈ u^{μ} : spacetime dependence of u is essential for Fermi acceleration… E and B have disappeared from the calculation…

→ **calculation scheme**:

(a) follow spatial coordinates in global frame, momenta in comoving frame…

(b) Lorentz force acts on angular part of momentum in comoving frame, not on energy, eventually leading to spatial diffusion with typical scattering time t_s.

(c) integrate over typical trajectory to obtain $\langle \Delta p^{\alpha}/\Delta t\rangle$ and $\langle \Delta p^{\alpha}\Delta p^{\beta}/\Delta t\rangle$, e.g.

$$\left\langle\frac{\Delta p^t}{\Delta t}\right\rangle = \lim_{\Delta t\to+\infty}\frac{1}{\Delta t}\left\langle e^{t}{}_{\hat{a}}(\Delta\tau)\int_0^{\Delta\tau}\mathrm{d}\tau_1\,\frac{\mathrm{d}\hat{p}^{\hat{a}}}{\mathrm{d}\tau_1}\right\rangle + \dots$$

Example: stochastic unipolar induction

→ wind in spherical geometry: $u^\mu(r,\theta) = \left[u^t(r,\theta), u^r(r,\theta), 0, 0\right]$

→ connection coefficients: $\hat{\Gamma}^{\hat{0}}_{\hat{1}\hat{0}} = \frac{u^r\, {u^r}_{,r}}{u^t},\ \hat{\Gamma}^{\hat{0}}_{\hat{1}\hat{1}} = {u^r}_{,r},\ \hat{\Gamma}^{\hat{0}}_{\hat{1}\hat{2}} = \frac{{u^r}_{,\theta}}{r u^t},\ \hat{\Gamma}^{\hat{0}}_{\hat{2}\hat{2}} = \hat{\Gamma}^{\hat{0}}_{\hat{3}\hat{3}} = \frac{u^r}{r}$

→ evolution of energy in comoving frame: $\frac{\mathrm{d}\hat{p}^{\hat{0}}}{\mathrm{d}\tau} = -\hat{\Gamma}^{\hat{0}}_{\hat{b}\hat{c}} \frac{\hat{p}^{\hat{b}}\hat{p}^{\hat{c}}}{m}$

→ mean drift (1st order moment):

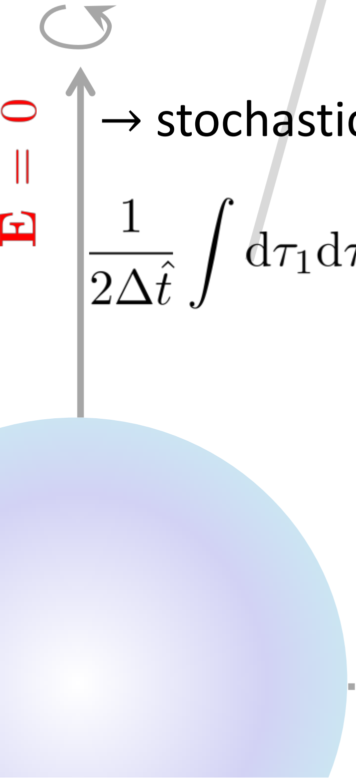
$$\left\langle \frac{\mathrm{d}\hat{p}^{\hat{0}}}{\mathrm{d}\tau} \right\rangle = -\frac{1}{3}\frac{\hat{p}^2}{m}\langle \boldsymbol{\nabla}\cdot\mathbf{u}\rangle$$

describing cooling through expansion…

→ stochastic term (2nd order moment):

$$\frac{1}{2\Delta\hat{t}}\int \mathrm{d}\tau_1 \mathrm{d}\tau_2 \left\langle \frac{\mathrm{d}\hat{p}^{\hat{0}}}{\mathrm{d}\tau_1}\frac{\mathrm{d}\hat{p}^{\hat{0}}}{\mathrm{d}\tau_2} \right\rangle - \left\langle \frac{\mathrm{d}\hat{p}^{\hat{0}}}{\mathrm{d}\tau_1} \right\rangle \left\langle \frac{\mathrm{d}\hat{p}^{\hat{0}}}{\mathrm{d}\tau_2} \right\rangle = \frac{\hat{p}^2 \hat{t}_{\mathrm{scatt}}}{18 r^2}\left[4{u^r}^2 + 2(r{u^r}_{,r})^2 + \frac{{{u^r}_{,\theta}}^2}{{u^t}^2} + 6\frac{(r{u^r}_{,r})^2}{{u^t}^2}\right]$$

describing heating/acceleration through shear and acceleration terms…



$$\mathbf{E} = -\mathbf{v}\times\mathbf{B}$$

Example: acceleration in black hole environments...

Stochastic gravito-centrifugo-shear acceleration near a (Schwarzschild) BH:

→ circular flow at angular velocity Ω(r): $u^\mu = \gamma_u(r)\,[1, 0, \Omega(r), 0]$

→ flow carries turbulence, which provides a source of scattering...
acceleration takes place through interaction with sheared flow...
(e.g. Rieger+Mannheim 02)

→ global frame: $\mathcal{R}_{\rm L}$= frame in which central object at rest..

→ at the equator,

$$D_{\hat p\hat p} = \hat p^2 \frac{\hat t_{\rm scatt}}{3r^2\left[1+\left(1-\frac{r_{\rm H}}{r}\right)\ell^2/r^2\right]^2} \times \left\{\left[\left(1-\frac{r_{\rm H}}{r}\right)^{3/2}\frac{\ell^2}{r^2} - \left(1-\frac{r_{\rm H}}{r}\right)^{-1/2}\frac{r_{\rm H}}{r}\right]^2 + \frac{\sqrt{2}}{6}\,r^4{\Omega_{,r}}^2\right\}$$

centrifugal term
$$\ell = \frac{r^2\,\Omega}{1-\frac{r_{\rm H}}{r}}$$

gravitational term

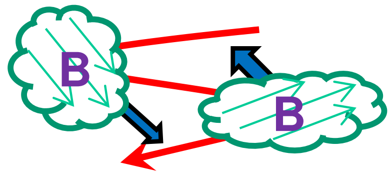
shear term

Turbulent acceleration

(e.g. Bykov, Chandran, Jokipii, Lazarian, Matthaeus, Petrosian, Ptuskin, Schlickeiser, and coworkers…)

Fermi model for acceleration:

… particle interaction with random moving scattering centers…
… acceleration becomes stochastic with diffusion coefficient:



$$\left\langle \frac{\Delta p^t \Delta p^t}{2\Delta t} \right\rangle \sim \beta_u^2 \frac{p^2}{t_{\rm int}}$$

… what is $t_{\rm int}$? β_u ?

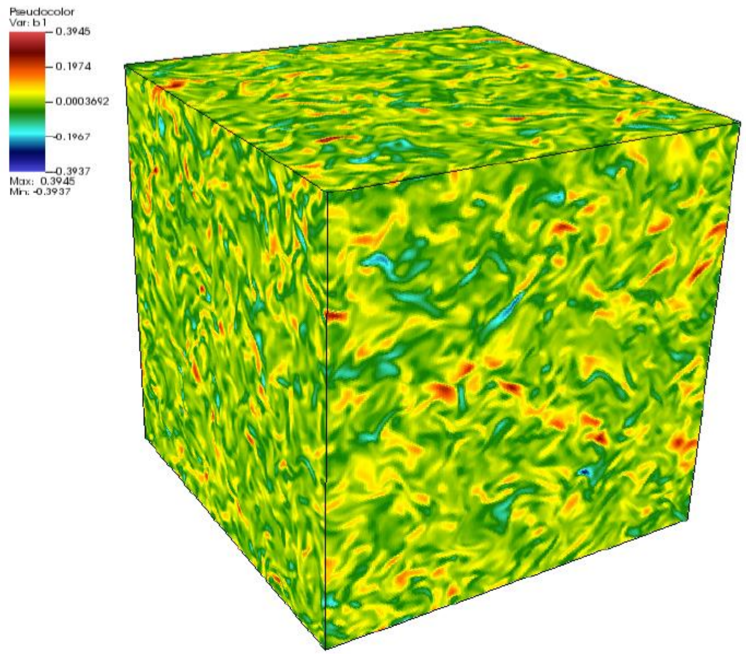
Wave picture: particles diffuse and gain energy through *resonant interactions* with the waves of the plasma (e.g. Alfvén, magnetosonic) , $t_{\rm int} \sim t_{\rm scatt}$ and $\beta_u \sim \beta_A$…

Current questions:

… do resonances exist in realistic (GS) turbulence theories (Chandran 00, Yan+Lazarian 02) **… ??**

… do waves provide a faithful representation of (strong) turbulence?

… in relativistic (high β_A) regime??



B_x in relativistic MHD turbulence (256^3)
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Non-resonant acceleration in strong turbulence

<u>Non-perturbative description:</u> follow transport of particle in momentum space in a continuous sequence of (non-inertial) local plasma rest frames, where the electric field vanishes at each point... (M.L. 19)

... evolution of energy in local plasma rest frame $$\frac{\mathrm{d}\hat{p}^{\hat{0}}}{\mathrm{d}\tau} = e^{\beta}{}_{\hat{b}}\, e^{\gamma}{}_{\hat{c}}\, \frac{\partial}{\partial x^{\gamma}} e^{\hat{0}}{}_{\beta}\, \frac{\hat{p}^{\hat{b}}\hat{p}^{\hat{c}}}{m}$$

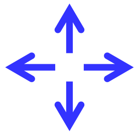
with $e^{\hat{a}}{}_{\mu}$ vierbein to comoving plasma frame

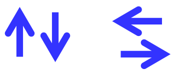
... specify the statistics of the velocity field (→ statistics of the vierbein):

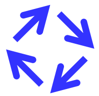
$$U^i = \cancel{\langle U^i \rangle} + u^i$$

u^i : here, of fixed magnitude, random orientation with coherence length and time k^{-1}

$$\frac{\partial u^i}{\partial x^{\alpha}} = u^i{}_{,\alpha} = \frac{1}{3}\theta\delta^i{}_{\alpha} + \sigma^i{}_{\alpha} + \omega^i{}_{\alpha} + a^i{}_{\alpha}$$

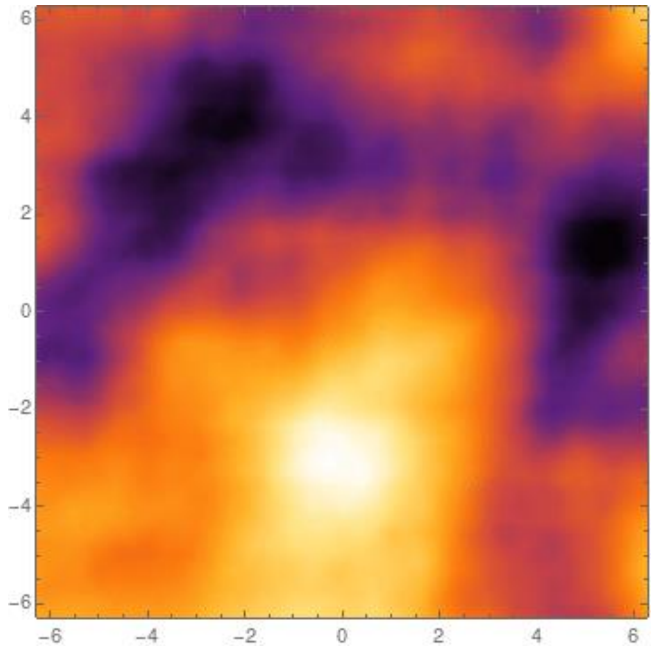
***θ*: expansion scalar** 

***σ*: shear tensor** 

***ω*: vorticity tensor** 

***a*: acceleration term** 

... each defined by a power spectrum of fluctuations



Non-resonant acceleration in strong turbulence

For a near-monochromatic spectrum with length scale k_{min}^{-1}...

... then, to lowest order in $\mathbf{u}^2$ (sub- to mildly-relativistic flow):

$$\begin{cases} \left\langle \dfrac{\Delta p^t \Delta p^t}{2\Delta t} \right\rangle = \dfrac{\sqrt{2}}{3} p^2 t_{\rm scatt} \left[\langle \theta^2 \rangle + \dfrac{3}{5} \langle \sigma^2 \rangle + \dfrac{3}{\sqrt{2}} \langle a^2 \rangle \right] & \left(t_{\rm scatt} \lesssim k_{\rm min}^{-1} c^{-1} \right) \\ \left\langle \dfrac{\Delta p^t \Delta p^t}{2\Delta t} \right\rangle = \dfrac{4}{\pi} \dfrac{p^2}{k_{\rm min}^2 t_{\rm scatt}} \left[\langle \theta^2 \rangle + \dfrac{3}{5} \langle \sigma^2 \rangle + \dfrac{3}{2} \langle a^2 \rangle \right] & \left(t_{\rm scatt} \gtrsim k_{\rm min}^{-1} c^{-1} \right) \end{cases}$$

Interpretation: $\langle \theta^2 \rangle \sim \left\langle (\boldsymbol{\nabla} \cdot \mathbf{u})^2 \right\rangle \sim k_{\rm min}^2 \delta u_{\rm comp.}^2$

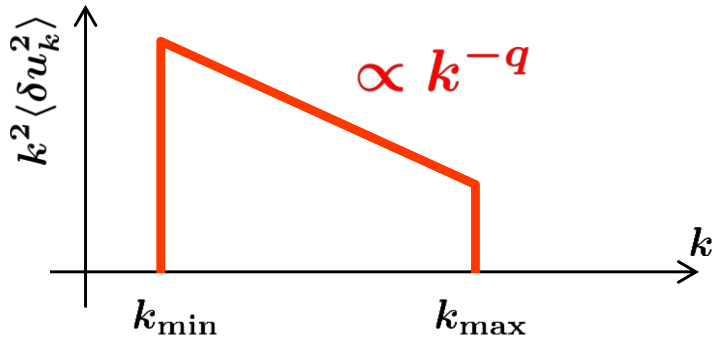
⇒ for $t_{scatt} << k_{min}^{-1}$, it takes a time ~ k_{min}^{-2} / t_{scatt} to travel diffusively across a decorrelation length k_{min}^{-1} of the turbulence and gain energy ~ δu^2 p...

⇒ for $t_{scatt} >> k_{min}^{-1}$, it takes a time ~ t_{scatt} to decorrelate the trajectory and gain an energy ~ δu^2 p ...

Non-resonant acceleration in relativistic turbulence

Generalisation to broadband turbulent spectrum:

→ particle sees large-scale modes ($t_{scatt} \ll k^{-1}$), and small-scale modes ($t_{scatt} \gg k^{-1}$) …



$$D_{pp}^{\rm non-res.} \simeq p^2\,\langle\delta u^2\rangle \begin{cases} k_{\rm min}\,\left(t_{\rm scatt}k_{\rm min}\right)^{q-2} & \left(k_{\rm max}^{-1}\lesssim t_{\rm scatt}\lesssim k_{\rm min}^{-1}\right) \\ t_{\rm scatt}^{-1} & \left(k_{\rm min}^{-1}\lesssim t_{\rm scatt}\right)\end{cases}$$

[here, δu symbolizes the compressive/shear/vorticity/acceleration combination]

Demidem, ML, Casse 19

Hints for the scattering timescale:

→ line wandering implies $\delta\theta \sim \pm\delta B/B$ over a coherence cell k_{min}^{-1}, so :

$$t_{\rm scatt} \sim k_{\rm min}^{-1} \quad \text{at } \delta B\sim B$$

→ at low turbulence amplitudes, hints from quasilinear theory …
(must include realistic account of turbulence anisotropy, resonance broadening etc.)

Summary

A generalized scheme for Fermi acceleration:

→ follow the particle trajectory in mixed phase space coordinates:

$\mathbf{x}^{\mu}(\tau)$ lives in the lab (global) reference frame

$\hat{\mathbf{p}}^{\hat{\mathbf{a}}}(\tau)$ lives in a locally inertial frame that is comoving with the electromagnetic structure, so that **E** vanishes there…

→ non-perturbative calculation of energy gain as a function of scattering time t_{scatt} … in possibly non-trivial flow structures / non-trivial geometries…

→ and its application to relativistic turbulence:

- **full characterization of non-resonant effects (compressive / shear / vorticity …)**
- **non-resonant acceleration timescale**

$$t_{\rm acc} \sim p^{\epsilon}\,\langle\delta u^2\rangle^{-1}\eta^{-1}\,\left(k_{\rm min}c\right)^{-1}$$